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Functional-Difference Equations and Their Link with Perturbations of the Mehler Operator

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Abstract. This work deals with the spectral properties of the functional-difference equations, that arise in a number of applications in the diffraction of waves and quantum scattering. Their link with some of the spectral properties of perturbations of the Mehler operator is addressed. The latter naturally arise in studies of functional-difference equations of the second order with a meromorphic potential which depend on a characteristic parameter. In particular, this kind of equations is frequently encountered with in the asymptotic treatment of eigenfunctions of the Robin Laplacians in wedge- or cone-shaped domains. The unperturbed selfadjoint Mehler operator is studied by means of the modified Mehler–Fock transform. Its resolvent and spectral measure are described. These results are obtained by use of some additional analysis applied to the known Mehler formulas. For a class of compact perturbations of this operator, sufficient conditions of existence and finiteness of the discrete spectrum are then discussed. Applications to the functional-difference equations are also addressed. An example of a problem leading to the study of the spectral properties for a functional-difference equation is considered. The corresponding eigenfunctions and characteristic values are found explicitly in this case.

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1. INTRODUCTION

1.1. Functional-Difference (FD) Equations of the Second Order and Perturbations of the Mehler Operator

In the recent years, an obvious interest to the description of spectral properties of Laplacians with δ or δ'-interactions supported on some surfaces [1, 2] occurred. In the classical formulation, the Robin-type boundary conditions are exploited in the description of the corresponding selfadjoint operators, see, e.g., [1, 9, 13] and the references therein. Traditionally, general methods [4] for such operators are applied in order to describe their spectra qualitatively. However, provided that a model admits a separation of variables partly, due to some symmetry, one could obtain additional information about the spectral properties. For instance, the asymptotic behavior of the eigenfunctions have been studied in [12, 13]. To this end, by making use of the incomplete separation of variables, the corresponding problems are reduced to a functional-difference equation with a meromorphic potential

$$H(\nu+1) - H(\nu-1) - 2\mathrm{i}\Lambda\, W(\nu)\, H(\nu) \,=\, 0\,, \quad \nu \in \mathbf{C}, \tag{1}$$

where, by definition, Λ is a characteristic parameter (so that $\mu = \Lambda^{-1}$ is the spectral parameter),[1] and W is the meromorphic potential belonging to a special class.

In [12], eigenfunctions and their asymptotic behavior at large distances are studied for the Laplace operator with singular potential whose support is placed on a circular conical surface in three-dimensional space. Within the framework of incomplete separation of variables, an integral representation of the Kontorovich–Lebedev (KL) type for the eigenfunctions is obtained in terms of the solution of an auxiliary functional-difference equation (see (1)) with a meromorphic potential. Solutions of the functional-difference equation are studied by reducing it to an integral equation with a bounded selfadjoint integral operator. The latter integral operator is exactly a perturbation of the Mehler operator.

In the recent paper [13], we have studied the eigenfunctions that describe eigenoscillations of acoustic waves in angular domains with ‘semitransparent’ boundary conditions. For some values of the spectral parameter in the boundary-value problem, we have considered essential and discrete spectra of the equations and described

[1]In our applications, this characteristic parameter is directly connected with the spectral parameter E for the corresponding Laplacian, e.g., $\Lambda = \frac{\gamma}{\sqrt{-E}}$, γ is the Robin parameter.

properties of the corresponding solutions. The study is based on the reduction of the functional-difference equations to integral equations of the Mehler type with a symmetric kernel.

A similar kind of equations is also encountered with in the problems of quantum scattering [16, 17, 18]. For convenience of the reader, in the next section we give a simple example of derivation of such a functional-difference equation and study the discrete spectrum of the problem explicitly.

An important observation is that the Fourier transform along the imaginary axis reduces equation (1) to a homogeneous integral equation and then to an integral equation with the perturbed Mehler operator depending on a characteristic parameter Λ, see also [12, 13]. As a result, the study of the spectral properties of the perturbed Mehler operator (2) leads to a description of the corresponding results for equation (1). Finally, an additional analysis based on the Kontorovich–Lebedev and Sommerfeld–Malyuzhinets integral representations enabled one to obtain asymptotics of the eigenfunctions for some models [12, 13], which is not considered in this paper.

It is obvious that the description of some spectral properties of the perturbed and unperturbed Mehler operators becomes a crucial step in the study of the spectra of the corresponding functional-difference equations. We notice, however, that the exhaustive study of the spectral properties of the Mehler operator is not a basic goal of the present work. It is natural and possible to give elsewhere their complete description which is similar to that obtained for the perturbations of the Carleman operator [23]. The other point is that we study only a particular class of perturbations of the Mehler operator that is connected with the FD equations, although more general results are also available.

1.2. Some Definitions and the Adopted Terminology

We study the FD equation (1) in a special class of meromorphic functions described below. The coefficient $W(\nu)$ is called the potential and is a meromorphic function in a certain class. The class of potential is motivated by applications (see, e.g., [12, 13]). The parameter Λ is called the characteristic parameter. Its values are responsible for existence of nontrivial solutions of equation (1) which belong to the prescribed class. Below we define the so-called characteristic and essential values of this parameter.

We also formally define the perturbed Mehler operator by the expression

$$[Ku](x) := \frac{1}{\pi}\int_0^1 \frac{w(x,y)}{x+y}\, u(y)\mathrm{d}y \tag{2}$$

in the space $L_2(0,1)$, where, usually in applications, the function w is continuous on $[0,1)\times[0,1)$, symmetric ($w(x,y)=w(y,x)$), and square integrable. (Note that the point $(0,0)$ is implied to be a singular point of the kernel; thus, the operator K is bounded but need not be compact in the general case.) The name of this operator is due to the fact that, in the majority of interesting cases, we can write

$$w(x,y) = 1 + [w(x,y)-1]$$

with $w(x,y)-1 = o(1)$ as $(x,y)\to(0,0)$, so that the operator K in (2) can be represented as follows:

$$K = M + V, \tag{3}$$

where

$$[Mu](x) := \frac{1}{\pi}\int_0^1 \frac{u(y)}{x+y}\,\mathrm{d}y \tag{4}$$

is the (unperturbed) Mehler operator that is bounded (see also [20, 3], where this operator is attributed to Dixon's integral equation)[2] and the perturbation $V = [K-M]$,

$$[Vu](x) := \frac{1}{\pi}\int_0^1 \frac{v(x,y)}{x+y}\, u(y)\mathrm{d}y \tag{5}$$

is often of the Hilbert-Schmidt class S_2 due to the properties of $v(x,y) := w(x,y)-1$. In our case of the FD equation (1) the potential of the perturbation has a very particular form with $v(x,y) := \sqrt{w_0(x)w_0(y)}-1$.

[2]This operator has been recently encountered with in diffraction theory [3, 15], Chapter 5, see also Dixon integral equation in [20]. We follow D. R. Yafaev who proposed to the author, to use this terminology for such an operator.

Note that the Mehler operator is formally analogous to a Hankel operator

$$[Hu](x) := \int_0^\infty h(x+y)\,u(y)\mathrm{d}y,$$

where $h(t) = t^{-1}/\pi$ and the integration is restricted to $[0, 1]$. In the paper [23], perturbations of the Carleman operator ($h(t) = t^{-1}$ for $t \in (0, \infty)$) by the Hankel operators are carefully studied.

1.3. Structure of the Paper

In the second section we give an example of derivation of a functional equation that belongs to the class discussed in this paper. It is remarkable that this equation is a solvable model, .e., the corresponding eigenfunctions can be found explicitly. A further analysis is devoted to more general situations when potentials belong to the aforementioned class of meromorphic functions. The corresponding models are not explicitly solvable (see, e.g., [12, 13]).

In the third section we describe a class of functional-difference equations (1), specifying a set of meromorphic potentials which are motivated by applications. We make use of the Fourier transform and reduce the equation to an integral one with integration along the imaginary axis. After some additional reduction, we arrive at a homogeneous integral equation with a perturbed Mehler operator (3) and a characteristic parameter. To our knowledge ,some of the results of the following sections are new in general and only partly considered in our papers [14, 13, 12] and some other works, see, e.g., [3]. The so-called dual singular integral equation to a Mehler-type integral equation is obtained in Appendix.

The fourth section deals with applications of the modified Mehler–Fock transform (mMF-transform) to the study of the unperturbed Mehler operator (4). To this end, we derive some useful formulas dealing with mMF-transform, and then compute the resolvent of the Mehler operator; we show that it has an essential (absolutely continuous) spectrum $\sigma_e(M)$ that coincides with the segment $[0, 1]$, and derive the 'eigenfunctions' of the continuous spectrum and the spectral resolution. These studies are based on the classical Mehler formulas.

The fifth section is devoted to the study of the spectrum of the perturbed Mehler operator K, $K = M+V$, where V is defined by (5) and is assumed to be compact. The Weyl theorem leads to preserving of the essential spectrum, so that the main attention is paid to the existence of the discrete spectrum $\sigma_d(K)$. We obtain some sufficient conditions for this. An application of the Birman–Schwinger principle and the discussion of finiteness of the discrete spectrum is then considered.[3]

In the sixth section, we apply the results of the previous sections to the functional-difference equations (1), which enables us to give a definition and a precise description of the characteristic sets for the class of potentials under consideration. Some examples are also discussed. In Conclusion, we underline some important aspects of the study and consider some further prospects.

