Asymptotic Analysis of Free Vibrations of Thin Cylindrical Shells

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Abstract: An algorithm for the solution of boundary value problems involving vibrations of thin cylindrical shells by means of symbolic computation is presented. The algorithm is based on the method of asymptotic integration of the shell equations. A linear shell theory of the Kirchhoff - Love type is used. The equations describing the vibrations of the shell contain several parameters, the main of which is the small parameter of the shell thickness. Formal asymptotic solutions in different domains of the space of the parameters are obtained by constructing the convex hull of points set. The constructed solutions are used for studying the free vibration spectra of the shells.

Résumé: Un algorithme de résolution des problèmes aux limites par des méthodes de calcul symbolique est présenté pour le cas des vibrations de coques minces. L'algorithme est basée sur la méthode d'intégration assymptotique de l'équations des coques. Une théorie linéaire du type Kirchhoff - Love est utilisée. Les vibrations des coques sont régies par plusieurs paramètres, dont le principal est le petit paramètre d'épaisseur de coque. Des solutions assymptotiques formelles dans les différent domaines de l'espace des paramètres sont obtenues par la construction de l'enveloppe convexe de l'ensemble des points. Les solutions construites sont utilisées pour l'analyse

Introduction: The vibrations of thin cylindrical shells are analyzed in this paper by applying the method of asymptotic integration developed by Goldenveizer, Lidsky and Tovstik [1]. A detailed review of their work as well as a reference list may be found in [1, 2, 3, 4]. The aim of this study is to develop an algorithm permitting symbolic integration of the equations governing the free vibrations of the shells, for any range of values of the parameters that these equations contain. The study is limited to the cases for which the asymptotic representation of the solution is the same in the entire domain of integration, and solutions are linearly independent (no turning points, no multiple roots). Axisymmetric as well as non-axisymmetric vibrations are considered. Some preliminary results of this work have been reported in [5].

Formulation of the Problem: A thin cylindrical shell having the thickness t, the length L and the radius R is considered. The system of orthogonal coordinates s, j that defines the position of a point on the neutral surface of the shell is employed, where s is the length of the generatrix, $0 \le s \le L$, and φ is the longitudinal angle, $0 \le j \le 2p$. The shell is limited by two parallels $s = 0$ and $s = L$. The cylindrical shell is considered to be thin if its relative thickness t/R is small. A local orthogonal system of coordinates $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{n}$ is also introduced, where $\mathbf{e}_1$ and $\mathbf{e}_2$

n is the normal unit vector (**n** = $\mathbf{e_1} \times \mathbf{e_2}$). Let *u*, *v*, and *w* be the components of the displacement ***U*** in the directions $\mathbf{e_1}$, $\mathbf{e_2}$ and **n**.

Using the shell equations of 2D Kirchoff-Love theory one obtains, after separating the variables in circumferential and axial directions, the equations of vibrations for thin cylindrical shells in the form

$$\frac{dU}{ds} = A(m,m,l)U\,, \tag{1}$$

where ***A*** is a square [8×8] matrix and $\boldsymbol{Y}(s) = \{y_1(s), \ldots, y_8(s)\}$ is a vector function of [8×1] size, *m* is the parameter of the shell thickness, *m* is the number of waves in circumferential direction and *l* is the frequency parameter. In this study the parameter *m*, which is proportional to $\sqrt{t/R}$, is considered as the main small parameter. The boundary conditions are imposed as

$$B_i U_i(s_i) = 0, \quad i = 1,\ 2 \tag{2}$$

where $s_1 = 0$ and $s_2 = L$, and $\boldsymbol{B}_i$ are matrices of the size [4×8].

When employing the Kirchoff-Love theory it is most convenient to use the following variables: $\boldsymbol{Y} = (u, v, w, T_1, S_{21}, N_1, M_1, g_1)$, where T_1, S_{21} are the stress resultants, N_1 is the transverse shear resultant, M_1 is the moment resultant, and γ_1 is the angle of rotation of the normal [4]. For such variables, boundary conditions of the following form are considered: $u = 0$ or $T_1 = 0$, $v = 0$ or $S_{21} = 0$, $w = 0$ or $N_1 = 0$, $\gamma_1 = 0$ or $M_1 = 0$, at $s = 0$ or $s = L$.

