

Dimensional transmutation and nonconventional scaling behavior in a model of self-organized criticality

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This paper addresses two unusual scaling regimes (types of critical behavior) predicted by the field-theoretic renormalization group analysis for a self-organized critical system with turbulent motion of the environment. The system is modeled by the anisotropic stochastic equation for a “running sandpile” introduced by Hwa and Kardar in [*Phys. Rev. Lett.* **62**, 1813 (1989)]. The turbulent motion is described by the isotropic Kazantsev–Kraichnan “rapid-change” velocity ensemble for an incompressible fluid. The original Hwa–Kardar equation allows for independent scaling of the spatial coordinates x_{\parallel} (the coordinate along the preferred dimension) and x_{\perp} (the coordinates in the orthogonal subspace to the preferred direction) that becomes impossible once the isotropic velocity ensemble is coupled to the equation. However, it is found that one of the regimes of the system’s critical behavior (the one where the isotropic turbulent motion is irrelevant) recovers the anisotropic scaling through “dimensional transmutation.” The latter manifests as a dimensionless ratio acquiring nontrivial canonical dimension. The critical regime where both the velocity ensemble and the nonlinearity of the Hwa–Kardar equation are relevant simultaneously is also characterized by “atypical” scaling. While the ordinary scaling with fixed IR irrelevant parameters is impossible in this regime, the “restricted” scaling where the times, the coordinates, and the dimensionless ratio are scaled becomes possible. This result brings to mind scaling hypotheses modifications (Stell’s weak scaling or Fisher’s generalized scaling) for systems with significantly different characteristic scales.

Keywords: Self-organized criticality; critical phenomena; turbulence; renormalization group.

1. Introduction

The Hwa–Kardar differential stochastic equation^{1,2} is a continuous model of self-organized criticality. Unlike equilibrium systems that require “fine-tuning” of control

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parameters (e.g. critical temperature in second-order phase transitions) to become critical,³ systems with self-organized criticality arrive at critical state in the natural course of their evolution.⁴ Such systems are widespread^{5–8} with examples ranging from neural systems to online social networks with many other systems in between.

The Hwa–Kardar model is expected to describe the infrared (IR), i.e. large-scale and long-time asymptotic behavior of the system with self-organized criticality and to yield the power laws that characterize the behavior.

The system is an anisotropic “running” sandpile with an average flat profile that has a constant tilt along the preferred direction. The new sand keeps entering the system resulting in avalanches that cause other sand to exit the system. Eventually, the system arrives at a steady state with a self-similar profile that is anisotropic (due to the tilt).

The height of the profile is described by the scalar field $h(x) = h(t, \mathbf{x})$ where t and a d -dimensional \mathbf{x} stand for the time-space coordinates. Note that $h(x)$ is measured from the average tilted profile, i.e. it is a deviation of the profile height from its average value. The tilt corresponds to the preferred direction defined by the unit constant vector \mathbf{n} . Any vector \mathbf{q} can be decomposed as $\mathbf{q} = \mathbf{q}_\perp + \mathbf{n}q_\parallel$ where $(\mathbf{q}_\perp \cdot \mathbf{n}) = 0$. The spatial derivative $\partial = \partial/\partial x_i, i = 1, \dots, d$, is then “split” into two derivatives: a derivative $\partial_\perp = \partial/\partial x_i, i = 1, \dots, (d - 1)$, in the subspace orthogonal to the vector \mathbf{n} , and a one-dimensional derivative $\partial_\parallel = (\mathbf{n} \cdot \partial)$.

The stochastic differential equation that describes the evolution of the system is

$$\partial_t h = \nu_{\perp 0} \partial_\perp^2 h + \nu_{\parallel 0} \partial_\parallel^2 h - \partial_\parallel h^2/2 + f. \tag{1}$$

Here, it is assumed that $\partial_t = \partial/\partial t$ and $\partial_\perp^2 = (\partial_\perp \cdot \partial_\perp)$. The factors $\nu_{\parallel 0}$ and $\nu_{\perp 0}$ stand for the diffusivity coefficients, and $f(x)$ is a Gaussian random noise with a zero mean and the correlation function

$$\langle f(x)f(x') \rangle = D_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \quad D_0 > 0. \tag{2}$$

Critical behavior of a system is sensitive to various external disturbances including, in particular, motion of the environment.^{9–11} Thus, it is desirable to consider the effect of the motion on a system with self-organized criticality.

The coupling with the velocity field $\mathbf{v}(x)$ is achieved through a “minimal substitution” $\partial_t \rightarrow \nabla_t = \partial_t + (\mathbf{v} \cdot \partial)$.

Let us ascribe the following statistic to the velocity field $\mathbf{v}(x)$:

$$\begin{aligned} \langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}') \rangle &= \delta(t - t') D_{ij}(\mathbf{x} - \mathbf{x}'), \\ D_{ij}(\mathbf{r}) &= B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P_{ij}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}), \quad B_0 > 0. \end{aligned} \tag{3}$$

This is the Kazantsev–Kraichnan “rapid-change” velocity ensemble.¹² Here, $k \equiv |\mathbf{k}|$ stands for the wave number while $P_{ij}(\mathbf{k})$ is the transverse projector defined in a standard way: $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$. The velocity field is, thus, incompressible, i.e. $(\partial \cdot \mathbf{v}) = 0$. The IR regularization is provided by the sharp cutoff $k > m$. The parameter ξ is nonnegative and smaller than 2.