2. EIGENFUNCTIONS OF A ROBIN LAPLACIAN AND FUNCTIONAL-DIFFERENCE EQUATIONS

Consider an angular domain $\Omega_* = \{r \geqslant 0,\ 0 < \varphi < \Phi\}$ with the opening $\Phi < \pi/2$ and the boundary S which consists of two half-lines $S_+ = \{r > 0,\ \varphi = \Phi\}$, and $S_- = \{r \geqslant 0,\ \varphi = 0\}$ with the same origin O, and

$$X = r\cos\varphi, \quad Y = r\sin\varphi$$

are the interconnected Cartesian and polar coordinates.

We are looking for solutions of the homogeneous problem in Ω_* with the spectral parameter E

$$\mathcal{A}_\gamma U \,=\, E\,U \tag{6}$$

for a formally symmetric operator A_γ defined in the 'classical' terms of differential equations and boundary conditions,

$$-\triangle U(r,\omega) = EU(r,\omega), \quad (r,\varphi) \in \Omega_*\,, \tag{7}$$

[3]This result is analogous to that discussed in [23] for perturbations of the Carleman operator.

$\triangle = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$. The boundary conditions on $S = S_+ \cup S_-$ read

$$\left.\frac{\partial U}{\partial n}\right|_{S_+} = \gamma U|_{S_+}\,, \qquad \left.\frac{\partial U}{\partial n}\right|_{S_-} = 0, \tag{8}$$

where $\gamma > 0$ is the Robin parameter, the normal n is directed from Ω and $\frac{\partial}{\partial n}\big|_{S_+} = \frac{1}{r}\frac{\partial}{\partial \varphi}\Big|_{\varphi=\Phi}$. A precise definition of the selfadjoint operator $\mathcal{A}_\gamma$ is discussed in [9, 14]. The spectrum of $\mathcal{A}_\gamma$ is essential for $E \geqslant -\gamma^2$ and is discrete for $E < -\gamma^2$ with a finite number of eigenvalues. This model is explicitly solvable, see [14]. In particular, the eigenfunctions and eigenvalues can be found in a closed form [14].

We look for the eigenfunctions as solutions in $H^1(\Omega_*)$ of equation (6) with conditions (7), (8) in the form of the Kontorovich–Lebedev integral

$$U(r,\varphi) = \frac{1}{\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \sin(\pi\nu)\, K_\nu(\kappa r)\, u_\nu(\varphi) \mathrm{d}\nu\,, \tag{9}$$

where $\kappa = \sqrt{-E}$, $E < 0$, K_ν stands for the Macdonald function. After the substitution into the equation, we find

$$u_\nu(\varphi) = H_*(\nu)\,\frac{\cos(\nu\varphi)}{\cos(\nu\Phi)}$$

which satisfies the second boundary condition in (8) because u_ν is even with respect to φ.

From the first boundary condition and (9) (see, e.g., [13] for similar calculations), for

$$D(\nu) = H_*(\nu)\tan(\Phi\nu),$$

we arrive at the functional-difference equation

$$D(\nu+1) - D(\nu-1) - 2\mathrm{i}\Lambda\, W(\nu) D(\nu) = 0 \tag{10}$$

with

$$W(\nu) = \mathrm{i}\cot(\Phi\nu)\,, \quad \Lambda = \frac{\gamma}{\kappa}.$$

Equation (10) gives an example of a second-order functional-difference equation (1) with meromorphic potential. We omitted the details of justification, see [13], for the derivations in a similar problem. These solutions should be even, exponentially vanishing on the imaginary axis, and holomorphic in its neighborhood. In this case, the Kontorovich–Lebedev integral representation converges and solutions are understood in classical sense.

Note that the functional-difference equation (10) of the second order with meromorphic potential $W(\nu) = \mathrm{i}\cot(\nu\Phi)$ is a direct analog of the FD equation in the Maryland model [5]. Alternative examples of FD equations under consideration are discussed in [12, 13]. We can solve this equation (10) explicitly and determine the eigenfunctions and characteristic values in an explicit form. We compute $D = D_n$ and $\Lambda = \Lambda_n = \gamma/\kappa_n$, ($\kappa_n = \sqrt{-E_n}$) satisfying the equation.

By the direct substitution, we see that

$$D_n(\nu) = 2\,\frac{\sin\Phi\nu}{\sin\pi\nu}\sum_{m=0}^{n-1} C_m \cos(2m\Phi\nu)$$

and $\Lambda_n = \sin(\Phi t_n)$, $t_n = 2n-1$, $\Phi t_n < \pi/2$, the coefficients C_m are recurrently determined and are given below. The following estimate is obvious:

$$|D_n(\nu)| \,<\, C\, \left|\mathrm{e}^{\mathrm{i}\nu[\pi - \Phi t_n]}\right|\,,$$

as $\nu \to \mathrm{i}\infty$, $\nu \in \mathrm{iR}$. The values Λ_n can naturally be called characteristic values, and D_n the eigenfunctions of the equation, $n = 1, 2\ldots, N$.

Having D_n in hand, we substitute H_n into (2) and then into the Kontorovich–Lebedev integral (9) and make use of the known integral representation 6.795(1) in [7],

$$\mathrm{e}^{-a\cosh(b)} = \frac{2}{\pi}\int_0^\infty \mathrm{d}x\, \cos(bx)\, K_{\mathrm{i}x}(a)\,,$$

$a > 0$, $|\mathrm{Im} b| < \pi/2$. After some reductions, one has the expressions for the eigenfunctions (see also [14]):

$$U_n(r,\varphi) = \sum_{m=0}^{n-1} C_m \left(\mathrm{e}^{-\kappa_n r\cos[2\Phi m-\varphi]} + \mathrm{e}^{-\kappa_n r\cos[2\Phi m+\varphi]}\right). \tag{11}$$

The constants C_m in (11) have the form $C_0 = 1$,

$$C_m = \frac{1}{2}\prod_{k=1}^{m}\frac{\sin(\Phi[2k-1]) - \gamma/\kappa_n}{\sin(\Phi[2k-1]) + \gamma/\kappa_n}\,.$$

The corresponding eigenvalues of A_γ are $E_n = -\frac{\gamma^2}{\Lambda_n^2} = -\frac{\gamma^2}{\sin^2(\Phi[2n-1])}$, $n = 1, 2, \dots N$. The total number N of eigenvalues is finite and depends on the opening of the angle Ω_* having the value Φ. If $\Phi \in [\pi/6, \pi/2)$, then there is only one eigenvalue $E_1 = -\frac{\gamma^2}{\sin^2\Phi}$, $N = 1$. For $\Phi \in [\pi/10, \pi/6)$, there are two eigenvalues, $N = 2$. In general, we have $N = \mathrm{ent}\left(\frac{1}{2}\left(\frac{\pi}{2\Phi} - 1\right)\right) + 1$ ($\mathrm{ent}(\cdot)$ is the entire part of a number which is assumed to be continuous from the right).

3. FUNCTIONAL-DIFFERENCE EQUATIONS AND INTEGRAL EQUATIONS WITH THE PERTURBED MEHLER OPERATOR

In the situations when the corresponding FD equation (1) cannot be solved explicitly (see, e.g. [12, 13]), we intend to consider the following questions. First, we specify a class of functions for which an equation has nontrivial solutions for some values Λ. Then we expect to describe the sets of such values, giving them the title of the so-called characteristic set C_d, and the set C_e of essential values. The finiteness of the discrete part is also of interest. [4] For each value in the set of characteristic values, we hope to construct a corresponding eigenfunction in the desired space. (Note that the construction of the generalized eigenfunctions corresponding to Λ in the set C_e of essential values is also possible, although it is omitted herein.) To this end, we establish a natural link of the FD equation with the corresponding integral equation. The operator in the latter equation is the perturbed Mehler operator which is bounded and selfadjoint. Some of its spectral properties are studied and then exploited to answer the aforementioned questions. In particular, the discrete part of of the spectrum of the perturbed Mehler operator is connected with the characteristic set C_d of the FD equation in hand. As was noticed, the finiteness of this set is analyzed by use of the Birman–Schwinger principle applied to the perturbed Mehler operator. We study the corresponding eigenfunctions and exploit this information in order to construct the corresponding nontrivial solutions of the FD equation (1). Finally, the results thus obtained are expected to be applicable to several examples originating from applications.

We turn to the equation (1) and discuss first the class $\mathcal{W}$ of potentials to be considered. We suppose that $W \in \mathcal{W}$ is a meromorphic odd function of the complex variable $\nu \in \mathrm{C}$. It is holomorphic in some strip $\Pi_\delta := \{\nu \in \mathrm{C} : \ |\mathrm{Re}\,\nu| < \delta\}$ for some positive δ with possible exception of zero, where it may have a simple pole. Moreover, $W(\nu) > 0$ for $\nu \in \mathrm{iR}_+$, i.e., on the positive part of the imaginary axis, and $W \to 1$ as $\nu \to \mathrm{i}\infty$ there.[5] On the one hand, this class $\mathcal{W}$ of potentials arises from natural applications, e.g., for the Robin Laplacians [14, 13, 12]; however, on the other hand, they lead to selfadjoint perturbations of the Mehler operator corresponding to equations (1), so that some traditional machinery can be applied for their study. A concrete example of the potential has been considered above, $W(\nu) = \mathrm{i}\cot(\Phi\nu)$; however, the other ones are also discussed below.

It is worth describing properties of the solutions H to the equation (1) which are natural in applications and, in particular, ensure correspondence to the characteristic set of the equation. We introduce a class of functions $\mathcal{M}$ consisting of meromorphic functions H such that

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[4] It is remarkable that each charateristic value $\Lambda_m \in C_d$ generates the corresponding eigenvalue E_m of the corresponding operator $\mathcal{A}_\gamma$ (see, e.g., [12, 13]).

