CALCULATION OF THE THERMOELASTIC STRAIN OF A MIRROR

S. M. Bauer and A. L. Smirnov

Vestnik Sankt-Peterburgskogo Universiteta
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We study the axisymmetric stressed-strained state of a uniformly heated circular mirror with a supporting ring. The deformations of the reflecting mirror surface are caused by differences in the mechanical characteristics of the mirror and the ring. The results obtained with the thin plate theory are compared with those obtained by using the finite-element method.

It is well-known that even a slight deformation of the reflecting surface of an optical mirror, which is an essential element of a telescope, substantially affects the performance of the instrument. Thermal effects acting upon the mirror in the course of operation are a major cause of deformations. The normal deflection of the reflecting surface of the mirror is a particular cause for concern.

We study the axisymmetric stressed-strained state of a uniformly heated round mirror with thickness h_1. The cross section of the mirror by a plane parallel to the reflecting surface at $h < h_0$ is a ring with inner radius r_0 and outer radius r_1. At $h_0 < h < h_1$ it is a circle of radius r_1. A supporting ring is inserted into the bottom annular part of the mirror. It is conjugated with the mirror on radius r_0 at height $0 < h < h_2$. The supporting ring and mirror are made of different materials, and the difference between the thermal expansion coefficients of the mirror α_1 and reference ring α_2 causes deformation of the reflecting surface of the mirror when the temperature varies by T degrees.

To derive equations which describe the strained state of the mirror, we divide the structure into three components: annular plate I with thickness h_1, outer radius r_1, and inner radius r_0; supporting ring II with thickness h_2, outer radius r_0, and inner radius r_2; and round plate III with thickness $h_3 = h_1 - h_0$ and radius r_0.

We investigate the stressed-strained state of structural components I, II, and III in terms of the theory of thin plates.

1. We first consider round plate III. If we treat the median surface of the plate as the reduction surface, we can write for definition of the elastic component of radial displacement $-u_3^{(y)}$, normal plate deflection w_3, radial force $-M_3^R$, and radial moment $-N_3^R$ in polar coordinates, taking into account that the plate is closed at the center [1]:

$$u_3^{(y)} = C_1 r, \quad w_3 = -\frac{C_2}{4} r^2 + C_3,$$
$$N_3^R = B_3 C_1 (1 + \nu_3), \quad M_3^R = \frac{C_2}{2} D_3 (1 + \nu_3) \qquad (1)$$

Here, $B_3 = E_1 \cdot h_3/(1 - \nu_1^2)$, $D_3 = E_1 \cdot h_3^3/12(1 - \nu_1^2)$ are the plate's rigidity coefficients; E_1 and ν_1 are the elastic modulus and Poisson's ratio of the mirror; C_1, C_2, and C_3 are coefficients to be found from the boundary conditions or in this case from the conjugation conditions.

Rotation angle ψ_3 of the normal can be found from relation

$$\psi_3 = -\frac{dw_3}{dr} = \frac{C_2}{2} r.$$

When the temperature changes by T degrees, total radial displacement u_3 is defined as the sum of the elastic component and the net thermal expansion:

$$u_3 = u_3^{(y)} + \alpha_1 T r.$$

2. Consider annular plate I and supporting ring II. A similar problem was solved in [2] for an annular anisotropic two-layer mirror with variable thickness. The equations for forces and moments were derived proceeding from a hypothesis of a straight normal for the entire plate, with the lower base used as the reduction plane. The system derived in [2] for a constant-thickness isotropic plate is

$$\frac{du_1^{(y)}}{dr} = \left[\nu_1 (L_1^2 - D_1 B_1)\frac{u_1^{(y)}}{r} + D_1 N_1^R - L_1 M_1^R\right]\frac{1}{(B_1 D_1 - L_1^2)},$$
$$\frac{d\psi_1}{dr} = \left[\nu_1 (L_1^2 - B_1 D_1)\frac{\psi_1}{r} - L_1 N_1^R + B_1 M_1^R\right]\frac{1}{(B_1 D_1 - L_1^2)}, \qquad (2)$$
$$\frac{dN_1^R}{dr} = B_1 u_1^{(y)} (1 - \nu_1^2)\frac{1}{r^2} + L_1 \psi_1 (1 - \nu_1^2)\frac{1}{r^2} - N_1^R (1 - \nu_1)\frac{1}{r},$$
$$\frac{dM_1^R}{dr} = L_1 u_1^{(y)} (1 - \nu_1^2)\frac{1}{r^2} + D_1 \psi_1 (1 - \nu_1^2)\frac{1}{r^2} - M_1^R (1 - \nu_1)\frac{1}{r},$$