In this paper, we apply field-theoretic renormalization group (RG) analysis to problem (1)–(3) and find out that it is the coupling of the anisotropic Eq. (1) and the isotropic velocity ensemble (3) that causes emergence of “nonconventional” scaling behavior. In RG analysis, stochastic problem (1)–(3) is substituted with a field theory. Critical exponents from the power laws that describe the IR asymptotic behavior of the system can be found from RG equations for the field theory. In particular, each IR attractive fixed point of the RG equations is connected to a regime of asymptotic behavior, i.e. to a universality class. It was shown in Ref. 13 that the system’s critical behavior is divided into four universality classes. Two of them correspond to ordinary isotropic scaling in which IR irrelevant parameters are kept fixed and the coordinates x_{\parallel} and \mathbf{x}_{\perp} are not scaled independently. The other two regimes, however, involve nonconventional or anisotropic scaling. The aim of this paper is to explore these two regimes.

2. Renormalization and Scaling of Pure Hwa–Kardar Model

Let us begin by deriving scaling of the Hwa–Kardar equation without the turbulent motion of the environment. This “pure” scaling was first analyzed by RG in its Wilsonian form in Refs. 1 and 2. We reproduce those results here (expressed in terms of the field-theoretic RG) so that we can compare them with the scaling regimes of the model with turbulent advection.

Instead of the stochastic problem (1) and (2), one can consider the equivalent field theory³ with the doubled set of fields $\Phi = \{h, h'\}$ and the action

$$S(\Phi) = \frac{1}{2} h' D_0 h' + h' \left(-\partial_t h + \nu_{\parallel 0} \partial_{\parallel}^2 h + \nu_{\perp 0} \partial_{\perp}^2 h - \frac{1}{2} \partial_{\parallel} h^2 \right). \quad (4)$$

The integrations over the space–time coordinates $x = \{t, \mathbf{x}\}$ and the summations over the vector indices are implied throughout.

As model (1), (2) is anisotropic, it involves two independent spatial scales L_{\parallel} and L_{\perp} instead of a single scale L . Generally, a quantity’s canonical dimension in dynamical models is described by the momentum dimension d_F^k and the frequency dimension d_F^{ω} related to the spatial scale L and the temporal scale T . Since there are two independent spatial scales, an arbitrary quantity F is described by three canonical dimensions:

$$[F] \sim [T]^{-d_F^{\omega}} [L_{\parallel}]^{-d_F^k} [L_{\perp}]^{-d_F^k}. \quad (5)$$

Table 1 contains the canonical dimensions of all fields and parameters for the theory (4) obtained from the condition that each term of the action (4) be dimensionless. We denoted a total canonical dimension d_F as $d_F = d_F^k + 2d_F^{\omega}$ and a total momentum dimension d_F^k as $d_F^k = d_{\parallel}^k + d_{\perp}^k$. The coupling constant g_0 is defined as $D_0 = g_0 \nu_{\perp 0}^{3/2} \nu_{\parallel 0}^{3/2}$.

We can see from Table 1 that $g_0 \sim \Lambda^{\varepsilon}$ where Λ is a characteristic ultraviolet momentum scale and $\varepsilon = 4 - d$, therefore, the logarithmic dimension is $d = 4$. Let us

Table 1. Canonical dimensions for the theory (4); $\varepsilon = 4 - d$.

F	h'	h	D_0	$\nu_{\parallel 0}$	$\nu_{\perp 0}$	u_0	g_0	g	μ, m, Λ
d_F^ω	-1	1	3	1	1	0	0	0	0
d_F^\parallel	2	-1	-3	-2	0	-2	0	0	0
d_F^\perp	$d - 1$	0	$1 - d$	0	-2	2	ε	0	1
d_F^k	$d + 1$	-1	$-d - 2$	-2	-2	0	ε	0	1
d_F	$d - 1$	1	$4 - d$	0	0	0	ε	0	1

also define the parameter $u_0 = \nu_{\parallel 0}/\nu_{\perp 0}$ that we will need later. Note that it possesses nontrivial momentum canonical dimensions d_u^\parallel and d_u^\perp while its total canonical dimension d_u is zero.

