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MATHEMATICS

Generalization of the Selfish Parking Problem

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Abstract—The work is devoted to the study of a new model of random filling of a segment of large length with intervals of smaller length. Two new formulations of the problem are considered. In the first case, we consider a model in which unit intervals are placed on the segment in such a way that with each subsequent placement of an interval, there should be a free space (from the left and from the right) of length no smaller than a fixed value. In the second model, intervals with a length of 2 are located randomly and no two intervals should neighbor each other. In both cases, the behavior of the average number of located intervals depending on the length of the filled segment is investigated.

Keywords: random filling, “parking” problem, asymptotic behavior

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INTRODUCTION

For the first time, the problem of the random filling of a segment was presented by A. Rényi in [1]. The interval $(t, t + 1)$ is randomly placed on the segment $[0, x]$ for some fixed $x > 1$, thereby dividing the initial segment into two segments: $[0, t]$ and $[t + 1, x]$. If any of them has a length of smaller than 1, it is excluded from further consideration. The remaining segments continue to be filled in accordance with the above rule. The term “randomly” means that t is a random variable uniformly distributed on $[0, x - 1]$, which is independent of other analogous random variables. This process ends at the moment when no segments with a length of at least 1, remain. Then the total number of intervals located in the initial segment is calculated and is denoted by N_x. For $0 \leqslant x < 1$, N_x is taken to be zero.

Rényi in [1] shows that for any $n \geqslant 1$,

$$E\{N_x\} = \lambda x + \lambda - 1 + o(x^{-n}) \quad (x \mapsto +\infty). \tag{1}$$

For the constant λ, the following expression is obtained:

$$\lambda = \int_0^\infty e^{-2\int_0^t \frac{1-e^{-u}}{u} du} dt. \tag{2}$$

In [2], which is devoted to the discrete selfish parking problem for the mathematical expectation of the number of placed unit intervals, it is found that

$$EX_n = \frac{2n-1}{3} \quad \text{for} \quad n \geq 2 \quad \text{and} \quad EX_n = 0 \quad \text{for} \quad n < 2. \tag{3}$$

Other models of the parking problem are considered in [3–7].

In this paper, we consider two new models of the parking problem.

MODEL I

We suppose n and k are non-negative integers, $i = 1, 2, \ldots, n - 1$. We randomly place a unit interval on the segment $[0, n]$ according to the following rule. If $n \leq k$, then we say that the unit interval is not placed and the segment $[0, n]$ remains unfilled. Otherwise, we place the interval $(i, i + 1)$ on the segment $[0, n]$, where i is a random variable that takes the values of $0, 1, \ldots, n - 1$ with equal probability in such a way that to the right or left of the interval there is a free space no smaller than k. After placing the first interval, two unoccupied segments are formed, namely, $[0, i]$ and $[i + 1, n]$, which in turn are filled independently of each other according to the same rule. When the lengths of all unoccupied segments become no larger than k, the process of filling the segment stops. We suppose X_n is the number of unit intervals placed on the segment $[0, n]$.

Theorem 1. *For the random variable X_n defined above, the following equalities hold*:

$$\begin{aligned} EX_n &= 0 \quad \text{for} \quad 0 \le n \le k, \\ EX_n &= H_{n-k} \quad \text{for} \quad k < n \le 2k, \end{aligned} \tag{4}$$

$$EX_n = \frac{H_k + 1}{2k + 1}(n+1) - 1 \quad \text{for} \quad n > 2k, \tag{5}$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

Proof. 1. If $0 \le n \le k$, then according to the rule of filling, $X_n = 0$ and $EX_n = 0$.

2. If $k < n \le 2k$, then according to the rule of filling, the left end of the first placed interval cannot take values from $[n - k, k)$. Suppose the first interval occupies the place $(i, i + 1)$. We assume that $X_{n,\,i}$ is the number of unit intervals placed on the segment $[0, n]$, provided that the left end of the first interval occupies the place i. Then,

$$X_{n,i} = X_i + X_{n-i-1} + 1. \tag{6}$$

We introduce the notation $E_n = EX_n$. Taking into account equality (6) and the fact that the random variable i with equal probability can take values of $0, 1, \ldots, n - k - 1, k, k + 1, \ldots, n - 1$, we obtain

$$\begin{aligned} E_n &= \frac{1}{2(n-k)}\left(\sum_{i=0}^{n-k-1} E_i + \sum_{i=k}^{n-1} E_i\right) + \frac{1}{2(n-k)}\left(\sum_{i=0}^{n-k-1} E_{n-i-1} + \sum_{i=k}^{n-1} E_{n-i-1}\right) + 1 \\ &= \frac{1}{(n-k)}\left(\sum_{i=0}^{n-k-1} E_i + \sum_{i=k}^{n-1} E_i\right) + 1. \end{aligned}$$