[5] In many cases, we actually have $W(\nu) = 1 + O(\exp(-a\mathrm{Im}\,\nu))$, $a > 0$; however, the power-law convergence (e.g., $W(\nu) = 1 + O(\nu^{-1})$) is also of interest in applications.

- $H(\nu) = H(-\nu)$ is even,
- H is holomorphic in the strip $\Pi_{1+\delta} = \{\nu \in \mathrm{C}: \ |\mathrm{Re}\,\nu| < 1+\delta\}$ for some $\delta > 0$,
- $|H(\nu)| \ < \ \mathrm{Const}\,|\exp(\mathrm{i}\nu[\pi/2+\delta_0])|, \ \ \nu \to \mathrm{i}\infty, \ \ \nu \in \Pi_{1+\delta}$ as $\delta_0 \in (0, \pi/2)$.

Remark that in the latter estimate $\delta_0 = 0$ corresponds to the solutions of the so-called essential characteristic set which can be similarly considered.

Taking into account that $H \in M$, we can write equation (1) as two equations (i.e., with only upper or lower signs)

$$H(\nu \pm 1) - H(-\nu \pm 1) = \pm 2\mathrm{i}\Lambda\, W(\nu)\, H(\nu)\,.$$

The difference operator on the left-hand side can be 'inverted' by using the following assertion.

Lemma 3.1. *Let $q(\nu)$ be holomorphic for $\nu \in \Pi_\delta$ and $|q(\nu)| \leqslant c_q \mathrm{e}^{-\varkappa|\nu|}$, $|\nu| \to \infty$, $\varkappa > 0$ in this strip, $q(\nu) = -q(-\nu)$. Then an even solution $s(\nu)$ of the equations*

$$s(\nu \pm 1) - s(-\nu \pm 1) = \mp 2\mathrm{i}\, q(\nu)\,,$$

which is regular (holomorphic) in the strip $\nu \in \Pi_{1+\delta}$ and exponentially vanishes as $|\nu| \to \infty$ there, is given by

$$s(\nu) = \frac{1}{2}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, q(\tau)\, \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu}\,, \quad \nu \in \Pi_{1+\delta}\,.$$

The proof is given in [14, 15] and is based on the Fourier transform along the imaginary axis. Making use of this lemma, we arrive at

$$H(\nu) = -\frac{\Lambda}{2}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, \frac{W(\tau)\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu}\, H(\tau)\,, \quad \nu \in \Pi_{1+\delta}\,, \tag{12}$$

where $W \in \mathcal{W}$.

It is worth noticing that, provided $H(\cdot)$ is known on the imaginary axis on the right-hand side of (12), the left-hand side is defined and holomorphic on the strip $\Pi_{1+\delta}$. Having specified $H(\cdot)$ on this strip, one can continue the function H to the whole complex plane as a meromorphic function. Due to this simple observation, it is sufficient to find H on the imaginary axis, and the aforementioned procedure enables one to determine the corresponding solution $H \in \mathcal{M}$ of the functional-difference equation. In order to derive H on the imaginary axis, we let $\nu \to \mathrm{iR}$ and consider (12) as an integral equation with the characteristic parameter Λ. We are looking for solutions of this equation.

To this end, we reduce the integral equation (12) to an equivalent form. We make use of the new variables

$$x = \frac{1}{\cos \pi\nu}\,, \quad y = \frac{1}{\cos \pi t}, \quad \frac{\mathrm{d}y}{\pi} = \frac{\sin \pi t}{\cos^2 \pi t}\mathrm{d}t,$$

and new unknown

$$h(x) = \cos \pi\nu\, H(\nu)|_{x=\frac{1}{\cos \pi\nu}}\,,$$

$x, y \in [0, 1]$,

$$h(x) - \frac{\Lambda}{\pi}\int\limits_0^1 \mathrm{d}y\, \frac{w_0(y)}{x+y}\, h(y) \ = 0\,, \tag{13}$$

where

$$w_0(y) = W(t)|_{y=\frac{1}{\cos \pi t}} > 0$$

and $w_0(y) = 1 + o(1)$ as $y \to 0$. Below, in examples, we specify the behavior of $w_0(y)$ as $y \to 0$ more explicitly. Finally, by (13), the desired form of the integral equation with a symmetric integral operator reads

$$\rho(x) - \frac{\Lambda}{\pi}\int\limits_0^1 \mathrm{d}y\, \frac{\sqrt{w_0(x)w_0(y)}}{x+y}\, \rho(y) \ = 0\,, \tag{14}$$

where $\rho(x) = \sqrt{w_0(x)}h(x)$.

In accordance with our definition in the introduction, the integral operator in (14) is a perturbed Mehler operator with $w(x, y) = \sqrt{w_0(x)w_0(y)}$. The perturbation V of the Mehler operator M in (4) is specified by (5) with $v(x, y) = \sqrt{w_0(x)w_0(y)} - 1$. Concluding, we observe from the derivations in this section that solutions of the functional-difference equation for $H \in \mathcal{M}$ are directly connected with solutions of the integral equation (14) with a perturbed Mehler operator. However, before studying the perturbations of the Mehler operator and its spectral properties, we consider the unperturbed Mehler operator. In the appendix, a singular integral equation dual to (12) is derived; it can be used to study some further properties of solutions to the FD equation (1).

4. MODIFIED MEHLER–FOCK TRANSFORM AND SPECTRAL PROPERTIES OF THE MEHLER OPERATOR

In [23], properties of the Carleman operator are investigated by means of the Mellin transform that 'diagonalizes' this operator. In our case of the Mehler operator, the modified form of the Mehler–Fock transform (mMF-transform) plays a similar role.

It should be stressed that the results of this section rely upon the Mehler formulas of 1881 [21] (see also Chap. 7 of [19]), and we only give their modern interpretation, explicitly describing the resolvent and spectral measure of the Mehler operator, exploiting also the results of [22]. The formulas of the (modified) Mehler–Fock transform can be derived directly from the Mehler results of 1881. However, unitary property of the transform requires an additional discussion which is given in this section.

In addition, we study the behavior of the resolvent on the cut along the essential (absolutely continuous) spectrum of the Mehler operator discussing also some estimates. The latter are further exploited for the investigation of the existence and finiteness of the discrete spectrum.

4.1. Diagonalizability of the Mehler operator M

It is useful to clarify the possibility to diagonalize the Mehler operator M by use of the so-called modified Mehler–Fock transform (mMF-transform) which is defined in this section (see also the original transform in Chap. 7 of [19, 6, 11, 8]) .

First we observe that the Mehler operator M is directly connected with the bounded selfadjoint operator M_1 in $L_2(1, \infty)$ which is defined by

$$[M_1 h](t) = \frac{1}{\pi}\int_1^\infty \frac{h(\tau)\mathrm{d}\tau}{t+\tau}\,. \tag{15}$$

Indeed, introduce the unitary operator ϕ: $L_2(1, \infty) \to L_2(0, 1)$,

$$[\phi\, w](x) = \frac{1}{x} w\left(\frac{1}{x}\right).$$

The operator ϕ establishes an isometric isomorphism between $L_2(1, \infty)$ and $L_2(0, 1)$,

$$(\phi\, w, \phi\, w)_{L_2(0,1)} = (w,\, w)_{L_2(1,\infty)}\,.$$

Then we directly prove, using the definition of M_1 in (15) and of the Mehler operator M, that

$$M = \phi^* M_1\, \phi$$

because

$$[M\, u](x) = \frac{1}{x}\frac{1}{\pi}\int_1^\infty \frac{\frac{1}{t}u\left(\frac{1}{t}\right)}{\frac{1}{x}+t}\,\mathrm{d}t = [\phi^*\, M_1\, \phi\; u](x),$$

which means that the operators M and M_1 are unitary equivalent.

Now we make use of an important observation (see the discussion in [22], Sect. 2, 3) that the operator M_1 and the selfadjoint Legendre operator L commute, where L is defined by

$$L = -\frac{\mathrm{d}}{\mathrm{d}t}\,(t^2 - 1)\,\frac{\mathrm{d}}{\mathrm{d}t}$$

with the domain $\mathrm{Dom}(L) = \{f \in H^2_{loc}(1,\infty)$ such that $\lim_{t\to 1} f(t)$ exists, $f'(t) = o([t-1]^{-1/2})$, $Lf \in L_2(1,\infty)$, $f \in L_2(1,\infty)\}$. As discussed in [22], M_1 is a function in $\mathrm{Dom}(L)$, $M_1 = 1/\cosh(\pi\sqrt{L-1/4})$, noticing that the spectrum of L is simple and occupies the half-line $\lambda \in [1/4,\infty)$, nd

$$\psi_p(t) = \sqrt{p\tanh(\pi p)}P_{\mathrm{i}p-1/2}(t)$$

are the 'eigenfunctions', $\lambda = p^2 + 1/4$,

$$L\,\psi_p \;=\; \lambda(p)\,\psi_p\,.$$

Recall that the Legendre function with $x = \cosh\alpha$ has the representation [7], 8.715,

$$P_{\mathrm{i}\tau-1/2}(\cosh\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos(\tau t)\mathrm{d}t}{\sqrt{\cosh\alpha - \cosh t}}\,.$$

As a consequence of the commutativity, one has

$$M_1\,\psi_p \;=\; \mu(p)\,\psi_p\,, \quad \mu(p) = \frac{1}{\cosh(\pi p)}\;.$$

Both the operators M_1 and L have simple absolutely continuous spectra with the eigenfunctions $\psi_p(t)$, $\mu \in [0,1]$, $\lambda \in [1/4,\infty)$. It is useful to recall that, in the modern terminology, F. G. Mehler in his paper [21] of 1881 could diagonalize the operator M_1 by proving the latter formula, so that it looks reasonable to call the operators M_1 and M the Mehler operators.