Canonical dimensions' analysis augmented with symmetry analysis reveals that the model (4) is multiplicatively renormalizable. It involves the sole nontrivial renormalization constant Z_{ν_1} . The only parameters that require renormalization are $\nu_{\parallel 0}$ and g_0 . They are related to their renormalized counterparts in the following way (μ is the renormalization mass):

$$\nu_{\parallel 0} = \nu_{\parallel} Z_{\nu_1}, \quad g_0 = g \mu^\varepsilon Z_g. \tag{6}$$

The canonical differential equations for a renormalized Green function $G^R = \langle \Phi \dots \Phi \rangle$ read

$$\left(\sum_i d_i^\omega \mathcal{D}_i - d_G^\omega \right) G^R = 0, \tag{7}$$

$$\left(\sum_i d_i^\perp \mathcal{D}_i - d_G^\perp \right) G^R = 0, \quad \left(\sum_i d_i^\parallel \mathcal{D}_i - d_G^\parallel \right) G^R = 0.$$

The index i enumerates the arguments of G^R which are $\omega, k_\perp, k_\parallel, \mu, \nu_\perp$ and ν_\parallel (or u depending on our choice of parameters). The operator \mathcal{D}_a is $\mathcal{D}_a = a \partial_a$ for any a .

To calculate the critical exponents (or dimensions) that stand in scaling power laws, we need to supplement Eq. (7) with the differential RG equation

$$(\mathcal{D}_\mu + \beta_g \partial_g - \gamma_{\nu_1} \mathcal{D}_{\nu_1} - \gamma_{\nu_\perp} \mathcal{D}_{\nu_\perp} - \gamma_G) G^R = 0. \tag{8}$$

Here, γ and β are RG functions (anomalous dimensions and a β -function, respectively) for the parameters of the system and coupling constant g . Note that quantities that are not renormalized have vanishing anomalous dimensions, i.e. $\gamma_{\nu_\perp} = 0$ in the model (4), but we keep this term for the future.

Universality classes of asymptotic (critical) behavior are determined by fixed points of the RG equations, i.e. by the zeroes of the β functions, $\beta(g^*) = 0$. Real parts of all the eigenvalues λ_i of the matrix $\Omega_{ij} = \partial_{g_i} \beta_{g_j}$ (here $g = \{g_i\}$ is a full set of the charges) must be positive for a fixed point to be IR attractive.

The substitution $g \rightarrow g^*$ and, hence, $\gamma_F \rightarrow \gamma_F^*$ turns Eq. (8) into equation with constant coefficients, i.e. into equation of the same type as Eq. (7). Each of these

equations describes a certain independent scaling behavior of the function G^R , in which some of its variables are scaled and some are kept fixed. A parameter is scaled if the corresponding derivative enters the differential operator; otherwise the parameter is fixed.

We are interested in the critical scaling behavior where the frequencies and momenta (or, equivalently, times and coordinates) are scaled, while the IR irrelevant parameters (namely, μ , ν_\perp and ν_\parallel) are kept fixed. Thus, we combine all these equations to exclude the derivatives with respect to all the IR irrelevant parameters and arrive at the critical scaling equation for a given fixed point:

$$(\mathcal{D}_{k_\perp} + \mathcal{D}_{k_\parallel} \Delta_\parallel + \Delta_\omega \mathcal{D}_\omega - \Delta_G) G^R = 0 \tag{9}$$

with $\Delta_\parallel = 1 + \gamma_{\nu_\parallel}^*/2$ and $\Delta_\omega = 2 - \gamma_{\nu_\perp}^*$.

As we will see below, such an exclusion is not always possible: it requires some balance between the numbers of IR relevant and IR irrelevant parameters and the number of independent scaling equations.

The model (4) has two fixed points. The Gaussian (free) fixed point with $g^* = 0$ is IR attractive for $\varepsilon < 0$; the corresponding critical dimensions coincide with canonical ones. The nontrivial fixed point with $g^* = 32\varepsilon/9 + O(\varepsilon^2)$ is IR attractive for $\varepsilon > 0$. The corresponding canonical dimensions read

$$\Delta_h = 1 - \varepsilon/3, \quad \Delta_{h'} = 3 - \varepsilon/3, \quad \Delta_\omega = 2, \quad \Delta_\parallel = 1 + \varepsilon/3. \tag{10}$$

3. Renormalization and Fixed Points of the Model with Turbulent Advection

As was stated above, inclusion of turbulent advection in the Hwa–Kardar model is achieved by replacing ∂_t with ∇_t in the action (4) and adding Gaussian averaging over the velocity ensemble with the correlation function (3). This leads to the following action functional:

$$S(\Phi) = \frac{1}{2} h' D_0 h' + h' \left(-\nabla_t h + \nu_{\parallel 0} \partial_\parallel^2 h + \nu_{\perp 0} \partial_\perp^2 h - \frac{1}{2} \partial_\parallel h^2 \right) + S_v, \tag{11}$$

$$S_v = -\frac{1}{2} \int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t, \mathbf{x}) D_{ij}^{-1}(\mathbf{x} - \mathbf{x}') v_j(t, \mathbf{x}'). \tag{12}$$

Here, $D_{ij}^{-1}(\mathbf{x} - \mathbf{x}')$ is the kernel of the inverse operator D_{ij}^{-1} for the integral operator D_{ij} from (3) as S_v describes Gaussian averaging over the field \mathbf{v} with the correlation function (3).

