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FREE VIBRATIONS OF THE CIRCULAR CYLINDRICAL SHELLS OF VARIED THICKNESS

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Introduction.

This paper is devoted to the problems of free vibrations of thin circular cylindrical shells of varied thickness. The mathematical theory of thin cylindrical shells of constant thickness is well developed. The different approaches to this problem are known and only to name all the works devoted to it seems to be impossible.

Several of the most successful approaches are two dimensional theories of the Kirchhof-Love type. Using Novozhilov shell theory, which is of this type, A.L.Goldenveiser, V.B.Lidsky, P.E.Tovstik have developed the theory of asymptotic integration of the equations of shell vibrations [1-3]. This theory allows one to estimate and in some cases to find analytical solutions for the eigenvalues. The main results of this theory are presented by A.L.Goldenveiser et al. in a monograph [2] (1979).

The aim of this paper is to apply asymptotic method to the solution of the eigenvalue problem for a cylindrical shell of varied thickness. We use Novozhilov's two-dimensional shell theory to obtain the equations for the vibrations of the shell. Then we apply the theory of asymptotic integration of the differential equations to solve the eigenvalue problem for these equations.

Low-frequencies vibrations of a shell.

Consider the thin cylindrical shell of thickness h(s). To describe the shell geometry we introduce an orthogonal curvilinear coordinate system, connected with the meridians and parallels of a shell. A position of a point on the neutral surface of a shell is defined by a longitudinal angle φ and the length of the arc of meridian s. The shell is limited with two parallels s=0 and s=l. The thickness of the shell is characterized by the function h(s). At each point on the neutral surface we introduce a local system of cartesian coordinates, the axes of which are the tangent to the meridian, the tangent to the parallel and the initial normal. u is a displacement vector with components u,v,w in the local coordinate system.

We use the following notation:

- E - Young's modulus.
- ν - Poisson's ratio.
- ρ - density of shell material.
- l - length of cylindrical shell.
- R - radius of cylindrical shell.
- h_0 - characteristic thickness of shell.

We will use the non-dimensional variables. It means, for example, that $l^*=l/R$, $u^*=u/R$ and so on. Later we omit the sign "*".

The entire system describing the vibrations of shells is represented in [1]:

$$\frac{d}{ds}\left[\frac{Eh}{1-\nu^2}(u' + \nu(mv-w))\right] - m\frac{Eh}{1+\nu}\left[\frac{v-mu}{2}\right] - Eh\lambda u = 0,$$

$$\frac{d}{ds}\left[\frac{Eh}{1+\nu}\,\frac{v-mu}{2}\right] - m\frac{Eh}{1-\nu^2}(mv - w + \nu u') + Eh\lambda v = 0,$$

$$\frac{Eh}{1-\nu^2}(mv - w + \nu u') + \frac{d^2}{ds^2}\left[\frac{Eh^3}{12(1-\nu^2)}(w - \nu m(mv - w''))\right] +$$
$$+ m^2\frac{Eh^3}{12(1-\nu^2)}\left[(m(v-mw) + \nu w'')\right] +$$
$$+ 2m\frac{d}{ds}\left[\frac{Eh^3}{12(1+\nu)}(v' - mw')\right] + Eh\lambda v = 0, \qquad (1)$$

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Here λ = [illegible] – parameter of eigenvalue, m – number of waves in x direction.

To get the equations of vibrations for a cylindrical shell of varied thickness we use the method proposed by [illegible] et al. in [illegible]. The main idea of this method is the representation of all variables in the form of asymptotic series with respect to small parameter ε

$$w = \exp\left\{\int_0^s p(s)ds\right\} \sum_{k=0}^{\infty} \varepsilon^k w_k(s),$$

$$\lambda = \sum_{k=0}^{\infty} \varepsilon^k \lambda_k \qquad (2)$$

(here w means (u, v, w)). Later we consider only low frequencies vibrations for which [illegible]. Substituting representation (2) into (1) and equalising the terms of the equal orders we obtain in the zero approximation the equation of low frequencies vibrations of the shell

$$\frac{1}{h}\frac{d^2}{ds^2}\left[h\frac{d^2 w_0}{ds^2}\right] + \left[\frac{m^8 h^6}{12(1-\nu^2)} - \lambda m^4\right] w_0 = 0 \qquad (3)$$

Because of we consider only the first term of the expansion the correctness of this equation has the order of $\sqrt{h_0}$.