As a result, due to the unitary equivalence of the selfadjoint operators M and M_1, we see that M has simple absolutely continuous spectrum $\sigma_a(M) = [0,1]$ with the 'eigenfunctions'

$$\mathcal{P}_p(x) := [\phi^*\,\psi_p](x) \;=\; \frac{\sqrt{p\tanh(\pi p)}}{x}\,P_{\mathrm{i}p-1/2}(1/x)$$

having the asymptotics (see [7], 8.772(1))

$$\mathcal{P}_p(x) = \frac{\sqrt{p\tanh(\pi p)}}{x}\left(\frac{\Gamma(-\mathrm{i}p)}{\Gamma(-\mathrm{i}p+1/2)}\left[\frac{x}{2}\right]^{1/2-\mathrm{i}p} + \frac{\Gamma(\mathrm{i}p)}{\Gamma(\mathrm{i}p+1/2)}\left[\frac{x}{2}\right]^{1/2+\mathrm{i}p}\right)\left(\frac{1}{\sqrt{\pi}} + O(x^2)\right),$$

$x \to 0+$, $p > 0$, and $\mathcal{P}_p(x) = O(1)$ as $p \to \infty$, $1 \geqslant x > 0$. The functions $\mathcal{P}_p(x)$ are real for $p \geqslant 0$, in particular, $\mathcal{P}_0(x) > 0$.

By means of 'completeness and orthogonality' of the set $\{\psi(t)\}$, due to formulas by Mehler, we obtain

Theorem 4.1. *The modified Mehler–Fock transform given by the formulas*

$$F(x) = \int_0^\infty \mathcal{P}_p(x)\,F^*(p)\mathrm{d}p\,, \tag{16}$$

$$F^*(p) = \int_0^1 \mathcal{P}_p(x)\,F(x)\mathrm{d}x\,, \tag{17}$$

is a unitary mapping $\mathcal{U}\colon \mathcal{L}_\in(\prime,\infty) \to \mathcal{L}_\in(\prime,\infty)$. *The mMF-transform diagonalizes the Mehler operator M,*

$$\frac{1}{\pi}\int_0^1 \frac{\mathcal{P}_p(y)}{x+y}\,\mathrm{d}y = \frac{\mathcal{P}_p(x)}{\cosh(\pi p)}\,, \tag{18}$$

or, equivalently,

$$[M\mathcal{P}_p](x) = \mu(p)\mathcal{P}_p(x), \quad \mu \in [0,1]\,,$$

where

$$\mu = \mu(p) = \frac{1}{\cosh(\pi p)}\,, \quad p = p(\mu) = \frac{1}{\pi}\log([1+\sqrt{1-\mu^2}]/\mu) \geqslant 0\,.$$

The operator M is absolutely continuous with the simple spectrum $\sigma_a(M) = [0,1]$ and the 'eigenfunctions' $\mathcal{P}_p(\cdot)$.

Note also that

$$\frac{1}{\pi\,(x+y)} \;=\; \int_0^\infty \frac{\mathcal{P}_p(x)\,\mathcal{P}_p(y)}{\cosh(\pi p)}\mathrm{d}p\,. \tag{19}$$

4.2. Resolvent of the Mehler operator

We apply the mMF-transform to the equation $[Mu](x) - \mu u(x) = f(x)$ and, taking into account (17), obtain $\left(\frac{1}{\cosh(\pi p)} - \mu\right) u^*(p) = f^*(p)$ and

$$u^*(p) = f^*(p)\frac{1}{\frac{1}{\cosh(\pi p)} - \mu} = f^*(p)\left(-\frac{1}{\mu} - \frac{1}{\mu^2}\frac{1}{\cosh(\pi p) - \mu^{-1}}\right).$$

We make use of (16) ($\mu \notin \sigma_a(M) = [0,1]$),

$$\begin{aligned} u(x) &= \int_0^\infty \mathcal{P}_p(x)\, f^*(p)\left(-\frac{1}{\mu} - \frac{1}{\mu^2}\frac{1}{\cosh(\pi p) - \mu^{-1}}\right) \mathrm{d}p \\ &= -\frac{1}{\mu}\int_0^\infty \mathcal{P}_p(x)\, f^*(p)\mathrm{d}p - \frac{1}{\mu^2}\int_0^\infty \mathrm{d}p\frac{\mathcal{P}_p(x)}{\cosh(\pi p) - \mu^{-1}}\int_0^1 \mathcal{P}_p(y)\, f(y)\mathrm{d}y \\ &= -\frac{1}{\mu}\left\{f(x) + \frac{1}{\pi}\int_0^1 a(x,y;\mu)\, f(y)\mathrm{d}y\right\}, \end{aligned} \tag{20}$$

where

$$a(x,y;\mu) = \pi\int_0^\infty \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\mu\cosh(\pi p) - 1}\mathrm{d}p. \tag{21}$$

Thus,

$$u(x) = [M-\mu]^{-1}f(x) = -\frac{1}{\mu}\left\{I + A_\mu\right\} f(x) \tag{22}$$

and A_μ is an integral operator in $L_2(0,1)$ defined by

$$[A_\mu f](x) = \frac{1}{\pi}\int_0^1 a(x,y;\mu)\, f(y)\mathrm{d}y \tag{23}$$

with the kernel (21).

Remark. The kernel (21) is represented in the form

$$a(x,y;\mu) = \pi\int_0^\infty \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\mu\cosh(\pi p) - 1}\mathrm{d}p = \frac{\pi}{\mu}\int_0^\infty \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\cosh(\pi p)}\mathrm{d}p + \frac{\pi}{\mu}\int_0^\infty \frac{\mathcal{P}_p(x)}{\cosh(\pi p)}\frac{\mathcal{P}_p(y)}{[\mu\cosh(\pi p) - 1]}\mathrm{d}p. \tag{24}$$

We take into account (19) for the first summand on the right-hand side of (24) and the Parseval relation for the mMF-transform for the second summand and obtain

$$\mu\, a(x,y;\mu) = \frac{1}{x+y} + \frac{1}{\pi}\int_0^1 \frac{a(y,z;\mu)}{z+x}\,\mathrm{d}z. \tag{25}$$

The kernel $a(x,y;\mu)$ solves the integral equation (25) for all $\mu \notin \sigma(M)$.

From the representation (21), one can describe properties of the resolvent kernel $a(x,y;\mu)$ as a function of the complex variable μ. It is obvious that a is a holomorphic function of μ acting from $\mathbf{C}\setminus[0,1]$ to $C((0,1]\times(0,1])$, because $\mathcal{P}_p(x)\mathcal{P}_p(y)$ is continuous as a function of $(x,y)\in(0,1]\times(0,1]$. The integral in (24) uniformly converges with respect to μ belonging to any compact subset of $\mathbf{C}\setminus[0,1]$. For any $\mu\in\mathbf{C}\setminus[0,1]$, the operator $A_\mu : L_2(0,1)\to L_2(0,1)$ is bounded. We have the following assertion.

Theorem 4.2. *The resolvent of the Mehler operator is a holomorphic on* $\mathbf{C}\setminus[0,1]$ *operator function in* $L_2(0,1)$. *It is represented by formulas* (22) *and* (23). *The kernel of the integral operator* A_μ *solves equation* (25).

Now we turn to the behavior of the kernel $a(x,y;\mu)$ on the sides of the branch-cut $[0,1]$. We introduce the following notation:

$$a_\pm(x,y;\tau) := \lim_{\epsilon\to 0+} a(x,y;\tau\pm \mathrm{i}\epsilon), \quad \tau\in(0,1)$$

and show the existence of the limits. In order to describe $a_\pm$ explicitly, we consider zeros of the denominator $\cosh(\pi p) - 1/\mu$ in

$$a(x,y;\mu) = \frac{\pi}{2\mu}\int\limits_{-\infty}^{\infty} \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\cosh(\pi p)-1/\mu}\mathrm{d}p. \tag{26}$$

as $\mu\in\mathbf{C}\setminus[0,1]$. From the equation $\cosh(\pi p)-1/\mu = 0$, we formally find its solution $p = \frac{1}{\pi}\log([1/\mu + \sqrt{1/\mu^2-1}])$. We must properly define the branch of the analytic function on the right-hand side of the latter formula.

To this end, we consider $\sqrt{\mu^2-1}$ in the complex plane cut along $[-1,1]$ with its branch fixed by the condition $\sqrt{\mu^2-1}>0$ for $\mu>1$. We then make use of the function $\frac{1-\mathrm{i}\sqrt{\mu^2-1}}{\mu}$ that takes no positive values (see also [23], Sec. 2). This enables us to set $\arg\left(\frac{1-\mathrm{i}\sqrt{\mu^2-1}}{\mu}\right)\in(0,2\pi)$. As a result, the function $\mathbf{p}(\mu)$ given by

$$\mathbf{p}(\mu) := \frac{1}{\pi}\log\left(\frac{1-\mathrm{i}\sqrt{\mu^2-1}}{\mu}\right)$$

is holomorphic on $\mathbf{C}\setminus[-1,1]$. By the direct substitution, we see that $\cosh(\pi\mathbf{p}(\mu)) - 1/\mu = 0$. All solutions of this equation are

$$\pm\mathbf{p}(\mu) + 2im, \quad m = 0,\pm1,\pm2,\dots,$$

because $\cosh(\cdot)$ is even and $2\mathrm{i}\pi$-periodic.