Since that velocity ensemble is isotropic, it is no longer possible to define two independent spatial scales in this model. Thus,

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k} \tag{13}$$

and $d_F = d_F^k + 2d_F^\omega$. Canonical dimensions of the fields and parameters of the model are presented in Table 2. We note that for the quantities that are present in the pure

Table 2. Canonical dimensions for the theory (12); $\varepsilon = 4 - d$.

F	h'	h	D_0	$\nu_{\parallel 0}$	$\nu_{\perp 0}$	g_0	v	B_0	x_0	u_0, u	g, x	μ, m, Λ
d_F^ω	-1	1	3	1	1	0	1	0	0	0	0	0
d_F^k	$d+1$	-1	$-2-d$	-2	-2	ε	-1	$\xi-2$	ξ	0	0	1
d_F	$d-1$	1	$4-d$	0	0	ε	1	ξ	ξ	0	0	1

Hwa–Kardar model with the action (4) these dimensions coincide with their counterparts in Table 1.

In contrast to the pure Hwa–Kardar model and its modifications with anisotropic velocity ensembles^{14–16} (where the diffusivity coefficients $\nu_{\parallel 0}$ and $\nu_{\perp 0}$ have different momentum dimensions d^{\parallel} and d^{\perp}), only total momentum dimension d^k can be defined for the theory (11). From Table 2, it follows that the ratio $u_0 = \nu_{\parallel 0}/\nu_{\perp 0}$ is completely dimensionless, that is, dimensionless with respect to the frequency and momentum dimensions separately. Therefore, according to the general rules, u_0 should be treated as an additional charge. The third coupling constant is $x_0 \sim \Lambda^\xi$ related to the amplitude B_0 of the correlation function (3) as $B_0 = x_0 \nu_{\perp 0}$. The theory is logarithmic when $\varepsilon = 0$ (i.e. $d = 4$) and $\xi = 0$.

The theory is renormalized by introducing the renormalization constants Z_i :

$$\nu_{\parallel 0} = \nu_{\parallel} Z_{\nu_{\parallel}}, \quad \nu_{\perp 0} = \nu_{\perp} Z_{\nu_{\perp}}, \quad g_0 = Z_g g \mu^\varepsilon, \quad x_0 = Z_x x \mu^\xi, \quad u_0 = Z_u u. \quad (14)$$

The anomalous dimensions calculated from the Feynman graphs have the following one-loop expressions (we omit details of calculations for brevity; see Ref. 17 for similar graphs):

$$\gamma_{\nu_{\parallel}} = \frac{3}{8} \frac{x}{u} + \frac{3}{16} g, \quad \gamma_{\nu_{\perp}} = \frac{3}{8} x. \quad (15)$$

The anomalous dimensions for the coupling constants read

$$\gamma_g = -\frac{3}{2}(\gamma_{\nu_{\parallel}} + \gamma_{\nu_{\perp}}), \quad \gamma_x = -\gamma_{\nu_{\perp}}, \quad \gamma_u = \gamma_{\nu_{\parallel}} - \gamma_{\nu_{\perp}}. \quad (16)$$

Similarly to the case of the pure Hwa–Kardar model (see Sec. 2), the anomalous dimensions for the fields h, h' and \mathbf{v} vanish due to the absence of their renormalization: $\gamma_h = \gamma_{h'} = \gamma_v = 0$.

From explicit results for the anomalous dimensions (15), it follows that the one-loop expressions for the β functions have the following form:

$$\beta_g = g \left(-\varepsilon + \frac{9}{32} g + \frac{9}{16} \frac{x}{u} + \frac{9}{16} x \right), \quad \beta_x = x \left(-\xi + \frac{3}{8} x \right), \quad (17)$$

$$\beta_u = u \left(-\frac{3}{16} g - \frac{3}{8} \frac{x}{u} + \frac{3}{8} x \right).$$

Analysis of the system (17) reveals two possible IR attractive fixed points: the Gaussian point FP1 with the coordinates $g^* = 0, x^* = 0$ and arbitrary u^* , and the

fixed point FP2 with the coordinates $g^* = 0, x^* = 8\xi/3, u^* = 1$ which corresponds to the regime of simple turbulent advection (the nonlinearity of the Hwa–Kardar equation is irrelevant in the sense of Wilson).

The point FP1 is IR attractive for $\varepsilon < 0, \xi < 0$, whilst the point FP2 is IR attractive for $\xi > \varepsilon/3, \xi > 0$. This is the full set of fixed points with finite and nonzero value of u^* ; there are no fixed points with $g^* \neq 0$ in the set, i.e. the Hwa–Kardar universality class is not realized for such values of u .

However, the system (17) may lose possible solutions with $u^* = 0$ or $1/u^* = 0$. In order to explore those exceptional values, we have to pass to new variables: $w = x/u$ (to study the case $u^* = 0$) and $\alpha = 1/u$ (to study the case $u^* \rightarrow \infty$).