Sometimes it is more convenient to express the resultants and the angle of rotation as functions of the displacements and get the system of equations for $\boldsymbol{U} = (u, v, w)$

$$m^4 L_m(U,m,m) + L(U,m,m) + lU = 0, \tag{3}$$

where L_m and L are linear differential operators of the 8th order and of the fourth order, respectively. In this case the boundary conditions must be formulated in terms of *u*, *v*, *w* and their derivatives.

To solve the boundary value problem (1) - (2) the method of asymptotic integration described in [2, 3, 4] is applied. For this, one needs to construct the formal asymptotic solution for equation (1) and then impose that boundary conditions (2) be satisfied.

The solution of equation (1) is sought in the form

$$U(s,m) = \sum_{i=1}^{8} \sum_{k=0}^{\infty} C_i U_k^i m^{kk_i} e^{p_i s}\,, \tag{4}$$

where C_i are arbitrary constants, $\boldsymbol{Y}^i_k$ is the matrix of the amplitude vectors, and k_i depends on the order of p_i with respect to *m*. For example, if $p_i \sim m^{-1}$ then $k_i = 1$

Substituting solution (4) into equation (1) leads to the characteristic equation for p_i

$$\left|A(m,m,l) - pI\right| = 0, \tag{5}$$

where ***I*** is the identity matrix. In this study only the cases when all p_i are simple roots of equation (5) are considered, i.e. when $p_i \neq p_j$ for all $i \neq j$. In these cases all solutions are linearly independent, and their linear combination provides the general solution of the initial equation.

For different relations between the parameters solutions (4) have different forms. To construct formal asymptotic solutions for different values of the parameters *m*, *l* and *m*, symbolic computation is used.

The order of the function $|p|$ in *m* is called the index of variation of the solution. The solution is exponentially increasing away from the edge $s = 0$ if $\hat{A}p_i > 0$. Such integral is called the integral of the edge effect near end $s = L$. The solution is exponentially decreasing away from the edge $s = 0$ if $\hat{A}p_i < 0$. Such integral is called the edge effect integral near the end $s = 0$. The solution is oscillating if $\hat{A}p_i = 0$ and $\acute{A}p_i \neq 0$.

If $p_i \equiv 0$ the solution is called slowly varying. In solving the boundary value problem with an error of the order e^{-c/m^d}, where *c* and *d* are some positive constants, one may take the value of the edge effect integrals to be equal to zero at the other end.

After constructing the formal asymptotic solutions, the boundary conditions should be imposed to find the frequency parameter *l*. Substituting (4) into (2) one obtains a system of linear equations in C_i that has nonzero solutions if its determinant vanishes

$$D(l,m) = 0. \tag{6}$$

This is an equation of the eighth degree that can be solved numerically. In some cases this equation may be simplified.

Concomitant with the problem for $m \neq 0$ (perturbed problem) the problem with $m = 0$ (imperturbed problem) is considered. If all p_i are different from 0 and not pure imaginary, then

$$lim_{m®0} D(l,m) \neq D(l,0), \tag{7}$$

and

$$l = l_0 + m l_1 + \ldots, \tag{8}$$

where l_0 is the frequency for the imperturbed system, i.e. $\Delta(l_0, 0) = 0$.

Of special interest are the cases of regular degeneracy [6]. Let the perturbed system have the order *n*, and the imperturbed system have the order *m*. Let the perturbed system have $l = n - m$ additional roots, such that l_1 of them have negative real parts and l_2 have positive real parts, where l_1 is the number of additional boundary conditions on the left edge and l_2 is the number of case the solution may be constructed using an iterative method.

The existence of pure imaginary roots makes the problem more difficult. As a rule in this case, the function $\Delta(l,m)$ has a limit point at $m = 0$ and $\lim_{m\to 0}\Delta(l,m) \neq \Delta(l,0)$.