Consider $\mu = \tau+\mathrm{i}\epsilon$, $\tau\in(0,1)$ and some small $\epsilon>0$; then $\mathbf{p}(\tau+\mathrm{i}\epsilon)$ goes to the positive part of the real axis as $\epsilon\to0$ from the lower half-plane $\operatorname{Im}\mu<0$. As a result, the contour of integration in (26) near the point $\mathbf{p}(\tau)>0$ is to be deformed into the upper half-plane $\operatorname{Im}\mu>0$ in order to pass the pole $\mathbf{p}(\tau)$ of the integrand from above. Similarly, the pole at $-\mathbf{p}(\tau)$ is to be passed from below. We denote the contour of integration deformed in such a way by $\mathcal{L}_+$. In the same way, we consider $\mathbf{p}(\tau-\mathrm{i}\epsilon)$ as $\epsilon\to0$ and introduce the contour $\mathcal{L}_-$, which is the complex conjugate of the contour $\mathcal{L}_+$, $\mathcal{L}_- = \mathcal{L}_+^*$. We thus find the limiting values of the kernel a on the sides of the branch-cut,

$$a_\pm(x,y;\tau) = \frac{\pi}{2\tau}\int\limits_{\mathcal{L}_\pm} \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\cosh(\pi p)-1/\tau}\mathrm{d}p, \quad \tau\in(0,1)\,. \tag{27}$$

From formula (27), we conclude that $a_+(x,y;\tau) = \overline{a_-(x,y;\tau)}$.

To study the perturbed Mehler operator, we also need to describe the behavior of $a(x,y;\mu)$ as $\mu\to1$.[6] It is obvious that $a(x,y;\mu)$ is continuous at $\mu=1$ and

$$a(x,y;1) := \lim_{\mu\to1} a(x,y;\mu) = \frac{\pi}{2}\int\limits_{-\infty}^{\infty} \frac{\mathcal{P}_p(x)\mathcal{P}_p(y)}{\cosh(\pi p)-1}\mathrm{d}p\,,$$

where $p=0$ is a removable singularity of the integrand.

Consider some small vicinity $\mathcal{B}_1$ of the point $\mu=1$, where the points of the cut are excluded. It is important to estimate the kernel $a(x,y;\mu)$ as $(x,y)\in(0,1]\times(0,1]$ and, in particular, as $(x,y)\to(0,0)$ with $\mu\in B_1$. We exploit the estimate for the Legendre function

$$|P_{\mathrm{i}t-1/2}(1/x)| \leqslant P_{-1/2}(1/x) \leqslant C\sqrt{x}\log(1+1/x),$$

where C is a positive constant and $x\in(0,1]$, $t\in\mathrm{R}$ (or $t\in\mathcal{L}$, see below). The latter estimate directly follows from the integral representation for $P_{\mathrm{i}t-1/2}$ given above (see also [7], 8.715 and [11]).

Now we turn to the corresponding estimate in $\mathcal{B}_1$. When μ varies in this small domain $\mathcal{B}_1$, we can locally deform the integration contour $(-\infty,\infty)$ in the integral for $a(x,y,\mu)$ in such a way that the denominator

[6] In this work we do not consider $a(x,y;\mu)$ as $\mu\to0$, although its behavior can be also clarified.

$\cosh(\pi p) - 1/\mu$ in (26) is not zero for any $\mu \in \mathcal{B}_1$. Denote the corresponding deformed contour by $\mathcal{L}$. Thus we have

$$|a(x,y,\mu)| \leqslant \frac{\pi}{2xy} \int\limits_{\mathcal{L}} \left| \frac{p\tanh(\pi p)}{\mu\cosh(\pi p) - 1} \right| |P_{\mathrm{i}p-1/2}(1/x)||P_{\mathrm{i}p-1/2}(1/y)||\mathrm{d}p| \\ \leqslant C_1 \tfrac{\pi}{2\sqrt{xy}} \int_{\mathcal{L}} \left| \tfrac{p\tanh(\pi p)}{\mu\cosh(\pi p)-1} \right| |\mathrm{d}p| \log(1+1/x) \log(1+1/y) \leqslant C \tfrac{|\log(2/x)\log(2/y)|}{\sqrt{xy}} \tag{28}$$

for $(x,y) \in (0,1] \times (0,1]$, where the constant C is independent of $\mu \in \mathcal{B}_1$

Theorem 4.3. *The kernel* (26) *of the operator A_μ has the limiting values $a_\pm$ (see* (27)*) on the sides of the branch-cut, and it satisfies the bound* (28) *for $\mu \in \mathcal{B}_1$. The limiting value $a(x,y;1)$ at $\mu = 1$ also exists and admits the bound*

$$|a(x,y;1)| \leqslant C\frac{|\log(2/x)\log(2/y)|}{\sqrt{xy}}, \quad (x,y) \in (0,1] \times (0,1]\,, \tag{29}$$

which follows from (28).

4.3. Resolution of identity for the operator M

Having the resolvent in hand, one can compute the spectral measure of the Mehler operator and the resolution of identity by means of the formula

$$E_t g(x) = \lim_{\epsilon\to 0+} \frac{1}{2\pi\mathrm{i}} \int\limits_{C_t^\epsilon} \mathrm{d}\mu\, [M-\mu]^{-1} g(x) = \lim_{\epsilon\to 0+} \frac{1}{2\pi\mathrm{i}} \int\limits_{C_t^\epsilon} \mathrm{d}\mu\, \frac{-1}{\mu} \left\{I + A_\mu\right\} g(x)\,, \tag{30}$$

where we have used (22), $g \in L_2(0,1)$. The integration contour C_t^ϵ begins at the point $t - \mathrm{i}\epsilon$, then goes below the spectrum $\sigma(M) = [0,1]$, bypasses it from the left, and arrives at the point $t + \mathrm{i}\epsilon$ over the spectrum.

From (30) and (23), one has

$$E_t g(x) = g(x) + \lim_{\epsilon\to 0+} \frac{1}{2\pi\mathrm{i}} \int\limits_{C_t^\epsilon} \mathrm{d}\mu\, \frac{1}{(-\mu)} \frac{1}{\pi} \int\limits_0^1 a(x,y;\mu)\, g(x)\mathrm{d}y\,, \tag{31}$$

where $a(x,y;\mu)$ is given by (26). We change the orders of integration, which is justified, and compute the most inner integral with respect to μ by taking its limit as $\epsilon \to 0+$, $t \in (0,1)$,

$$\lim_{\epsilon\to 0+} -\frac{1}{2\pi\mathrm{i}} \int\limits_{C_t^\epsilon} \frac{\mathrm{d}\mu}{\mu\,[\mu\cosh(\pi p) - 1]} = 0\,, \quad \text{as } t > \frac{1}{\cosh(\pi p)}$$

and

$$\lim_{\epsilon\to 0+} -\frac{1}{2\pi\mathrm{i}} \int\limits_{C_t^\epsilon} \frac{\mathrm{d}\mu}{\mu\,[\mu\cosh(\pi p) - 1]} = -1\,, \quad \text{as } t < \frac{1}{\cosh(\pi p)}.$$

Thus we obtain

$$\begin{aligned} &E_t g(x) = 0, \;\; t \leqslant 0\,, \\ &E_t g(x) = g(x) + \frac{1}{\pi}\int\limits_0^1 e(x,y;t)\, g(y)\mathrm{d}y\,, \quad t \in (0,1]\,, \\ &E_t g(x) = g(x), \;\; t > 1\,, \end{aligned} \tag{32}$$

with

$$e(x,y;t) = -\pi \int\limits_0^\infty \mathcal{P}_p(x)\mathcal{P}_p(y) H\left(\frac{1}{\cosh(\pi p)} - t\right) \mathrm{d}p, \tag{33}$$

where $H(\cdot)$ is the Heaviside unit-step function. In the integral (33), we change the integration variable by $\tau = \frac{1}{\cosh(\pi p)}$ and $p(\tau) = \frac{1}{\pi}\log\left(\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 1}\right)$ implying that $p(\tau) \geqslant 0$, $p(\tau) \to \infty$ as $\tau \to 0+$,

$$e(x,y;\mu) = -\int\limits_0^1 \frac{\mathrm{d}\tau}{\tau} \frac{H(\tau-\mu)}{\sqrt{1-\tau^2}} \mathcal{P}_{p(\tau)}(x)\mathcal{P}_{p(\tau)}(y)\,, \;\; \mu \in (0,1). \tag{34}$$

The representation (34) can be also rewritten as

$$e(x,y;\mu) = -\int\limits_{\mu}^{1} \frac{\mathrm{d}\tau}{\tau\sqrt{1-\tau^2}} \mathcal{P}_{p(\tau)}(x)\mathcal{P}_{p(\tau)}(y)\,, \quad \mu \in (0,1).$$

Lemma 4.1. *For all $x, y \in (0,1)$, the kernel $e(x,y;\mu)$ of the operator E_μ is differentiable with respect to μ and*

$$\frac{\mathrm{d}\, e(x,y;\mu)}{\mathrm{d}\mu} = \frac{\mathcal{P}_{p(\mu)}(x)\mathcal{P}_{p(\mu)}(y)}{\mu\sqrt{1-\mu^2}}\,, \quad \mu \in (0,1).$$

Spectral properties of the Mehler operator are exploited in the study of its perturbations.

5. ESSENTIAL AND DISCRETE SPECTRA OF COMPACT PERTURBATIONS OF THE MEHLER OPERATOR

In this section, we assume that the Mehler operator M in (4) is perturbed by a selfadjoint operator V in (5), which is compact in $L_2(0,1)$. Making use of the Weyl theorem on preserving of the essential spectrum under compact perturbations, we arrive at the following assertion.

Lemma 5.1. *The essential spectrum $\sigma_e(M+V)$ of the perturbed Mehler operator, where V is selfadjoint and compact, coincides with the interval* $[0,1]$.