For the first case, we obtain no IR attractive fixed points. In the second case, we have two more fixed points: the point FP3 with the coordinates $g^* = 32\varepsilon/9, x^* = 0, \alpha^* = 0$ and the point FP4 with $g^* = 32\varepsilon/9 - 16\xi/3, x^* = 8\xi/3, \alpha^* = 0$. The point FP3 corresponds to the regime of critical behavior where only the nonlinearity of the Hwa–Kardar equation is relevant, and the point FP4 corresponds to the regime where both the nonlinearity and the turbulent advection are relevant. The point FP3 is IR attractive for $\varepsilon > 0, \xi < 0$, and the point FP4 is IR attractive for $\xi < \varepsilon/3, \xi > 0$.

4. Scaling Regimes and Critical Dimensions in the Model with Turbulent Advection

Unlike model (12), the original Hwa–Kardar model (4) allows to introduce two independent momentum dimensions. Thus, it involves three equations (7) related to the canonical scale invariance. On the other hand, the model with turbulent advection contains only two such equations. Combining them with differential RG equation

$$(\mathcal{D}_\mu + \beta_g \partial_g + \beta_x \partial_x + \beta_u \partial_u - \gamma_{\nu_\perp} \mathcal{D}_{\nu_\perp} - \gamma_G) G^R = 0 \tag{18}$$

taken at a fixed point, we arrive at

$$(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + \Delta_\omega \mathcal{D}_\omega - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^*) G^R = 0, \tag{19}$$

where $\Delta_\omega = 2 - \gamma_{\nu_\perp}^*$. This is the critical scaling equation; note that it does not involve derivatives over μ and ν_\perp . The critical dimension Δ_F of a quantity F now reads

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*. \tag{20}$$

For the fixed points, FP2 critical dimensions are

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_\omega = 2 - \xi. \tag{21}$$

These results are perturbatively exact due to the fact that $\gamma_{\nu_\perp}^*$ is known exactly while $\gamma_h = \gamma_{h'} = \gamma_v = 0$.

The fixed point FP1 is Gaussian so the corresponding critical dimensions coincide with the canonical ones.

Surprisingly, the fixed points FP3 and FP4 are related to scaling that is majorly different from the scaling described by the points FP1 and FP2. The difference arises when one sets $\beta_{\{g_i^*\}} = 0$ in the RG equation (18). The Green functions have well-defined finite limits at $g \rightarrow 0$ and $x \rightarrow 0$, thus, the β -functions at the points FP1 and FP2 can be set to zero without the need for any further analysis. However, such a straightforward substitution cannot be performed for β_α at the fixed points with $\alpha^* = 0$ (or alternatively at the fixed points with $u \rightarrow \infty$). Instead, the first nontrivial order of the expansion of β_α around such exceptional fixed points should be retained in (18). More detailed discussion of this important issue is given in App. A. As a result, the critical scaling equation at the fixed points FP3 and FP4 takes on the following form:

$$(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + \Delta_\omega \mathcal{D}_\omega - \lambda^* \mathcal{D}_\alpha - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^*) G^R = 0, \tag{22}$$

where $\lambda = \partial\beta_\alpha/\partial\alpha$ taken at $\alpha = 0$, $\Delta_\omega = 2 - \gamma_{\nu_\perp}^*$ and $\lambda^* = \lambda(g^*, x^*)$.

The point FP3 corresponds to the regime where only the nonlinearity of the Hwa–Kardar equation is relevant; its scaling equation takes the following form:

$$\left(\mathcal{D}_{k_\parallel} + \mathcal{D}_{k_\perp} + 2\mathcal{D}_\omega - \frac{2\varepsilon}{3} \mathcal{D}_\alpha - d_G^k - 2d_G^\omega - \gamma_G^* \right) G^R = 0. \tag{23}$$

This equation corresponds to scaling where the momenta k_\parallel , \mathbf{k}_\perp , the frequency ω , and the ratio α are scaled. However, in this special case where only the nonlinearity is relevant, the model (11) coincides with the pure Hwa–Kardar model (4). It means that additional canonical symmetry arises and turns d^\parallel and d^\perp into independent dimensions, see Eq. (7).

The general solution of the system of equations that includes two canonical invariance equations, the RG equation (18) taken at a fixed point, and homogeneous counterpart of Eq. (23) is an arbitrary function of three independent variables chosen here for definiteness as

$$z_1 = \frac{\omega}{\nu_\perp k_\perp^2}, \quad z_2 = \frac{k_\parallel}{k_\perp} \quad \text{and} \quad z_3 = \alpha \left(\frac{k_\perp}{\mu} \right)^{2\varepsilon/3}. \tag{24}$$

Additional symmetry requires the variables z_1 , z_2 and z_3 to be dimensionless with respect to Table 1. While the variables z_2 and z_3 do not satisfy this requirement, it is possible to introduce a new variable $z_0 = z_2 z_3^{-1/2}$ with needed canonical dimensions that serves as the second solution (along with z_1) of the homogeneous part of Eq. (9):

$$z_0 = \frac{k_\parallel}{k_\perp^{\Delta_\parallel} \alpha^\varphi} \mu^{\varepsilon/3} \quad \text{with} \quad \varphi = \frac{3}{2\varepsilon} (\Delta_\parallel - 1), \tag{25}$$

where $\Delta_\parallel = 1 + \varepsilon/3$ is in agreement with expressions (10). This implies that the fixed point FP3 allows for the scaling where the coordinates x_\parallel and \mathbf{x}_\perp (or momenta k_\parallel and \mathbf{k}_\perp) are scaled simultaneously with nontrivial $\Delta_\parallel \neq 1$, while all the IR irrelevant parameters (including α) are kept fixed. Thus, the results derived for the pure Hwa–Kardar model (4) are reproduced.