It follows from item 1 that $E_i = 0$ for all $i = 0, 1, \ldots, k$. Then we obtain

$$E_n = \frac{1}{(n-k)} \sum_{i=k+1}^{n-1} E_i + 1. \tag{7}$$

In order to solve this equation, we also introduce the notation

$$S_n = \sum_{j=k+1}^{n} E_j. \tag{8}$$

We have

$$E_n = S_n - S_{n-1}. \tag{9}$$

Taking into account (7), (8), and (9), we obtain the recurrence relation

$$S_n = \frac{n-k+1}{n-k} S_{n-1} + 1, \tag{10}$$

or

$$S_{n+1} = c_n S_n + 1, \tag{11}$$

where $c_n = \dfrac{n-k+2}{n-k+1}$. Thus, taking into account equality $S_{k+1} = 1$, we have

$$S_{n+1} = c_n S_n + 1 = c_n(c_{n-1}S_{n-1} + 1) + 1 = \ldots = c_n c_{n-1} \ldots c_{k+1} + c_n c_{n-1} \ldots c_k + \ldots + c_n + 1,$$

or, since $c_n = \dfrac{n-k+2}{n-k+1}$, we can write

$$S_{n+1} = \sum_{i=k+1}^{n} \prod_{j=i}^{n} c_j + 1 = \sum_{i=k+1}^{n} \prod_{j=i}^{n} \frac{j-k+2}{j-k+1} + 1$$
$$= \sum_{i=k+1}^{n} \frac{n-k+2}{i-k+1} + 1 = (n-k+2) \sum_{i=2}^{n-k+1} \frac{1}{i} + 1.$$

Taking into account (7), we obtain

$$E_n = \frac{1}{(n-k)} S_{n-1} + 1 = \sum_{i=2}^{n-k-1} \frac{1}{i} + \frac{1}{n-k} + 1 = \sum_{i=1}^{n-k} \frac{1}{i} = H_{n-k},$$

where H_{n-k} is the partial sum of the first $n-k$ terms of a harmonic series.

3. We suppose $n > 2k$. Taking into account equality (6) and the fact that i is a random variable with equal probability taking values of 0, 1, ..., $n-1$, we obtain the equality

$$E_n = \frac{1}{n} \sum_{i=0}^{n-1} E_i + \frac{1}{n} \sum_{i=0}^{n-1} E_{n-i-1} + 1 = \frac{2}{n} \sum_{i=0}^{n-1} E_i + 1.$$

We introduce the notation $c = \sum_{i=0}^{2k} E_i$. Then the following equality holds:

$$E_n = \frac{2}{n} \sum_{i=0}^{n-1} E_i + 1 = \frac{2}{n}\left(c + \sum_{i=2k+1}^{n-1} E_i\right) + 1.$$

We assume that $T_n = \sum_{i=0}^{n} E_i$. Then $E_n = T_n - T_{n-1}$ and $T_{2k} = c$.

On the other hand, taking into account the equality

$$E_n = \frac{2}{n} \sum_{i=0}^{n-1} E_i + 1 = \frac{2}{n} T_{n-1} + 1,$$

we obtain

$$T_n = \frac{n+2}{n} T_{n-1} + 1$$

or

$$T_{n+1} = \frac{n+3}{n+1} T_n + 1.$$

If we introduce the notation $a_n = \dfrac{n+3}{n+1}$ and take into account the equality $T_{2k} = c$, we obtain

$$T_{n+1} = a_n T_n + 1 = a_n(a_{n-1} T_{n-1} + 1) + 1 = a_n(a_{n-1}(a_{n-2} T_{n-2} + 1) + 1) + 1 = \ldots$$
$$= a_n(a_{n-1}(a_{n-2}(\ldots a_{2k+1}(a_{2k} T_{2k} + 1) + 1)\ldots) + 1 =$$
$$= a_n a_{n-1} \ldots a_{2k} c + a_n a_{n-1} \ldots a_{2k+1} + \ldots + a_n + 1.$$

If we introduce the notation $p_n = \prod_{i=1}^{n} a_i$, then we obtain

$$T_{n+1} = c \frac{p_n}{p_{2k-1}} + \frac{p_n}{p_{2k}} + \ldots + \frac{p_n}{p_n} = c \frac{p_n}{p_{2k-1}} + \sum_{i=2k}^{n} \frac{p_n}{p_i}.$$