For instance, the bound $|v(x,y)| \leqslant C/|\log^\alpha(x+y)|$ as $(x,y) \to (0,0)$ with $\alpha > 1/2$, provided that v is continuous, is sufficient for the relation $V \in S_2$ (of the Hilbert–Schmidt class).

With the aim to apply our results to the functional-difference equations (1), we consider the existence problem for the discrete spectrum of $M+V$ on the right of $\sigma_e(M+V)$.[7] In order to obtain sufficient conditions for $\sigma_d(M+V)$ to be nontrivial, we use the following simple argumentation. Due to the minimax principle it is sufficient to submit a function u in $L_2(0,1)$ such that

$$([M+V]u, u) > 1 \tag{35}$$

that reads in our case in the form

$$\int\limits_0^1 \mathrm{d}x \int\limits_0^1 \mathrm{d}y\, \frac{\sqrt{w_0(x)w_0(y)}}{\pi(x+y)}\, u(y)\overline{u(x)} \;>\; 1\,.$$

We can take $u = h/\|h\|$ with $h(x) = \frac{1}{\sqrt{w_0(x)}}$; then inequality (35) is valid provided that

$$\frac{2\log 2}{\pi} > \int\limits_0^1 \frac{\mathrm{d}x}{w_0(x)}. \tag{36}$$

This sufficient condition has been applied in [13] and works for some parameters of the problem under consideration.

We can also consider an alternative condition which is not so simple as the previous one, but we expect that it is efficient for a wider class of potentials. The idea is as follows. We look for a sequence of test functions u_n such that, for some of them, the inequality $([M+V]u_n, u_n) > 1$ holds true. A natural choice is to take a normalized sequence in order to have $Mu_n \approx u_n$ and $(Mu_n, u_n) \approx 1$ for sufficiently large n and such that $([(M-I)+V]u_n, u_n) =: \delta_n > 0$.

To this end, consider a singular (Weyl) sequence $u_n \in L_2(0,1)$, $n = 1, 2, \dots$ corresponding to the point $\mu = 1$ of the essential spectrum of the Mehler operator M, for example, let $\|u_n\| = 1$ and let u_n be an orthogonal sequence, i.e., $u_n \rightharpoonup 0$ (weakly), and ley $\|Mu_n - u_n\| \to 0$ as $n \to \infty$. We obviously obtain

$$([M+V]u_n, u_n) = \|u_n\|^2 + ([M+V-I]u_n, u_n) = 1 + ([M+V-I]u_n, u_n)$$

and conclude from (35) that the condition that the inequality

$$([(M-I)+V]u_n, u_n) > 0, \tag{37}$$

[7] In particular, $M+V \geqslant 0$ in many practical cases.

holds for some n is a sufficient condition for the existence of a discrete spectrum for $\mu > 1$.

We further reduce condition (37) by using the spectral resolution $E_t := E(-\infty, t)$ for the Mehler operator discussed in the previous section. Consider a sequence $\varepsilon_n > 0$ ($\varepsilon_k > \varepsilon_{k+1}$) and $\varepsilon_n \to 0$ (e.g., $\varepsilon_n = 1/n$). Introduce

$$\mathcal{H} := L_2(0,1), \quad \delta_n = (1-\varepsilon_n, 1-\varepsilon_{n+1}), \quad |\delta_n| = \varepsilon_n - \varepsilon_{n+1}.$$

We choose an orthonormal sequence u_n such that $u_n \in E(\delta_n)\mathcal{H}$, noticing that $\dim E(\delta_n)\mathcal{H} = \infty$. From the Spectral Theorem we see that

$$([M-I]u_n, u_n) = \int_{-\infty}^{\infty} (t-1)\mathrm{d}(E_t u_n, u_n) \;=\; \int_{\delta_n} (t-1)\mathrm{d}(E_t u_n, u_n),$$

because $E_t u_n = \|E(\delta_n)h\|^{-1} E_t E(\delta_n)h = 0$ when $h \in \mathcal{H}$ and $\delta_n \cap (-\infty, t) = \emptyset$, recalling also that $\sigma(M) = [0,1]$. We have

$$([V+(M-I)]u_n, u_n) = \frac{1}{\pi}\int_0^1 \mathrm{d}x\, \overline{u_n}(x) \int_0^1 \mathrm{d}y\, \frac{v(x,y)}{y+x}\; u_n(y) \;+ \int_{\delta_n} (t-1)\mathrm{d}(E_t u_n, u_n)$$

$$\geqslant \frac{1}{\pi}\int_0^1 \mathrm{d}x\, \overline{u_n}(x) \int_0^1 \mathrm{d}y\, \frac{v(x,y)}{y+x}\; u_n(y) \;-\; \varepsilon_n (E(\delta_n)u_n, u_n)$$

noting that $1-t \leqslant \varepsilon_n$ for $t \in [1-\varepsilon_n, 1-\varepsilon_{n+1}]$ and

$$-\int_{\delta_n} (t-1)\mathrm{d}(E_t u_n, u_n) \;\leqslant\; \varepsilon_n (E(\delta_n)u_n, u_n)\,.$$

We take into account that $E(\delta_n)u_n = u_n$ and assume for some n that

$$\frac{1}{\pi}\int_0^1 \mathrm{d}x\, \overline{u_n}(x) \int_0^1 \mathrm{d}y\, \left(\frac{v(x,y)u_n(y)}{y+x} - \varepsilon_n\, \pi\, u_n(x) \right) > 0. \tag{38}$$

Introduce the notation

$$\omega_n(x,y) = v(x,y)u_n(y) - \varepsilon_n\; \pi(x+y)u_n(x)\,.$$

Exploiting (35),(37) and (38), we arrive at the following assertion.

Theorem 5.1. *Let V be selfadjoint and compact in $\mathcal{H} = L_2(0,1)$ (see (5)) and let, for some n, the inequality*

$$\frac{1}{\pi}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \frac{\omega_n(x,y)}{y+x}\; \overline{u_n}(x) \;>\; 0 \tag{39}$$

hold. Then the perturbed Mehler operator $M+V$ has a nontrivial discrete spectrum on the right from $\sigma_e(M+V) = [0,1]$.

It is worth commenting on the sufficient condition (39). We can take $u_n(x)$ to be real, $\overline{u_n}(x) = u_n(x)$. Then condition (39) can be written in the symmetric form

$$\frac{1}{\pi}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \frac{\Omega_n(x,y)}{y+x} \;>\; 0 \tag{40}$$

with

$$\Omega_n(x,y) = v(x,y)u_n(y)u_n(x) - \varepsilon_n\; \pi(x+y)\frac{u_n^2(x) + u_n^2(y)}{2}\,.$$

We recall that, for the functional-difference equations, one has $v(x,y) = \sqrt{w_0(x)w_0(y)} - 1$, and, therefore,

$$\Omega_n(x,y) = (\sqrt{w_0(x)w_0(y)} - 1)u_n(y)u_n(x) - \varepsilon_n\; \pi(x+y)\frac{u_n^2(x) + u_n^2(y)}{2}\,.$$

Condition (39) (and (40)) is written in terms of the perturbation V and some objects (the Weyl sequence for $\mu = 1$) dealing with the operator M. Having the spectral measure of M, one can construct this sequence efficiently. In practice, for any concrete $v(x, y)$, condition (39) could be verified, say, numerically. Condition (39) is connected with the values of $\omega_m(x, y)u_n(x)$ on the square $[0, 1] \times [0, 1]$. From (40), we then obtain a sufficient condition for the nonemptiness of the discrete component, also in the form

$$\frac{1}{\pi}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{\sqrt{w_0(x)w_0(y)} - 1}{y + x} \, u_n(y)\overline{u_n}(x) \, > \, \varepsilon_n \tag{41}$$

for some $n = 1, 2, \dots$, where u_n is a Weyl sequence for $\mu = 1$, $\|u_n\| = 1$, and $\varepsilon_n \to 0$ as $n \to \infty$.

However, to make conditions (39),(40) and (41) more constructive, it is useful to write out u_n explicitly, which is not difficult because the spectral resolution E_t is known. The sequence

$$u_n = w_n / \|w_n\|$$

is normalized and orthogonal, since $w_n = E(\delta_n)h$ with $h \in \mathcal{H}$,

$$w_n(x) = \frac{1}{\pi}\int_0^1 \mathrm{d}y \; \left(e(x, y; 1 - \varepsilon_{n+1}) - e(x, y; 1 - \varepsilon_n)\right) h(y)$$

$$= \frac{1}{\pi}\int_0^1 \mathrm{d}y \, h(y) \int_0^\infty \mathrm{d}p \left(H\left(\frac{1}{\cosh(\pi p)} - [1 - \varepsilon_n]\right) - H\left(\frac{1}{\cosh(\pi p)} - [1 - \varepsilon_{n+1}]\right)\right) \mathcal{P}_p(x)\mathcal{P}_p(y)$$

$$= \frac{1}{\pi}\int_0^1 \mathrm{d}y \, h(y) \int_{p_{n+1}}^{p_n} \mathrm{d}p \mathcal{P}_p(x)\mathcal{P}_p(y),$$

where

$$p_n = \frac{1}{\pi}\log\left(\frac{1}{[1 - \varepsilon_n]} + \sqrt{\frac{1}{[1 - \varepsilon_n]^2} - 1}\right) = O(\sqrt{\varepsilon_n}) \, .$$

We change the order of integration in the expression for w_n and obtain

$$w_n(x) = \frac{1}{\pi}\int_{p_{n+1}}^{p_n} \mathrm{d}p \, \mathcal{P}_p(x) \, h^*(p), \tag{42}$$

where $h^*(\tau)$ is the mMF-transform (17) of $h \in \mathcal{H}$. Note that h can be chosen in an optimal way and u_n is an orthonormal sequence by construction. We can always choose $h \in \mathcal{H}$ in such a way that $h^* > 0$. The function $\mathcal{P}_p(x)$ is positive for $p \in [p_{n+1}, p_n]$ for sufficiently small ε_n, because $P_{-1/2}(1/x) > 0$. As a result, if it is necessary for the analysis, we can always assume that $u_n(x) \geqslant 0$ (see (42)) for sufficiently large n, because h^* can be taken positive and such that $h(x) = \int_0^\infty \mathcal{P}_p(x) \, h^*(p)\mathrm{d}p \, , \quad h \in \, L_2(0, 1)$.