$$-dw_1/dr = \psi_1. \qquad (2')$$

Here, B_1, L_1, and D_1 are the rigidity coefficients of annular plate I, and

$$B_1 = E_1 \frac{h_1}{(1 - \nu_1^2)}, \quad L_1 = E_1 \frac{h_1^2}{2(1 - \nu_1^2)}, \quad D_1 = E_1 \frac{h_1^3}{3(1 - \nu_1^2)};$$

$u_1^{(y)}$ is the elastic component of the radial displacement; w_1 is normal deflection; ψ_1 is the rotation angle of the normal; N_1^R is the radial force; M_1^R is the bending moment of plate I. (Analogous quantities for supporting ring II will be denoted by $u_2^{(y)}$, w_2, ψ_2, N_2^R, M_2^R.)

It is convenient to pass to new variables in this equations system:

$$\Psi_1 = \psi_1 h_1 \quad N_1^r = \frac{N_1^R}{B_1} \quad M_1^r = \frac{M_1^R}{B_1 h_1}. \tag{3}$$

Now, considering that $L_1 = B_1h_1/2$ and $D_1 = B_1h/3$, we rewrite (2) as

$$\begin{aligned}
\frac{du_1^{(y)}}{dr} &= -\frac{u_1^{(y)}}{r}\nu_1 + 4N_1^r - 6M_1^r,\\
\frac{d\Psi_1}{dr} &= -\frac{\Psi_1}{r}\nu_1 - 6N_1^r + 12M_1^r,\\
\frac{dN_1^r}{dr} &= \frac{(1-\nu_1^2)}{r^2}\left(u_1^{(y)} + \frac{\Psi_1}{2}\right) - \frac{N_1^r}{r}(1-\nu_1),\\
\frac{dM_1^r}{dr} &= \frac{(1-\nu_1^2)}{r^2}\left(\frac{u_1^{(y)}}{2} + \frac{\Psi_1}{3}\right) - \frac{M_1^r}{r}(1-\nu_1).
\end{aligned} \tag{4}$$

We write equations system (4) as

$$dy/dr = A(r)y, \tag{5}$$

where $y = (u_1^{(y)}, \Psi_1, N_1^r, M_1^r)$. It can readily be seen that system (5) has the following linearly independent solutions:

$$y_1 = \left(r, 0, 1+\nu_1, \frac{(1+\nu_1)}{2}\right),\quad y_2 = \left(0, r, \frac{(1+\nu_1)}{2}, \frac{(1+\nu_1)}{3}\right),$$
$$y_3 = \left(-\frac{2}{3r}, \frac{1}{r}, \frac{(1-\nu_1)}{6r^2}, 0\right),\quad y_4 = \left(-\frac{1}{2r}, \frac{1}{r}, 0, -\frac{(1-\nu_1)}{12r^2}\right).$$

and its general solution should be sought as

$$Y = G_1y_1 + G_2y_2 + G_3y_3 + G_4y_4.$$

Normal deflection w_1 of the plate is found from (2'), i.e.,

$$w_1(r) = -G_2\frac{r^2}{2h_1} - \frac{(G_3+G_4)}{h_1}\ln r + G_5.$$

Equations system analogous to (4) can be written for supporting ring II. The solution of this system is formulated as

$$Y^{(k)} = Q_1y_1^{(k)} + Q_2y_2^{(k)} + Q_3y_3^{(k)} + Q_4y_4^{(k)}.$$

We thus have to find 12 constants to define the stressed-strained state of the entire structure: C_i, G_j, and Q_l, i = 1-3, j = 1-5, and l = 1-4.