The fixed point FP4 corresponds to the regime where both the nonlinearity of the Hwa–Kardar equation and the turbulent advection are relevant; its equation of critical scaling (22) reads

$$\left(\mathcal{D}_{k_{\parallel}} + \mathcal{D}_{k_{\perp}} + \Delta_{\omega} \mathcal{D}_{\omega} - \left(\frac{2}{3} \varepsilon - 2\xi \right) \mathcal{D}_{\alpha} - d_G^k - \Delta_{\omega} d_G^{\omega} - \gamma_G^* \right) G^R = 0, \quad (26)$$

where $\Delta_{\omega} = 2 - \xi$. Three combinations \hat{z}_1 , \hat{z}_2 and \hat{z}_3 are possible solutions of its homogeneous part:

$$\hat{z}_1 = \frac{\omega}{\nu_{\perp} k_{\perp}^2} \left(\frac{k_{\perp}}{\mu} \right)^{\xi}, \quad \hat{z}_2 = \frac{k_{\parallel}}{k_{\perp}}, \quad \hat{z}_3 = \alpha \left(\frac{k_{\perp}}{\mu} \right)^{2\varepsilon/3 - 2\xi}. \quad (27)$$

The variables (27) describe generalized critical scaling where α (i.e. the ratio of ν_{\perp} and ν_{\parallel}) is also scaled.

This resembles various modified similarity hypotheses for systems with different characteristic scales or different scaling laws.^{18–21}

Of course, it is not impossible that the Green functions have some peculiar dependence on the parameters (27) so that the ordinary critical scaling at fixed u , with some nontrivial scaling dimensions, takes place. However, as far as we can see, there is no anticipated reason for this situation to occur.

5. Conclusion

We studied nonconventional scaling behavior of the self-organized critical system in an isotropic turbulent environment. The system was modeled by the anisotropic Hwa–Kardar equation (1) and (2) while the turbulent advection was described by Kazantsev–Kraichnan “rapid-change” ensemble (3).

We analyzed a field theory equivalent to the stochastic problem and established that the theory is multiplicatively renormalizable. The coordinates of the four fixed points of RG equations were calculated in the one-loop approximation or found exactly.

Among four universality classes of critical behavior, two classes turned out to correspond to nonconventional scaling behavior. It was shown that a kind of dimensional transmutation takes place in one of those regimes, precisely, in the one where the nonlinearity of the Hwa–Kardar equation is relevant. The transmutation results in the ratio u of the two diffusivity coefficients ν_{\parallel} and ν_{\perp} acquiring a nontrivial canonical dimension. Due to this, the new canonical symmetry arises in the model (11) and allows canonical dimensions d^{\parallel} and d^{\perp} to be scaled independently. Thus, the scaling behavior involves simultaneous scaling of the coordinates x_{\parallel} and \mathbf{x}_{\perp} with a nontrivial relative exponent $\Delta_{\parallel} \neq 1$. The regime is in agreement with the predictions of the pure Hwa–Kardar model analysis (4).

In the regime, where both the advection and nonlinearity are relevant, the self-similar behavior necessarily involves some dilation of the ratio u . The ordinary critical behavior with fixed definite critical dimensions can be established only if the

correlation functions of the model have some special dependence on u . Otherwise, the IR behavior will remind of various generalized self-similarity hypotheses, like the weak scaling due to George Stell^{18–20} or the parametric scaling in the spirit of Michael Fisher²¹ for systems with different characteristic scales and modified scaling laws.

In the future, it would be interesting to consider advection by more realistic velocity ensembles and take into account finite correlation time, non-Gaussianity, anisotropy and so on. It is indeed a very interesting avenue to explore and sometimes introducing of “small” properties of turbulent motion leads to very unexpected resulting features, such as losing of universality¹⁶ or logarithmic corrections to ordinary scaling (instead of power ones) in anisotropic advection of vector impurity field.^{22–25} However, it is impossible to predict the result from general considerations before calculations will be done.

This work is partly in progress. Preliminary investigation^a of the advection by the stochastic Navier–Stokes equation with white stirring noise (model B of Ref. 26) has shown that the RG equation possess at least one IR attractive fixed point. It corresponds to the regime of simple turbulent advection, where the nonlinearity of the Hwa–Kardar equation is irrelevant in the sense of Wilson and where isotropy is restored. While the system of β functions is quite complicated even at one-loop level due to emergence of two new charges, existence of other fixed points seems to be a most likely possibility.