5.1. *Finiteness of the discrete component*

We turn to the discussion of finiteness of the discrete component. In this section, we follow the line of the paper [23] and make use of the Birman–Schwinger principle which takes the form

Theorem 5.2. *Let M_0 be a bounded and selfadjoint operator such that $M_0 \leqslant 1$. Let $V_0 \geqslant 0$ and $V_0 \in S_\infty$ (i.e., V_0 is compact). Then the total number (i.e., counted according to the multiplicity) of the eigenvalues of $K_0 = M_0 + V_0$ that are greater than μ ($\mu \geqslant 1$) is equal to the total number of the eigenvalues of the operator $B(\mu) = V_0^{1/2}[\mu - M_0]^{-1}V_0^{1/2}$.*

Making use of the representation (22) for the resolvent, one has

$$B(\mu) = \mu^{-1}(V + V^{1/2}A_\mu V^{1/2}) \, . \tag{43}$$

Introduce the operator Q,

$$[Qf](t) =< \log t > f(t),$$

with $< \log t >= \log(2/t)$, $f \in L_2(0, 1)$. It is easy to define Q^β , for $\beta \in \mathrm{R}$ in a natural way.

In accordance with (23), the operator A_1 is an integral operator with the kernel $a(x, y; 1)$ admitting the bound (28) and (29) with $\mu = 1$. We arrive at the following assertion.

Lemma 5.2. *Let $\alpha > 3/2$. Then*

$$\lim_{\mu\to 1} \|Q^{-\alpha}\,(A_\mu - A_1)\,Q^{-\alpha}\|_2 \,=\, 0 \tag{44}$$

in the Hilbert–Schmidt norm.

Indeed, we should prove that

$$\lim_{\mu\to 1} \int_0^1 \int_0^1 \, < \log t >^{-2\alpha} \, |a(x,t,\mu) - a(x,t,1)|^2 < \log x >^{-2\alpha} \, \mathrm{d}x\mathrm{d}t \,=\, 0\,.$$

Due to the Lebesgue dominated convergence theorem, one should represent an integrable majorant for the integrand. In view of the bounds (29) and(28), we find such a function in the form

$$C \, < \log t >^{-2\alpha} \, \frac{< \log t >^2 < \log x >^2}{t\,x} \, < \log x >^{-2\alpha}$$

which is obviously in $L_1((0,1)\times(0,1))$ for $2\alpha > 3$. Lemma 5.2 is followed by the next assertion.

Lemma 5.3. *Let $\alpha > 3/2$, let the operator $V \geqslant 0$, $V \in S_2$ be of the Hilbert–Schmidt class, and let $Q^\alpha V Q^\alpha \in S_\infty$ be compact. Then the operator $B(\mu)$ in (43) has the limit*

$$B(1) = V + V^{1/2} A_1 V^{1/2} \tag{45}$$

in the Hilbert–Schmidt norm $\|\cdot\|_2$ as $\mu \to 1$.

The proof follows from the sequence of equalities

$$\|B(\mu) - B(1)\|_2^2 = \|V^{1/2} Q^\alpha Q^{-\alpha}(A_\mu - A_1) Q^{-\alpha} Q^\alpha V^{1/2}\|_2^2 = \|Q^\alpha V Q^\alpha Q^{-\alpha}(A_\mu - A_1)Q^{-\alpha}\|_2^2,$$

where $\|A\|_2^2 =< A, A >_{S_2}$, $\quad < A, B >_{S_2} := \mathrm{Tr}(B^* A) = \mathrm{Tr}(AB^*)$.

Let $N(\mu)$ be the total number of eigenvalues of the operator $K = M + V$ located on the right from μ, $(\mu \geqslant 1)$. From the Birman–Schwinger principle, one has $N(\mu) \leqslant \|B(\mu)\|_2^2$; then, by Lemma 5.3, we arrive at the bound

$$N(1) \leqslant \|B(1)\|_2^2.$$

Then, using (45), we see that

$$\|B(1)\|_2 \leqslant (\|V\|_2 + \|Q^\alpha V Q^\alpha\| \|Q^{-\alpha} A_1 Q^{-\alpha}\|_2).$$

Taking into account properties of $a(x,t,1)$, we introduce

$$G_\alpha^2 := \int_0^1 \int_0^1 \, < \log t >^{-2\alpha} \, |a(x,t,1)|^2 < \log x >^{-2\alpha} \, \mathrm{d}x\mathrm{d}t\,.$$

This leads to the bound

$$N(1) \,\leqslant\, (\|V\|_2 \,+\, G_\alpha \|Q^\alpha V Q^\alpha\|)^2 \tag{46}$$

and to the following assertion.

Theorem 5.3. *Let $\alpha > 3/2$, let the operator $V \geqslant 0$, $V \in S_2$ be in the Hilbert–Schmidt class, and let $Q^\alpha V Q^\alpha \in S_\infty$ be compact. Then the total number $N(1)$ of eigenvalues of the operator $K = M + V$ that are greater than $\mu = 1$ is finite and satisfies the bound* (46).

If, in addition, the sufficient conditions (36) *or* (39), (40) *are satisfied, then the discrete spectrum of the operator K on the right of $\mu = 1$ is nonempty.*

The result is analogous to that obtained in [23]. Note that, if the condition $\alpha \leqslant 3/2$ holds instead of $\alpha > 3/2$ in Theorem 5.3, then the discrete component is infinite.

6. APPLICATIONS TO THE FUNCTIONAL-DIFFERENCE EQUATIONS AND EXAMPLES

Now we turn to nontrivial solutions of the functional-difference equation (1) for the potentials W belonging to $\mathcal{W}$. In this way, we exploit the results of the last two sections. Any nontrivial solution ρ of the equation $K\rho = \mu\rho$ corresponds to a solution of the functional-difference equation (1) for $\Lambda = \mu^{-1}$. The functional-difference equation (1) has a set of characteristic values $\Lambda_m = \mu_m^{-1}$ in (14), where $\mu_m \in \sigma_d(M+V)$ implying that the discrete spectrum is nonempty. We can specify the corresponding solutions of (1) by means of the eigenfunctions ρ_m of $M+V$, satisfying the equation

$$(M+V)\rho_m = \mu_m \rho_m.$$

We have

$$H_m(\nu) = \frac{\rho_m(x)|_{x=1/\cos\pi\nu}}{\cos\pi\nu\sqrt{W(\nu)}}, \quad \Lambda_m = \mu_m^{-1} \tag{47}$$

so that Λ_m and H_m are characteristic values and eigenfunctions of equation (1), and $\Lambda_m \in (0,1)$. Recall that H_m in (47) can be continued to the complex plane from the imaginary axis as a meromorphic function, as was described above. From $\rho \in L_2(0,1)$, one has

$$\int_0^{\mathrm{i}\infty} |d\tau|\,|H(\tau)|^2 W(\tau)|\sin\pi\tau| \,<\, \infty\,. \tag{48}$$

Note that the eigenfunctions H_m belonging to $\mathcal{M}$ satisfy the bound $|H(\nu)| \,<\, \mathrm{Const}\,|\exp(-\mathrm{i}\nu[\pi/2+\delta_0])|$, $\nu \to \mathrm{i}\infty,\ \nu \in \Pi_{1+\delta}$ for $\delta_0 \in (0,\pi/2)$, which implies (48).

The procedure of reconstruction of the meromorphic function $H_m(\nu)$ from $\mathcal{M}$ is as follows. Let one can construct an integrable (and then continuous) solution $H_m(\nu)$ of the integral equation (12) on $(\mathrm{i}\infty, \mathrm{i}\infty)$ for some Λ_m, which exponentially decays at infinity. This specifies $H_m(\nu)$ on the imaginary axis, and then it is continued to the strip Π_δ for some positive δ. The integral representation (12) enables one to compute values of $H_m(\nu)$ in the strip $\Pi_{1+\delta}$ of its regularity. Indeed, having $H_m(\nu)$ in hand in some vicinity Π_δ of the imaginary axis, we see that the integral on the right-hand side of (12) specifies a holomorphic function in the strip $\Pi_{1+\delta}$, because the denominator has no zeroes in this strip, whereas the integral converges exponentially and uniformly with respect to ν. Then $H_m(\nu)$ is continued as a meromorphic function to the complex plane C by means of the functional-difference equation (1).