Using the last equality, we have

$$E_n = T_n - T_{n-1} = \frac{p_{n-1}}{p_{n-1}} + \sum_{i=2k}^{n-2} \frac{p_{n-1} - p_{n-2}}{p_i} + c \frac{p_{n-1} - p_{n-2}}{p_{2k-1}}.$$

We note that

$$p_n = \prod_{i=1}^{n} \frac{i+3}{i+1} = \frac{(n+3)(n+2)}{6}.$$

Hence,

$$p_{n-1} - p_{n-2} = \frac{(n+2)(n+1)}{6} - \frac{(n+1)n}{6} = \frac{n+1}{3}.$$

Thus, from the last equalities it follows that

$$E_n = 1 + \sum_{i=2k}^{n-2} \frac{p_{n-1} - p_{n-2}}{p_i} + c\frac{p_{n-1} - p_{n-2}}{p_{2k-1}} = 1 + \frac{n+1}{3}\sum_{i=2k}^{n-2}\frac{1}{p_i} + c\frac{n+1}{3}\frac{1}{p_{2k-1}}$$

$$= 1 + \frac{(n+1)}{3}\sum_{i=2k}^{n-2}\frac{6}{(i+3)(i+2)} + c\frac{n+1}{3}\frac{6}{(2k+2)(2k+1)}$$

$$= 1 + 2(n+1)\sum_{i=2k}^{n-1}\left(\frac{1}{i+2} - \frac{1}{i+3}\right) + c\frac{n+1}{(k+1)(2k+1)}$$

$$= 1 + 2(n+1)\left(\frac{1}{2k+2} - \frac{1}{n+1}\right) + c\frac{n+1}{(k+1)(2k+1)} = 1 + \frac{n+1}{k+1} - 2 + c\frac{n+1}{(k+1)(2k+1)}$$

$$= \frac{(n+1)(2k+1) + c(n+1)}{(k+1)(2k+1)} - 1 = \frac{2k+1+c}{(k+1)(2k+1)}(n+1) - 1.$$

Taking into account the result of item 2, we find the constant c:

$$c = \sum_{i=0}^{2k} E_i = S_{2k} = (k+1)H_k - k,$$

where $H_n = \sum_{i=1}^{n}\frac{1}{i}$.

Thus, for all $n > 2k$, we obtain the expression for E_n

$$E_n = \frac{k+1+(k+1)H_k}{(k+1)(2k+1)}(n+1) - 1 = \frac{H_k+1}{2k+1}(n+1) - 1.$$

The theorem is fully proved.

Remark. For $k = 1$, our problem represents the problem of selfish parking discussed in [2]; here, we obtain $E_n = \frac{2n-1}{3}$. For $k = 2$, we have $E_n = \frac{n-1}{2}$; for $k = 3$, $E_n = \frac{17n-25}{42}$; and for $k = 4$, $E_n = \frac{37n-10}{108}$.

MODEL II

We suppose n is a non-negative integer. The process of filling of the segment $[0, n]$ with intervals of the length 2 occurs according to the following rule. If $n < 4$, then we say that an interval is not placed. Otherwise, we place the interval $(i, i+2)$ on the segment $[0, n]$, where i is a random variable that takes values of $1, 2, \ldots, n-3$ with equal probability (the left and right ends of the placed interval should be at a distance of at least 1 from the boundaries of the filled segment). After placing the first interval, we obtain two segments: $[0, i]$ and $[i+2, n]$, which are subsequently filled with intervals of the length 2 according to the same rule independently of each other. When the lengths of all unfilled segments are smaller than 4, the filling process stops and the total number of placed intervals is calculated; this number we denote by Y_n.

Theorem 2. *For the random variable Y_n, the following relation holds*:

$$\frac{EY_n}{n} \xrightarrow[n\to\infty]{} \lambda, \tag{12}$$

where $\lambda = e^{-3}\int_0^1 e^{t^2+2t}dt \approx 0.274551.$

Proof. We suppose that the first interval occupies the place $(i, i+2)$, where $i = 1, 2, \ldots, n-3$, and denote by $Y_{n,i}$ the number of intervals placed on the segment $[0, n]$, provided that the left end of the first interval occupies the place i. In this case, the number of intervals placed on the segments $[0, i]$ and $[i+2, n]$ are Y_i and Y_{n-i-2}, respectively. Then the following equality holds:

$$Y_{n,i} = Y_i + Y_{n-i-2} + 1.$$

We introduce the notation $E_n = EY_n$. Taking into account the fact that i is a random variable with equal probability with values of 0, 1, ..., $n-3$, we obtain the recurrence relation