Formal Asymptotic Solutions for the Equations of Cylindrical Shells: Let consider the equations describing the vibrations of thin cylindrical shells in terms of displacements [1]:

$$-\frac{\partial^2 u}{\partial s^2}-\frac{1-n}{2}\frac{\partial^2 u}{\partial j^2}-\left(1-n^2\right)l u-\frac{1+n}{2}\frac{\partial^2 v}{\partial s\partial j}+n\frac{\partial w}{\partial s}=0,$$

$$-\frac{1+n}{2}\frac{\partial^2 u}{\partial s\partial j}-\frac{1-n}{2}\frac{\partial^2 v}{\partial s^2}-\frac{\partial^2 v}{\partial j^2}+m^4\left(-2(1-n)\frac{\partial^2 v}{\partial s^2}-\frac{\partial^2 v}{\partial j^2}\right)-$$
$$\left(1-n^2\right)l v+\frac{\partial w}{\partial j}+m^4\left(-(2-n)\frac{\partial^3 w}{\partial s^2\partial j}-\frac{\partial^3 w}{\partial j^3}\right)=0,$$

$$-n\frac{\partial u}{\partial s}-\frac{\partial v}{\partial j}+m^4\left((2-n)\frac{\partial^3 v}{\partial s^2\partial j}+\frac{\partial^3 v}{\partial j^3}\right)+w-\left(1-n^2\right)l w+$$
$$m^4\left(\frac{\partial^4 w}{\partial s^4}+2\frac{\partial^4 w}{\partial s^2\partial j^2}+\frac{\partial^4 w}{\partial j^4}\right)=0. \qquad (9)$$

Here $m^4=\frac{t^2}{12R^2}$ is the main small parameter, $l=\frac{rw^2R^2}{E}$ is the frequency parameter, E is Young's modulus, n is Poisson's ratio, r is the shell mass density, and w is the natural frequency.

Separating the variables in s and φ in the expressions of the displacements

$$u(s, j) = U(s)\sin mj,$$
$$v(s, j) = V(s)\cos mj, \qquad (10)$$
$$w(s, j) = W(s)\sin mj$$

and substituting them in (9) the following system of ordinary differential equations is obtained

$$-\frac{\partial^2 U}{\partial s^2}+\frac{1-n}{2}m^2U-\left(1-n^2\right)lU+\frac{1+n}{2}m\frac{\partial V}{\partial s}+n\frac{\partial W}{\partial s}=0,$$

$$-\frac{1+n}{2}m\frac{\partial U}{\partial s}-\frac{1-n}{2}\frac{\partial^2 V}{\partial s^2}+m^2V+m^4\left(-2(1-n)\frac{\partial^2 V}{\partial s^2}+m^2V\right)-$$
$$\left(1-n^2\right)lV+mW+m^4\left(-(2-n)\frac{\partial^2 W}{\partial s^2}+m^3W\right)=0,$$

$$-n\frac{\partial U}{\partial s}+mV+m^4\left(-(2-n)\frac{\partial^2 V}{\partial s^2}+m^3V\right)+W-\left(1-n^2\right)lW+$$
$$m^4\left(\frac{\partial^4 W}{\partial s^4}+2m^2\frac{\partial^2 W}{\partial s^2}+m^4W\right)=0. \qquad (11)$$

To determine the structure of the asymptotic expansions, the solution is sought in the form

$$U=U_0\,e^{ps},\ V=V_0\,e^{ps},\ W=W_0\,e^{ps}. \qquad (12)$$

Substituting (12) in (11) gives the system of equations with respect to U_0, V_0 ,and W_0

$$-p^2U_0+\frac{1-n}{2}m^2U_0-\left(1-n^2\right)lU_0+\frac{1+l}{2}mpV_0+npW_0=0,$$
$$-\frac{1+n}{2}mpU_0-\frac{1-n}{2}p^2V_0+m^2V_0+$$
$$m^4\left(-2(1-n)p^2V_0+m^2V_0\right)-$$
$$\left(1-n^2\right)lV_0+mW_0+m^4\left(-(2-n)mp^2W_0+m^3W_0\right)=0, \qquad (13)$$
$$-npU_0+mV_0+m^4\left(-(2-n)mp^2V_0+m^3V_0\right)+W_0-$$
$$\left(1-n^2\right)lW_0+m^4\left(p^2-m^2\right)^2W_0=0.$$

The system (13) has nontrivial solutions if its determinant is equal to zero. So, one has the eighth order equation from which all p may be determined.