For the shell of constant thickness the similar equation has been obtained in [2]. The same equation has been obtained, for example, in [5],[6], where the authors applied the theory of flat shells.

If $h = h_0 = const$ we can evaluate the eigenvalue λ. The solutions of (3) for the constant thickness have forms of beam functions and for the parameter λ we get

$$\lambda = \lambda_0 + O(h_0^{[illegible]}), \qquad (4)$$

$$\lambda_0 = \frac{a^4}{m^4 l^4} + \frac{m^4 h_0^2}{12(1-\nu^2)}, \qquad (5)$$

where a depends on boundary conditions (for example, [illegible] for the shells of free supported edges).

The shell with small variability of the thickness

Generally the equation (3) may be solved only numerically. But it is interesting to consider the important case of a shell of varied thickness when h varied slightly along the generator. [illegible] $h = h_0(1+\varepsilon f(s))$, where [illegible]. It is clear that for the shell of constant thickness $\varepsilon = 0$.

We consider only low-frequencies vibrations for which $m \approx h_0^{-1/4}$, $\varepsilon \approx h_0$. We can try to find the correction in the formula for λ caused by the variety of the thickness.

First we consider the case $\varepsilon \approx h_0^{1/2}$. To find the eigenvalue parameter λ we expand λ and w into the power series and apply the well known method of disturbances.

Let

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots,$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots \qquad (6)$$

Substitution of the expressions (6) into equation (4) followed by comparison of the terms up to ε, leads to the formula for λ_1

$$\lambda_1 = \left[\frac{m^4 h_0^2}{4(1-\nu^2)} - \lambda_0\right] \frac{\int_0^1 f(s) w_0^2 ds}{\int_0^1 w_0^2 ds} + \frac{1}{m^4} \frac{\int_0^1 f(s) (w_0'')^2 ds}{\int_0^1 w_0^2 ds} \qquad (7)$$

Therefore the formula for eigenvalue parameter λ will be

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + O(h^{[illegible]}) \qquad (8)$$

Obviously the formula (8) for the shells of varied thickness has the same precision as that of the formula (4) for constant thickness shells.

Considering the general case ($\varepsilon \approx h^{\beta}, \beta > 0$) one can make the following statements [4]:

(i) if [illegible] $\le \beta \le$ [illegible] the formula (8) is valid and its precision is the same as that of the formula (4) for the shells with constant thickness.

(ii) if $0 < \beta <$ [illegible] we can not achieve the same precision without taking into account the term of the second order in expansion (6) for λ.

(iii) if $\beta \ge$ [illegible] the formula (4) is applicable for both shells of constant and varied thickness and the second term in the formula (8) may be neglected.

The partial cases

To apply the formulas obtained above we need to evaluate the functions $w_0(s)$. It is well known that the solution of the equation (4) for $h = const$ are beam functions $w_0^{(k)}(s) = X^{(k)}(a^{(k)} s)$, where $a^{(k)}$ and $X^{(k)}$ depend on the boundary conditions, for example, for the one clamped edge ($s=0$) and another free edge ($s=1$) $a=1.88$ and

$$X(x,\alpha)=\frac{\sin(\alpha(x-1/2))}{\sin(\alpha/2)}+\frac{ch(\alpha(x-1/2))}{ch(\alpha/2)} \qquad (9)$$

Any continuous functions may be approximated with the polynomial. That is why we may restricted with the polynomial case for h(s). Let, for example,

$$h(s)=A_0+A_1 s+A_2 s^2 \qquad (10)$$

In this case for the boundary conditions mentioned above

$$\lambda_1=\left(\frac{m^4 h_0^2}{6(1-\nu^2)}-\left[\frac{\alpha}{ml}\right]^4\right)(a_0+0.807a_1+0.675a_2)+$$

$$+\left[\frac{\alpha}{ml}\right]^4(a_0-0.193a_1+0.0615a_2) \qquad (11)$$

where $a_0=\frac{A_0-h_0}{h_0}$, $a_1=\frac{A_1 l}{h_0}$, $a_2=\frac{A_2 l^2}{h_0}$ and h_0 is some mean value of shell thickness. Here and later we shortly write λ_1 instead of $\varepsilon\lambda_1$ for the eigenvalue parameter correction.