3. For defining these constants we have four boundary conditions of a free edge at $r = r_3$ and $r = r_1$:

$$
\begin{aligned}
&N_1^r(r_1)=0,\ N_2^r(r_1)=0,\\
&M_1^r(r_1)=0,\ M_2^r(r_1)=0,
\end{aligned}
\tag{6}
$$

two conditions of equality of forces and moments in cross section $r = r_0$:

$$
\begin{aligned}
&N_1^r(r_0)=N_2^r(r_0)+N_3^r,\\
&M_1^r(r_0)=M_2^r(r_0)+M_3^r+N_3^r(h_1-h_3/2),
\end{aligned}
\tag{7}
$$

and five conjugation conditions linking the deflections and rotation angles of the normal:

$$
\begin{aligned}
&u_1^{(y)}(r_0)+z_1Tr_0=u_2^{(y)}(r_0)+z_2Tr_0,\\
&u_1^{(y)}(r_0)=u_3^{(y)}(r_0)-\varphi_3(r_0)(h_1-h_3/2),
\end{aligned}
\tag{8}
$$

$$
\varphi_1(r_0)=\varphi_2(r_0)=\varphi_3(r_0), \tag{9}
$$

$$
w_1(r_0)=w_3(r_0). \tag{10}
$$

Ten constants C_1, C_2, G_1-G_4, and Q_1-Q_4 can thus be found from the 10 linear equations in (6)-(9). Constants C_3 and G_5 characterize only the normal deflection and are connected with relation (10). The deflection of the plate with this formulation can be found only with the accuracy up to a constant.

4. Making use of boundary condition (6), we can express constants G_3 and G_4 in terms of constants G_1 and G_2 and express constants Q_3 and Q_4 in terms of Q_1 and Q_2. Proceeding from these relations and the conditions of connection of main variables (3) for which the conjugation conditions are written and the new variables used in (4), we obtain

$$
\begin{aligned}
&G_1=\frac{UU}{a_2},\quad G_2=\frac{\Psi}{a_2k_2},\quad Q_1=\left[UU-Tr_0(z_2-z_1)\right]\frac{1}{a_1},\quad Q_2=\frac{\Psi}{a_1},\\
&NN=c_{11}\left(u_3^{(y)}-\frac{\Psi}{k_2}(1-k_3)\right)+c_{12}\Psi+d_1,\\
&MM=k_2c_{12}\left(u_3^{(y)}-\frac{\Psi}{k_2}(1-k_3)\right)+c_{22}\Psi+d_1\frac{k_2}{2}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
&NN=N_3^r\frac{k_2}{B_1(1+\nu_1)}y,\quad MM=M_3^r+h_1\left(1-\frac{k_3}{2}\right)N_3^r\frac{k}{B_1h_1(1-\nu_1)}y,\\
&y=\frac{r_0^2}{(r_0^2-r_1^2)},\quad UU=u_3^{(y)}-h_1\left(1-\frac{k_3}{2}\right)\varphi_3,\quad \Psi=h_2\varphi_3,\\
&b=\frac{E_2(1-\nu_1)(r_0^2-r_1^2)}{E_1(1-\nu_2)(r_0^2-r_1^2)},\quad a_1=r_0-\frac{r_1^2(1-\nu_1)}{r_0(1-\nu_2)},\quad a_2=r_0-\frac{r_1^2}{r_0}\frac{1-\nu_1}{1-\nu_1},\\
&k_2=h_2/h_1,\quad k_3=h_3/h_1,\\
&c_{11}=\frac{1}{a_2}-b\frac{k_2}{a_1},\quad c_{12}=\frac{1}{2}\left(\frac{1}{a_2}k_2-b\frac{k_2}{a_1}\right)\\
&c_{22}=\left(\frac{1}{a_2}k_2-bk_2^2\frac{1}{a_1}\right),\quad d_1=Tr_0(z_2-z_1)b\frac{k_2}{a_1}.
\end{aligned}
\tag{11}
$$

Table 1

Maximum Deflection (10^{-4} mm) as a Function of the Number of Finite Elements

Distance from the center, cm	FEM, 24 elements	FEM, 15 elements	Theoretical
0.00	0.00	0.00	0.00
0.40	0.04	0.04	0.03
0.80	0.10	0.11	0.12
1.20	0.22	0.23	0.29
1.60	0.37	0.40	0.50
2.00	0.58	0.63	0.77
2.75	1.05	1.12	1.27
3.50	1.50	1.61	1.67
4.25	1.91	2.01	2.02
5.00	2.28	2.41	2.34
5.75	2.62	2.74	2.63
6.50	2.94	3.07	2.91