Axisymmetric Vibrations: For the case of axi-symmetric vibrations ($m = 0$), system (13) becomes

$$-p^2U_0-\left(1-n^2\right)lU_0+npW_0=0,$$
$$-\frac{1-n}{2}p^2V_0+m^4\left(-2(1-n)p^2V_0\right)-\left(1-n^2\right)lV_0=0, \qquad (14)$$
$$-npU_0+W_0-\left(1-n^2\right)lW_0+m^4p^4W_0=0.$$

The system of equations splits. The set of the first and third equations in (14) defines the transverse-axial vibrations, and the second equation defines the torsional vibrations. Only the case of transverse-axial vibrations is considered.

The characteristic equations is

$$\begin{vmatrix}-p^2-\left(1-n^2\right)l & np\\ -np & 1-\left(1-n^2\right)l+m^4p^4\end{vmatrix}=0, \qquad (15)$$

or

$$P(p;h,l)=l-l^2+l^2n^2+p^2-lp^2+$$
$$h^4lp^4-h^4ln^2p^4+h^4p^6=0, \qquad (16)$$

where $h^4 = \frac{m^4}{1-n^2}$.

Now, one must find the roots p_i of equation (16) for different values of the small parameter h«1. Equation (16) may be written in the form

$$P(p;h,l) = \sum_{i}^{6} a_i p^{k_i} h^{a_i} l^{b_i} = 0, \qquad (17)$$

where a_i are coefficients not depending on p, h and l, and i is the number of the term in (16). The points $M_i = \{k_i, a_i, b_i\}$ in the space p, h, λ are called representative points. Through $M^*_i = \{a_i, M_i\}$ we denote a point associated with the coefficient a_i, that is later called the weight of the point. For equation (16) we have M^*_i = {{1,{0, 0, 1}}, {{-1+n^2, {0, 0, 2}}, {{1, {2, 0, 0}}, {{-1, {2, 0, 1}}, {{1-n^2, {4, 4, 1}}, {{1, {6, 4, 0}}.

If the order of the parameter l is given, i.e. $l = l_0 h^{\kappa}$, where $l_0 \sim 1$ and k is known, equation (16) contains only one small parameter, h. To obtain the roots of such an equation Newton's diagram method may be used [3]. In this case the representative points lie in the plane (p, h) and have the form $M_i = \{k_i, a_I + b_i k\}$. The segments of the lower part of the convex hull of the set of points M_i, i. e. the segments that are visible from the point $(p, h) = (0, -¥)$, define the terms of equation (16) that should be kept to determine the main terms of the roots p_i.

Three cases are considered here, when k is equal to 1, 0 and -1 respectively. For the the case $k = 1$ Newton's diagram is plotted in Figure 1.

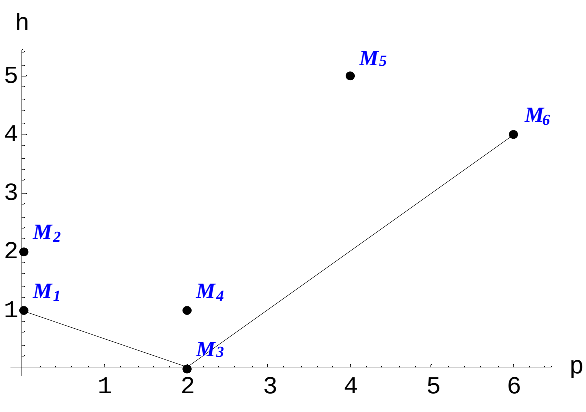

Figure 1. Newton's diagram for $k = 1$, $m = 0$.