$$E_n = \frac{1}{n-3}\sum_{k=1}^{n-3} E_k + \frac{1}{n-3}\sum_{k=1}^{n-3} E_{n-k-2} + 1 = \frac{2}{n-3}\sum_{k=1}^{n-3} E_k + 1 \tag{13}$$

with the initial data

$$E_1 = E_2 = E_3 = 0, \quad E_4 = E_5 = E_6 = 1, \quad E_7 = \frac{3}{2}.$$

We also introduce the notation $Z_n = \sum_{k=0}^{n} E_k$. Then we have

$$E_n = Z_n - Z_{n-1}. \tag{14}$$

We present Eq. (13) and the initial data using the new notation:

$$Z_n - Z_{n-1} = \frac{2}{n-3} Z_{n-3} + 1,$$

$$Z_0 = Z_1 = Z_2 = Z_3 = 0, \quad Z_4 = 1.$$

We multiply the resulting equation by $(n-3)$ and obtain

$$(n-3)Z_n - (n-3)Z_{n-1} - 2Z_{n-3} - (n-3) = 0.$$

Next, we multiply it by t^n and sum from 3 to ∞:

$$\sum_{n=3}^{\infty}(n-3)Z_n t^n - \sum_{n=3}^{\infty}(n-3)Z_{n-1}t^n - 2\sum_{n=3}^{\infty} Z_{n-3}t^n - \sum_{n=3}^{\infty}(n-3)t^n = 0. \tag{15}$$

We suppose $P(t) = \sum_{n=0}^{\infty} Z_n t^n$ is a generating function of the sequence Z_n. Then,

$$P'(t) = \sum_{n=1}^{\infty} nZ_n t^{n-1}.$$

We transform the first term of Eq. (15) with account for $Z_0 = Z_1 = Z_2 = Z_3 = 0$. We have

$$\sum_{n=3}^{\infty}(n-3)Z_n t^n = \sum_{n=3}^{\infty} nZ_n t^n - 3\sum_{n=3}^{\infty} Z_n t^n = t\sum_{n=0}^{\infty} nZ_n t^{n-1} - 3\sum_{n=0}^{\infty} Z_n t^n = tP'(t) - 3P(t).$$

Analogously, we transform the remaining terms:

$$\sum_{n=3}^{\infty}(n-3)Z_{n-1}t^n = \sum_{n=1}^{\infty}(n-1)Z_{n-1}t^n - 2\sum_{n=1}^{\infty} Z_{n-1}t^n = t^2P'(t) - 2tP(t),$$

$$\sum_{n=3}^{\infty} Z_{n-3}t^n = t^3P(t),$$

$$\sum_{n=3}^{\infty}(n-3)t^n = \sum_{n=3}^{\infty} nt^n - 3\sum_{n=3}^{\infty} t^n = t\left(\sum_{n=3}^{\infty} t^n\right)' - \frac{3t^3}{1-t}$$

$$= t\left(\frac{t^3}{1-t}\right)' - \frac{3t^3}{1-t} = t\left(\frac{3t^2}{1-t} + \frac{t^3}{(1-t)^2}\right) - \frac{3t^3}{1-t} = \frac{3t^3}{1-t} + \frac{t^4}{(1-t)^2} - \frac{3t^3}{1-t} = \frac{t^4}{(1-t)^2}.$$

Thus, Eq. (15) takes the form

$$P'(t)(t-t^2)+P(t)(-2t^3+2t-3)-\frac{t^4}{(1-t)^2}=0. \tag{16}$$

The resulting Eq. (16) is a linear inhomogeneous first-order differential equation for the generating function $P(t)$. In order to find its solution, we consider first the following homogeneous equation:

$$R'(t)(t-t^2)+R(t)(-2t^3+2t-3)=0.$$

We have

$$(\ln R'(t))'=\frac{R'(t)}{R(t)}=\frac{2t^3-2t+3}{t-t^2}=-2t-2+\frac{3}{t}+\frac{3}{1-t},$$

$$\ln R(t)=-t^2-2t+3\ln t-3\ln(1-t)+c_1, \quad c_1\in\mathbf{R}.$$

It follows that

$$R(t)=\frac{c_2t^3}{(1-t)^3}e^{-t^2-2t}, \quad c_2\in\mathbf{R}, \quad c_2>0.$$

Consider Eq. (15). We suppose $P(t)=R(t)T(t)$. Then, substituting our expression into (16) and taking into account that $R(t)$ is a solution of the homogeneous equation, we obtain

$$(t-t^2)R(t)T'(t)-\frac{t^4}{(1-t)^2}=0,$$

$$T'(t)=\frac{t^3}{(1-t)^3}\frac{1}{R(t)}=c_2e^{t^2+2t},$$

$$T(t)=c_2\int_0^t e^{\tau^2+2\tau}d\tau+c_3, \quad c_3\in\mathbf{R}.$$

Thus, we can present the general solution of Eq. (16)

$$P(t)=R(t)T(t)=\frac{t^3}{(1-t)^3}e^{-t^2-2t}\left(c_3+\int_0^t e^{\tau^2+2\tau}d\tau\right).$$