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Appendix A. Subtleties of RG Equations

Let us illustrate the reasoning behind the derivation of Eq. (22) using the simplified model as an example; cf. the discussion of the magnetohydrodynamical turbulence in Subsec. 3.7 of the monograph.²⁷

Let $D = D(k, \mu, \nu, g)$ be the renormalized equal-time pair correlation function of a certain renormalizable dynamic model with the diffusivity coefficient ν and the coupling constant g . From the dimensionality considerations, one can write

$$D = k^{d_D^k + 2d_D^\nu} \nu^{d_D^\nu} R(k/\mu, g), \tag{A.1}$$

where d_D^k and d_D^ν are the canonical dimensions of D and $R(\cdot)$ is a function of dimensionless arguments.

^aThe results were reported at the International Workshop on Statistical Physics (December 1–3, 2021; Antofagasta, Chile) by A. Yu. Luchin; see the poster “A self-organized critical system under the influence of turbulent motion of the environment” at <https://www.iwosp.cl/poster-session>.

The corresponding RG equation has the following form (here and below, see, e.g. the monograph³):

$$(\mathcal{D}_\mu + \beta(g)\partial_g - \gamma_\nu \mathcal{D}_\nu + \gamma_D)D = 0, \tag{A.2}$$

and its solution is

$$D = k^{d_\nu^k + 2d_\nu^z} \bar{\nu}^{d_\nu^w} R(1, \bar{g}) \exp \left\{ \int_g^{\bar{g}} \frac{\gamma_D(s) ds}{\beta(s)} \right\}, \tag{A.3}$$

where the functions $\bar{g} = \bar{g}(k/\mu, g)$, $\bar{\nu} = \bar{\nu}(k/\mu, g, \nu)$ are the RG-invariant counterparts of the parameters g, ν while $\gamma_\nu, \gamma_D, \beta$ are the RG functions.

If the RG equation has an IR attractive fixed point, $\beta(g^*) = 0, \omega = \beta'(g^*) > 0$ (the case $g^* = 0$ is allowed) in the IR asymptotic region $k/\mu \rightarrow 0$ one has

$$\bar{g} - g^* \rightarrow (k/\mu)^\omega, \quad \bar{\nu} \rightarrow \nu(k/\mu)^{-\gamma_\nu^*}, \quad \exp\{\dots\} \rightarrow (k/\mu)^{\gamma_D^*}. \tag{A.4}$$

The IR asymptotic expression for D is obtained by the substitution of (A.4) into (A.3). If the function $R(1, g)$ has a finite limit at $g \rightarrow g^*$, one can neglect the term $\sim (k/\mu)^\omega$ in the expression for $\bar{g} - g^*$, decaying for $k/\mu \rightarrow 0$, and simply substitute $\bar{g} \rightarrow g^*$. This gives

$$D \simeq k^{\Delta_D} R(1, g^*), \tag{A.5}$$

where the critical dimension Δ_D is given by the standard expression (20) with $\Delta_\nu = 2 - \gamma_\nu^*$ (here and in similar expressions below we do not display the dependence on the fixed parameters μ and ν).

The expressions (A.4) and (A.5) can be more easily derived from the simplified RG equation with constant coefficients, obtained from (A.2) by the replacement of the RG functions by their leading terms^b at $g \rightarrow g^*$, that is, $\gamma_F(g) \rightarrow \gamma_F^*, \beta(g) \rightarrow \omega(g - g^*)$:

$$(\mathcal{D}_\mu + \omega \mathcal{D}_{g-g^*} - \gamma_\nu^* \mathcal{D}_\nu + \gamma_D^*)D = 0, \tag{A.6}$$

where we used the identity $\partial/\partial g = \partial/\partial(g - g^*)$. Note that the asymptotic expressions for $\bar{g}, \bar{\nu}$ in (A.4) are solutions of the homogeneous analog of Eq. (A.6).

If the function $R(1, g)$ is finite at $g = g^*$ and one can neglect the right-hand side in the expression $\bar{g} - g^* \simeq (k/\mu)^\omega$ and the term coming from the β function in (A.6) can also be omitted; this is the usual situation.

If this is not the case, the IR behavior requires a more careful analysis. Assume, as an example, that R has the following form:

$$R(k/\mu, g) = (g - g^*)^a F(k/\mu, g), \tag{A.7}$$

where a is some exponent and F has a finite limit for $g \rightarrow g^*$. Then the naive substitution $\bar{g} \rightarrow g^*$ leads to the vanishing or divergence of the amplitude factor in (A.5). In order to find genuine IR behavior, one should take into account the way in which

^bAccounting for the higher-order corrections in $(g - g^*)$ would give only g -dependent nonuniversal amplitudes in scaling laws and corrections to the leading-order IR asymptotic expressions for the correlation functions of the type (A.5); see the monograph,³ Subsecs. 1.3 and 1.33.

the invariant coupling approaches its fixed point, namely: $\bar{g} - g^* \simeq (k/\mu)^\omega$, where $\omega = \beta'(g^*) > 0$. This gives

$$D \simeq k^{\Delta_D + a\omega} F(1, g^*). \tag{A.8}$$

Thus, the correct critical dimension of the function D appears to be $\Delta_D + a\omega$ rather than Δ_D itself.