It is important to have not only the bound (48) for $H_m(\nu)$ in (47) corresponding to $\Lambda_m =: \sin\tau_m$ with $\tau_m \in (0,\pi/2)$ but also to compute the asymptotics as $\nu \to \pm\mathrm{i}\infty$ in the strip $\Pi_{1+\delta}$. As is known, the asymptotics of a function at infinity on the complex plane can be computed by means of localization of the singularities of its Fourier transform. Exploiting this simple observation for H_m and studying singularities of the Fourier transform (see Appendix), after some tedious work, we can conclude that

$$H_m(\nu) = O(\exp(\mathrm{i}\nu[\pi-\tau_m])|,\ \ \nu \to \mathrm{i}\infty,\ \ \nu \in \Pi_{1+\delta}\,.$$

Now we can describe the so-called characteristic set $C_d \cup C_e$ of Λ for equation (1). The set of the characteristic values Λ_m is nonempty and finite if and only if the same holds for $\sigma_d(M+V)$. The corresponding theorem in the previous section gives a sufficient condition for this fact.[8] We say, by definition, that $\Lambda_m = 1/\mu_m$ belongs to the set C_d of characteristic values of equation (1) if $\mu_m = (\sin\tau_m)^{-1} \in \sigma_d(M+V)$. It is obvious that $C_d \subset [0,1]$. In the same way, $\Lambda \in C_e = [1,\infty)$, i.e., by definition, its belonging to the essential characteristic set means that $\mu = \Lambda^{-1} \in \sigma_e(M+V) = [0,1]$, where $M+V$ is the perturbation of the Mehler operator attributed to equation (1) with the potential $W \in \mathcal{W}$. In this case, $|H(\nu)| \,<\, \mathrm{Const}\,|\exp(\mathrm{i}\nu\pi/2)|$, $\nu \to \mathrm{i}\infty,\ \ \nu \in \Pi_{1+\delta}$.

By the results of the previous two sections, the following assertion holds.

Proposition 6.1. *The set C_d of characteristic values of equation* (1) *is nonempty if the potential $w_0(x) = W|_{x=1/\cos\pi\nu}$ in $v(x,y) = \sqrt{w_0(x)w_0(y)} - 1$ satisfies the sufficient conditions* (36) (*or* (40), (41)). *The corresponding solutions satisfy the bound* (48) *and belong to the class $\mathcal{M}$.*

The finiteness of the characteristic set C_d is described by Theorem 5.3.

[8] Solutions of (1) that correspond to $\Lambda = \mu^{-1}$ with $\mu \in \sigma_e(M+V)$, i.e., specified by the essential spectrum, can also be studied.

6.1. Examples

We turn to some examples of potentials that arise in applications, see, e.g., [13, 12]. We begin with the potential $W(\mathrm{i}t) = \mathrm{i}\cot(\mathrm{i}\Phi t)$ from the Sec. 2. We make use of a simpler sufficient condition (that follows directly from (36) and is also discussed in Sec. 3.2 of [13]) for the existence of the discrete spectrum

$$\pi \int_0^\infty \mathrm{d}t \frac{\sinh(\mathrm{i}\pi t)}{\cosh^2(\mathrm{i}\pi t) W(\mathrm{i}t)} < \frac{\log 2}{\pi} \tag{49}$$

that is satisfied for $0 < \Phi < \pi/2$, where Φ is also assumed to be small enough. In view of the bounds $w_0(x) = W(\mathrm{i}t)|_{x=1/\cosh(\pi t)} = 1 + O(x^{2\Phi/\pi})$ as $x \to 0$, it can be shown that the discrete spectrum is finite in this problem (see Theorem 5.3). This fact is also known from the literature (see, e.g., [14]).

The second example deals with the potential $W(\mathrm{i}t) = \frac{\mathrm{i}}{2}\left(\cot(\mathrm{i}\Phi t) + \cot(\mathrm{i}\bar{\Phi} t)\right)$, where $0 < \Phi < \pi/2$ and $\bar{\Phi} = \pi - \Phi$. This potential arises in the study of eigenfunctions for the Laplacian with δ'-potential supported on the common boundary of two angles in R^2_+, [13]. As in the previous example, the discrete spectrum of the corresponding perturbed Mehler operator $M + V$ exists and is shown to be finite. This means that the number of characteristic values of the functional-difference equation (1) is finite in accordance with Theorem 5.3.

In the next example, the potential W takes the form

$$W(\mathrm{i}t) = \frac{t}{2}\left(\frac{P_{\mathrm{i}t-1/2}(\cos\theta)}{\partial_\theta P_{\mathrm{i}t-1/2}(\cos\theta)} - \frac{P_{\mathrm{i}t-1/2}(-\cos\theta)}{\partial_\theta P_{\mathrm{i}t-1/2}(-\cos\theta)}\right),$$

Such a kind of potentials occurs in the study of the discrete spectrum of Laplacians in cone-shaped domains [12] in the framework of incomplete separation of variables. In this case, the potential has the asymptotics $W(\mathrm{i}t) = 1 + O(1/t)$ and

$$v(x, y) = O\left(\frac{1}{\log(x)}\right) + O\left(\frac{1}{\log(y)}\right), \quad (x, y) \to (0, 0).$$

The set of characteristic values is infinite for this problem (the condition $\alpha > 3/2$ in Theorem 5.3 fails to hold), and they accumulate at $\Lambda = 1$. These results about the discrete spectrum are known for problems in the cone-shaped domains [10], [12] from which this potential gives rise.

Remark that the potentials in our examples belong to the class $\mathcal{W}$. We expect that other potentials in this class can also be attributed to various applications of the functional-difference equation (1) in diffraction or quantum scattering theory.

7. CONCLUSION

In this paper, we studied some questions dealing with spectral properties of the functional-difference equation of the second order with meromorphic potential of a certain class. We reduced it to an integral equation with a characteristic parameter. The integral operator in the equation is an operator which is considered as a compact perturbation of the Mehler operator. The latter is an explicitly solvable model, so that some of its spectral properties are studied. The corresponding results are given in an explicit form and are actually based on the well-known Mehler formulas of 1881.

We then discussed the existence of the discrete spectrum of perturbations of the Mehler operator. In this way, we exploited some information about the spectral properties of the Mehler operator, in particular, about its resolvent and spectral measure. The problem of finiteness of the discrete spectrum is studied by means of the Birman–Schwinger principle. This problem is directly connected with the behavior of the potential $W(\mathrm{i}t)$ as $t \to \infty$, i.e., with the rate of decay of $|W(\mathrm{i}t) - 1|$. The results obtained here are then applied to some examples that occur in applications.

A further development of the approach might deal with new applications of the functional-difference equations in diffraction theory or in quantum scattering.

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8. APPENDIX. DUAL INTEGRAL EQUATION TO (12)

In this section, we consider a singular integral equation that is equivalent to equation (12). Together with equation (12) derived from the FD equation (1) and playing a principal role in the study, we consider its natural counterpart. Namely, a singular integral equation will be derived herein. Making use of the Fourier transform along the imaginary axis

$$\chi(\zeta) = \int_{\mathrm{iR}} \mathrm{e}^{\mathrm{i}\zeta\nu} h(\nu) \mathrm{d}\nu, \qquad h(\nu) = -\frac{v.p.}{2\pi} \int_{\mathrm{iR}} \mathrm{e}^{-\mathrm{i}\zeta\nu} \chi(\zeta) \mathrm{d}\zeta$$

for (1), we have

$$\sin\zeta\, \chi(\zeta) - \Lambda \int_{\mathrm{iR}} \mathrm{e}^{\mathrm{i}\zeta\nu} \operatorname{sign}(\mathrm{i}\nu)\, h(\nu)\, \mathrm{d}\nu + \Lambda \int_{\mathrm{iR}} \mathrm{e}^{\mathrm{i}\zeta\nu} \left[W(\nu) + \operatorname{sign}(\mathrm{i}\nu) \right] h(\nu)\, \mathrm{d}\nu = 0.$$

We take into account that

$$W(\nu) + \operatorname{sign}(\mathrm{i}\nu) = \mathbf{O}(\exp\{\pm \mathrm{i} q_* \nu\})$$

for $q_* > 0$ as $\nu \to \pm\mathrm{i}\infty$ along the imaginary axis. Exploit the formula

$$\int_{\mathrm{iR}} \mathrm{e}^{\mathrm{i}\zeta\nu} \operatorname{sign}(\mathrm{i}\nu)\, \mathrm{d}\nu = (-2\mathrm{i}) P\left(\frac{1}{\zeta}\right)$$

in the sense of distributions, where $P(1/\zeta)$ is the Cauchy principle value. We arrive at

$$\sin\zeta\, \chi(\zeta) - \Lambda \frac{v.p.}{\mathrm{i}\pi} \int_{\mathrm{iR}} \frac{\chi(\tau)}{\tau - \zeta}\, \mathrm{d}\tau - \frac{\Lambda}{2\pi} \int_{\mathrm{iR}} \mathbf{Q}_a(\zeta - \tau) \chi(\tau)\, \mathrm{d}\tau = 0, \tag{50}$$

where $\zeta \in \mathrm{iR}$,

$$Q(\zeta) = \int_{\mathrm{iR}} \mathrm{e}^{\mathrm{i}\zeta\nu} \left[W(\nu) + \operatorname{sign}(\mathrm{i}\nu) \right] \mathrm{d}\nu\,,$$

$\mathbf{Q}_a(\zeta)$ vanishes as $\zeta \to \pm\mathrm{i}\infty$. We also have $\|\chi(\zeta)\| < C\exp(-\alpha_0|\zeta|)$, $\alpha_0 > 1$ as $\Im\zeta \to \infty$, since $\mathbf{h}$ is holomorphic in the strip $\Pi(-1-\delta, 1+\delta)$.

Equation (50) is a classical singular integral equation that can be used for further studies. This equation can be regularized from the left by using the singular operator

$$R_l \chi_0(\zeta) := \sin\zeta\, \chi_0(\zeta) + \Lambda \frac{v.p.}{\mathrm{i}\pi} \int_{\mathrm{iR}} \frac{\chi_0(\tau)}{\tau - \zeta}\, \mathrm{d}\tau\,.$$

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