As an example we consider the vibrations of a cylindrical shell with linearly varied thickness for which

$$h(s)=h_0(1+\gamma(s-1/2)) \qquad (12)$$

and the formulas for the λ_1 will be

$$\lambda_1=0.614\,\gamma\,l\left[\frac{\pi^4 h^4}{12(1-\nu^2)}\left(\frac{\alpha}{ml}\right)^4\right] \qquad (13)$$

The effect of varied thickness on the eigenvalue parameter λ one can see on fig.1

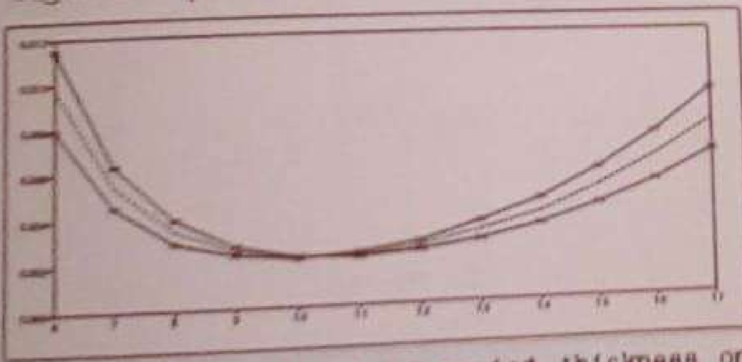

Fig.1 The effect of varied thickness on the eigenvalue parameter λ. (o for γ =-0.3, x for γ=0.3, - for γ=0).

It is interesting to examine the joint effect of the boundary conditions and the variability of the thickness on the eigenvalues.

It proves to be in the considered case that for $m<m_0$ the thickness of the edge with more rigid boundary conditions has more effect on the eigenvalues and vice versa for the frequencies corresponding to the modes with $m>m_0$ the influence of the free edge thickness is more important.

Applying the energy principle we can get the interpretation of this phenomena. For the vibrations with large value of wave number m the main part of energy is bending-twisting energy (4) which is concentrated near the most weak supported edge (in this case near the free edge). For the vibrations with small wave number m the main part of energy is vice versa is the elongation-shear energy which is larger near the clamped edge.

As a numerical example we considered the cylindrical shell with the relative thickness h_0=0.001, and γ=0.1, the boundary conditions are of clamped-free type. For such kind of a shell the lowest value of the parameter λ is equal to $1.84*10^{-3}$ and corresponds to the mode with m=10. The values of the correction λ_1 are presented in the table 1

Table 1 The effect of wave parameter m on exact and asymptotic values of λ_1

m	λ_1*10^4	$\lambda_1^n*10^4$
6	5.3	3.4
7	2.8	2.0
8	1.5	1.2
9	0.7	0.6
10	0.2	0.1
11	-0.3	-0.3
12	-0.7	-0.7
13	-1.2	-1.2
14	-1.8	-1.7
15	-2.5	-1.9
16	-3.2	-2.9
17	-4.2	-3.2

Here λ_1 is eigenvalue parameter correction calculated due to the formula (13) and λ_1^n is eigenvalue parameter correction determined as a difference between numerical solutions of the equations (1) for γ=0 and for γ=0.1.

Conclusion

The formulas for the eigenvalues of the cylindrical shells of varied thickness with different boundary conditions are obtained and some cases of the thickness variety are considered in the paper. The proposed asymptotic method allowed to estimate the effect of the variety of thickness and rigidity of supports on the eigenvalues.

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