Since $P(t)$ is a generating function of Z_n and $Z_0=Z_1=Z_2=Z_3=0$, we have

$$\left.\frac{P(t)}{t^3}\right|_{t=0}=Z_3=0.$$

It follows that $c_3=0$ and

$$P(t)=\frac{t^3}{(1-t)^3}e^{-t^2-2t}\int_0^t e^{\tau^2+2\tau}d\tau.$$

Consider equality (14). We multiply both sides of this equality by t^n and sum from 1 to ∞:

$$\sum_{n=1}^{\infty}E_nt^n=\sum_{n=1}^{\infty}Z_nt^n-\sum_{n=1}^{\infty}Z_{n-1}t^n.$$

If $Q(t)=\sum_{n=1}^{\infty}E_nt^n$ is a generating function of the sequence E_n, then

$$Q(t)=P(t)-tP(t)=(1-t)P(t)=\frac{t^3}{(1-t)^2}e^{-t^2-2t}\int_0^t e^{\tau^2+2\tau}d\tau.$$

We suppose $f(t)=t^2e^{-t^2-2t}\int_0^t e^{\tau^2+2\tau}d\tau$ and assume that $f(t)=\sum_{n=0}^{\infty}b_nt^n$.

We denote $f(1) = e^{-3}\int_0^1 e^{\tau^2+2\tau}d\tau$ by λ. Then,

$$Q(t) = \frac{t}{(1-t)^2} f(t) = \sum_{n=0}^{\infty} nt^n \sum_{n=0}^{\infty} b_n t^n.$$

From the last equality we have

$$E_n = \sum_{k=0}^{n} b_k(n-k) = \left(\sum_{k=0}^{n} b_k\right) n - \sum_{k=0}^{n} b_k k.$$

We note that

$$\sum_{k=0}^{n} b_k \xrightarrow[n\to\infty]{} \sum_{k=0}^{\infty} b_k = f(1) = \lambda.$$

We obtain

$$\sum_{k=0}^{n} b_k k \xrightarrow[n\to\infty]{} \sum_{k=0}^{\infty} b_k k = f'(1) = \frac{d}{dt}\left(t^2 e^{-(t^2+2t)}\int_0^t e^{\tau^2+2\tau}d\tau\right)\Bigg|_{t=1}$$
$$= \left((2t - 2t^3 - 2t^2)e^{-(t^2-2t)}\int_0^t e^{\tau^2+2\tau}d\tau + t^2\right)\Bigg|_{t=1} = 1 - 2\lambda.$$

Hence,

$$\frac{E_n}{n} \xrightarrow[n\to\infty]{} \lambda \approx 0.274551.$$

Thus, Theorem 2 is proved.

REFERENCES

1. A. Rényi, “On a one-dimensional problem concerning space-filling,” Publ. Math. Inst. Hung. Acad. Sci. **3**, 109–127 (1958).
2. S. M. Ananjevskii and N. A. Kryukov, “The problem of selfish parking,” Vestn. St. Petersburg Univ.: Math. **51**, 322–326 (2018). https://doi.org/10.3103/S1063454118040039
3. S. M. Ananjevskii and N. A. Kryukov, “On asymptotic normality in one generalization of the Renyi problem,” Vestn. St. Petersburg Univ.: Math. **52**, 227–233 (2019). https://doi.org/10.1134/S1063454119030026
4. M. P. Clay and N. J. Simanyi, “Rényi’s parking problem revisited. Mathematical theory” (2014). ArXiv 1406.1781v1 [math.PR]
5. L. Gerin, “The Page-Rényi parking process” (2014). ArXiv 1411.8002v1 [math.PR]
6. S. M. Ananjevskii, “Generalizations of the parking problem,” Vestn. St. Petersburg Univ.: Math. **49**, 299–304 (2016). https://doi.org/10.3103/S1063454116040026
7. S. M. Anan’evskii, “The "parking” problem for segments of different length,” J. Math. Sci. **93**, 259–264 (1999). https://doi.org/10.1007/BF02364808

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