Substituting (1) into (11), we obtain two equations for C_1 and C_2 with solution

$$C_1 = d_1 \frac{\left(\frac{k_2}{2} f_{12} - f_{22}\right)}{(f_{11} f_{22} - f_{12} f_{21})}, \quad C_2 = d_1 \frac{\left(f_{21} - f_{11} \frac{k_2}{2}\right)}{(f_{11} f_{22} - f_{12} f_{21})},$$

where

$$f_{11} = c_{11} r_0 - h_3 y, \quad f_{22} = (c_{12} k_2 - (1 - k_3) k_2 c_{12}) r_0 h_1 - \frac{k_1^2}{24 y h_1},$$

$$f_{12} = \frac{r_0 h_1}{2} (c_{12} k_2 - (1 - k_3) c_{11}), \quad f_{21} = k_2 c_{12} r_0 - (1 - k_3) y.$$

As noted above, we are primarily concerned with normal deflection of the mirror. Coefficients G_3, G_4, and G_2 can readily be expressed in terms of C_2. Setting in (1) $C_3 = 0$, we can find G_5 from condition (10). The expression for normal deflection of a specular surface thus appears as

$$\begin{aligned} w(r) &= -\frac{C_2}{4} r^2 \quad \text{at} \quad r_0 > r > 0, \\ w(r) &= -C_2 \frac{r_0}{2a_2} \left[\frac{(r_0^2 - r^2)}{2} + r_2^2 \frac{(1 + \nu_1)}{(1 - \nu_1)} \ln\left(\frac{r}{r_0}\right) - \frac{r_0^2}{4} \right] \quad \text{at} \quad r_2 > r > r_0. \end{aligned} \qquad (12)$$

Expressions (13) describe the deflection of the mirror surface with the assumption that axially symmetric mirror deformations satisfy thin plate equations. This assumption does not always hold for optical mirrors. However, the preceding formulas describe deformation of real mirrors with sufficient accuracy.

As a numeric example, we consider a mirror with these parameters: $r_0 = 20$ mm, $E_1 = E_3 = 1.6 \cdot 10^{11}$ Pa, $\alpha_1 = \alpha_3 = 2.8 \cdot 10^{-6}$, $\nu_1 = \nu_3 = 0.2$, $r_1 = 65$ mm, $E_2 = 1.2 \cdot 10^{11}$ Pa, $\alpha_2 = 2.5 \cdot 10^{-6}$, $\nu_2 = 0.3$, $r_2 = 16$ mm, $h_1 = 20$ mm.

Table 2

Deformation of the Surface of an Elastic Mirror (10^{-3} mm)

Distance from the center, cm	[illegible]									
	[illegible]		[illegible]		[illegible]		[illegible]		[illegible]	
	FEM	Theoretical	FEM	Theoretical	FEM	Theoretical	FEM	Theoretical	FEM	Theoretical
1.0	0.003	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.002	[illegible]	[illegible]	0.000
4.0	0.002	0.001	0.000	0.001	0.?04	[illegible]	0.003	[illegible]	0.004	0.00?
[illegible]	0.005	0.??5	0.001	0.003	[illegible]	0.01?	0.010	0.012	[illegible]	0.013
1?.0	[illegible]	0.011	0.00?	[illegible]	0.0?2	0.02?	0.?21	0.02?	0.016	[illegible]
1?.?	0.010	0.019	0.004	0.?23	[illegible]	[illegible]	0.03?	[illegible]	[illegible]	0.0?4
2?.0	0.016	[illegible]	[illegible]	0.031	[illegible]	0.077	0.055	0.07?	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	0.023	0.0?2	0.105	0.12?	[illegible]	[illegible]	0.?76	0.1?9
35.0	0.0?1	0.066	0.04?	0.06?	[illegible]	0.167	0.151	0.1?8	0.?18	[illegible]
42.?	0.06?	0.080	[illegible]	0.08?	0.?91	[illegible]	0.192	[illegible]	0.166	0.?21
50.?	0.07?	0.092	[illegible]	[illegible]	0.2??	0.234	[illegible]	0.2?4	0.207	0.?55
57.5	0.0?2	0.104	0.084	0.107	0.262	0.263	[illegible]	0.26?	0.?34	[illegible]
65.0	0.104	0.115	0.0??	0.119	0.2?4	[illegible]	0.295	0.292	[illegible]	[illegible]

Parameters h_2 and h_3 are treated as variable. A finite-element method (FEM) implemented in a program package entitled "Structural Analysis of Complex Designs," which is a modification of SAP IV software [3], was used to solve the initial problem. Quadrangular two-dimensionally strained elements were adopted as the finite elements for solution of the axially symmetric problem. A complete picture of the deformation was obtained by applying the FEM to these elements. Bulk expansion, deformation, and radial deformation were eliminated to be able to compare the calculation results using analytic formulas.