In ordinary cases, expression (A.8) determines one of the correction terms to the leading-order IR asymptotic expression (A.5); see Subsecs. 1.3 and 1.34 in the monograph.³ In our case, it becomes the leading-order term in itself. In order to derive it directly from the simplified RG equation, one has to substitute (A.1) and (A.7) into (A.6). This gives

$$(\mathcal{D}_\mu + \omega \mathcal{D}_{g-g^*} - \gamma_\nu^* \mathcal{D}_\nu + \gamma_D^* + a\omega)F = 0, \tag{A.9}$$

where the desired replacement $\gamma_D^* \rightarrow \gamma_D^* + a\omega$ is due to the term with \mathcal{D}_{g-g^*} in (A.6). Now, since the function F is finite at $g = g^*$, this term in Eq. (A.9) can be omitted, which immediately leads to representation (A.8). It is also worth noting that the functions R and F in (A.7) differ by the constant (k -independent) factor $(g - g^*)^a$ and not by the k -dependent factor $(\bar{g} - g^*)^a$, so that their scaling behavior is identical up to irrelevant nonuniversal amplitudes.

If the character of the singularity at $g = g^*$ is not known *a priori*, the operation \mathcal{D}_{g-g^*} should be retained in the RG differential operator. It is the situation that is encountered in Sec. 4 when analyzing the fixed points FP3 and FP4 with the coordinate $\alpha^* = 0$. That is, the term with the coefficient $\lambda^* = \beta'_\alpha(\alpha^*)$ in Eq. (22) must be retained.

References

1. T. Hwa and M. Kardar, *Phys. Rev. Lett.* **62**, 1813 (1989).
2. T. Hwa and M. Kardar, *Phys. Rev. A* **45**, 7002 (1992).
3. A. N. Vasiliev, *The Field Theoretic Renormalization Group in Critical Behaviour Theory and Stochastic Dynamics* (Chapman & Hall, Boca Raton, FL, USA, 2004).
4. G. Pruessner, *Self-Organized Criticality: Theory, Models and Characterisation* (Cambridge University Press, Cambridge, MA, USA, 2012).
5. N. W. Watkins, G. Pruessner, S. C. Chapman, N. B. Crosby and H. J. Jensen, *Space Sci. Rev.* **198**, 3 (2016).
6. M. A. Muñoz, *Rev. Mod. Phys.* **90**, 031001 (2018).
7. D. Marković and C. Gros, *Phys. Rep.* **536**, 41 (2014).
8. M. J. Aschwanden, *Self-Organized Criticality Systems* (Open Academic Press, Berlin, Warsaw, 2013).
9. A. Onuki, K. Yamazaki and K. Kawasaki, *Ann. Phys.* **131**, 217 (1981).
10. A. Aronowitz and D. R. Nelson, *Phys. Rev. A* **29**, 2012 (1984).
11. G. Satten and D. Ronis, *Phys. Rev. A* **33**, 3415 (1986).
12. G. Falkovich, K. Gawędzki and M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001).
13. N. V. Antonov, N. M. Gulitskiy, P. I. Kakin and G. E. Kochnev, *Universe* **6**, 145 (2020).
14. N. V. Antonov and P. I. Kakin, *EPJ Web Conf.* **108**, 02009 (2016).

15. N. V. Antonov, N. M. Gulitskiy, P. I. Kakin and V. D. Serov, *EPJ Web Conf.* **226**, 02002 (2020).
16. N. V. Antonov, N. M. Gulitskiy, P. I. Kakin and V. D. Serov, *Phys. Rev. E* **103**, 042106 (2021).
17. N. V. Antonov, P. I. Kakin and N. M. Lebedev, *J. Phys. A, Math. Theor.* **52**, 505002 (2019).
18. G. Stell, *Phys. Rev. Lett.* **24**, 1343 (1970).
19. G. Stell, *Phys. Rev. B* **2**, 2811 (1970).
20. G. Stell, “*Weak Scaling*” *Enrico Fermi School of “Critical Phenomena”* (Course LI, Ed. M. S. Green; Academic Press: New York, NY, USA, 1971).
21. M. Fisher, “*The Theory of Critical Point Singularities*” *Enrico Fermi School of “Critical Phenomena”* (Course LI, Ed. M. S. Green; Academic Press, New York, NY, USA, 1971).
22. N. V. Antonov and N. M. Gulitskiy, *Phys. Rev. E* **91**, 013002 (2015).
23. N. V. Antonov and N. M. Gulitskiy, *Phys. Rev. E* **92**, 043018 (2015).
24. N. V. Antonov and N. M. Gulitskiy, *AIP Conf. Proc.* **1701**, 100006 (2016).
25. N. V. Antonov and N. M. Gulitskiy, *EPJ Web Conf.* **108**, 02008 (2016).
26. D. Forster, D. R. Nelson and M. J. Stephen, *Phys. Rev. A* **16**, 732 (1977).
27. L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasiliev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon and Breach, London, 1999).