For this case the representative points for equation (16) are $M_1 = (0, 1)$, $M_2 = (0, 2)$, $M_3 = (2, 0)$, $M_4 = (2,1)$, $M_5 = (4, 5)$, $M_6 = (6, 4)$. Newton's diagram consists of 2 segments. The first segment is determined by points $M_1 = (0, 1)$, and $M_3 = (2, 0)$, and the second segment by points M_3 and M_6. Therefore, equation (16) has 2 groups of roots, the first of which is defined by the equation

$$l + p^2 = 0, \qquad (18)$$

while the second one may be found from the equation

$$p^2 + h^4 p^6 = 0. \qquad (19)$$

Hence, the roots are

$$p_{1,2} = \pm\sqrt{l}\, i \qquad (20)$$

and

$$p_j = \frac{e_j}{h}, \quad e_j = \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}, \quad j = 3,4,5,6. \qquad (21)$$

The orders of the variables p may be determined as the inclination angles between the segments and axis p.

One also can determine the relative orders of the eigenvectors U_0^i, W_0^i. For this, one keeps only the main terms in equations (14) and the expressions for the roots p_i are substituted into either the first or the third equation. The only limitation for the choice is that both coefficients of U_0^i and W_0^i are nonzero. The main terms for U_0^i and W_0^i are given in Table 1.

Table 1: Roots and eigenvectors for $m = 0$, $k = 1$.

	1	2	3	4	5	6
p	$\sqrt{l}\, i$	$-\sqrt{l}\, i$	$\frac{e_1}{h}$	$\frac{e_2}{h}$	$\frac{e_3}{h}$	$\frac{e_4}{h}$
U_0	p_1	p_2	n	n	n	n
W_0	$-nl$	$-nl$	p_3	p_4	p_5	p_6

For the second case, $k = 0$, the representative points for equation (16) are $M_1 = (0, 1)$, $M_2 = (0, 1)$, $M_3 = (2, 0)$, $M_4 = (2,0)$, $M_5 = (4, 4)$, $M_6 = (6, 4)$. In this case Newton's diagram consists of 2 segments (Figure 2).

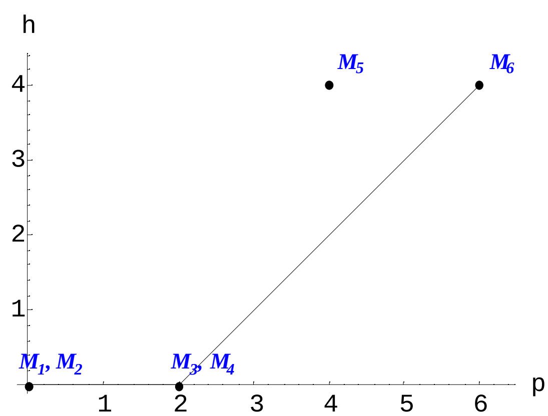

Figure 2. Newton's diagram for $k = 0$, $m = 0$.

The first segment is determined by M_1, M_2, M_3 and M_4 and the second one by points M_3, M_4 and M_6. Therefore, equation (16) has 2 groups of roots, the first one being defined by equation

$$l - l^2 + l^2 n^2 + p^2 - l p^2 = 0 \qquad (22)$$

and the second one by equation

$$p^2 - lp^2 + h^4 p^6 = 0. \tag{23}$$

Hence

$$p_{1,2} = \pm F(l), \quad F(l) = \sqrt{\frac{l - (1-n^2)l^2}{l-1}}, \tag{24}$$

and

$$p_{3,4,5,6} = \frac{(l-1)^{1/4}}{h}. \tag{25}$$

For this case the roots and the eigenvectors are shown in Table2.

Table 2: Roots and eigenvectors for $m = 0$, $k = 0$.

	1	2	3	4	5	6
p	$F(l)$	$-F(l)$	$\frac{(l-1)^{1/4}}{h}$	$-\frac{(l-1)^{1/4}}{h}$	$\frac{(l-1)^{1/4} i}{h}$	$-\frac{(l-1)^{1/4} i}{h}$
U_0	p_1	p_2	n	n	n	n
W_0	$\frac{ln}{l-1}$	$\frac{ln}{l-1}$	p_3	p_4	p_5	p_6

Note that the above results for k = 0 are valid when l is not too close to 1, otherwise the first negligible term for p has the same order as the main term [3].