We begin by studying the convergence of this method. Table 1 gives a comparison of calculation results for a grid with 15 and 24 elements. We can see that an increase in the number of elements yields only a slight improvement. In all other cases we operated with 24 elements.

Analytic results were compared with calculations using the finite-element method for various ratios $k_2 = h_2/h_1$ and are given in Table 2.

While for $k_2 = 0.375$ the results of analytic and numeric calculations virtually coincided, the differences were significant for smaller and larger k_2. The deformations computed from analytic formulas were slightly higher than the actual values. However, analytic formulas provided a qualitatively correct description of the deformation pattern. The relative error for the maximum deflection of the reflecting surface in no case exceeded 15%.

It is interesting to trace how deformation was affected by k_2 and k_3. Table 3 describes the dependence of the maximum deformation of the reflecting mirror surface on k_2 and fixed parameter k_3 ($k_2 + k_3 \leq 1$). As k_3 varied, the

Table 3

Dependence of Maximum Deflection (10^{-6} mm) on k_2

k_1	[illegible]	[illegible]	[illegible]	[illegible]
0.0	0.000	0.000	[illegible]	[illegible]
0.1	1.03[illegible]	1.187	1.1[illegible]	[illegible]
0.2	1.730	[illegible]53	1.[illegible]7	1.97[illegible]
0.3	2.190	1.972	1.137	2.353
0.4	2.440	1.7[illegible]0	1.0[illegible]0	2.1[illegible]0
[illegible]	2.[illegible]60	3.08[illegible]	3.113	3.131
[illegible]	2.480	[illegible].170	3.233	3.253
[illegible]	2.330	3.144	3.294	—
0.7	1.980	2.919	3.18[illegible]	—
0.75	1.715	2.754	3.057	—
[illegible]8	1.450	2.501	—	—
[illegible]	0.770	1.[illegible]4	—	—
1.[illegible]	0.000	—	—	—

Table 4

Dependence of Maximum Deflection (10^{-6} mm) on k_3

k_2	k_1					
	0.7	0.6	0.55	0.5	0.4	0.1
0.05	2.813	3.126	3.199	3.217	3.018	1.254
0.1	2.919	3.144	3.182	3.17	2.923	1.187
0.15	3.017	3.187	3.203	3.172	2.893	1.158
0.2	3.104	3.24	3.241	3.198	2.895	1.148
0.25	3.18	3.294	3.285	3.233	2.913	1.147
0.3	3.241	3.341	3.326	3.268	2.935	1.151
0.35	—	3.375	3.356	3.295	2.953	1.155
0.4	—	3.388	3.369	3.306	2.961	1.156
0.45	—	—	3.356	3.294	2.95	1.151
0.5	—	—	—	3.253	2.916	1.137
0.55	—	—	—	—	2.855	1.112
0.6	—	—	—	—	2.763	1.076
0.65	—	—	—	—	—	1.029
0.7	—	—	—	—	—	0.971
0.75	—	—	—	—	—	0.904
0.8	—	—	—	—	—	0.831
0.85	—	—	—	—	—	0.736
0.9	—	—	—	—	—	0.68

function maximum was shifted to the right and its value increased.

Table 4 also gives the maximum deformation of the reflecting surface of the mirror for various k_1 at fixed k_2. The relationship displays a distinct nonlinearity. However, at fixed $k_1 \ll 1$ and small k_3 the maximum deflection value decreases with increasing k_2. This follows also from analytic formulas, because, if we retain only first-order terms, we have

$$C_2 = 12bT(z_2 - z_1)\frac{h_2 a_2}{h_1 d_1}\left(1 - 2a_2 r_0 \frac{k^2}{(r_2^2 - r_0^2)}\right),$$

The formula also indicates that the maximum deflection for fixed k_3 and small k_2 increases with increasing k_2.

If $k_2 = 1 - \varepsilon$ and $\varepsilon \ll 1$, for fixed k_3 we have

$$C_2 = 12T(\dot{z}_1 - z_1)\frac{R}{n_1}\frac{a_1 a_2}{(a_1 - k a_2)^2}\left(1 + k a_2 \frac{r_0}{(r_2 - r_0)}\right)$$

This expression indicates that the maximum deflection increases in proportion to k_3.

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