Finally, for the third case, k = –1, and the representative points are $M_1 = (0, -1)$, $M_2 = (0, -2)$, $M_3 = (2, 0)$, $M_4 = (2,-1)$, $M_5 = (4, 3)$ and $M_6 = (6, 4)$ (Figure 3).

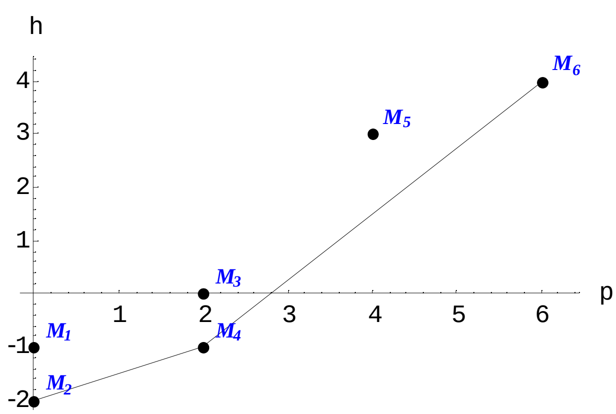

Figure 3. Newton's diagram for k = – 1, m = 0.

In this case Newton's diagram consists again of 2 segments. The first segment is determined by points M_2 and M_4, and the second one by points M_4 and M_6. Therefore, equation (16) has 2 groups of roots, the first one being defined by equation

$$-l^2 + l^2 n^2 - lp^2 = 0, \tag{26a}$$

and the second one defined by

$$-lp^2 + h^4 p^6 = 0. \tag{26b}$$

For this case, the roots and the eigenvectors are given in Table 3.

Table 3: Roots and eigenvectors for $m = 0$, $k = -1$.

	1	2	3	4	5	6
p	$\sqrt{(1-n^2)l}\,i$	$-\sqrt{(1-n^2)l}$	$\frac{l^{1/4}}{h}$	$-\frac{l^{1/4}}{h}$	$\frac{l^{1/4} i}{h}$	$-\frac{l^{1/4} i}{h}$
U_0	p_1	p_2	n	n	n	n
W_0	n	n	p_3	p_4	p_5	p_6

Note that for roots p_1 and p_2 the coefficient of U_0 in the first equation in (14) is equal to zero, and to determine U_0 and W_0 the third equation in (14) must be used.

The representative points move in the plane p, h as k changes. The cases (called separative) when the convex hull changes are of interest. These occur when one of the interior points reaches the convex hull, or two or more segments form a straight line.

By plotting the representative points in the 3D-space (p, h, λ), the separative cases may be determined by the 3D convex hull facets. For equation (16) the 3D convex hull is plotted in Figure 4.

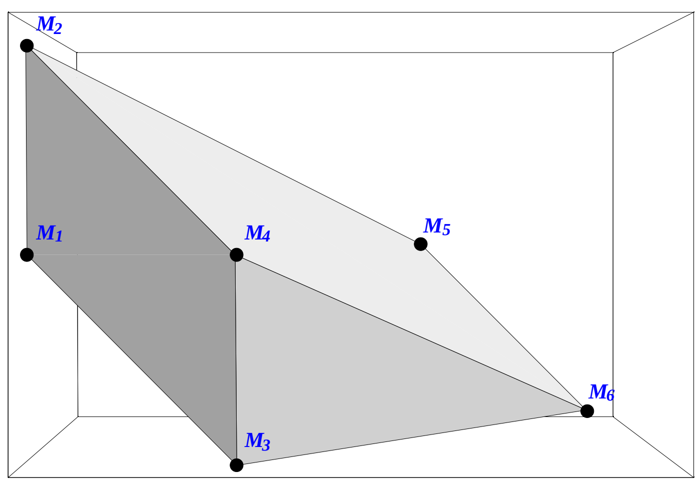

Figure 4. 3D Convex hull for m = 0.

Since one assumes that parameter h is small, only the facets that are visible from the point h = -¥ are needed. In the case considered, the 3D convex hull consists of 3 facets: 1: (M_1, M_2, M_3, M_4); 2: (M_3, M_4, M_6); 3: (M_2, M_4, M_5, M_6). Imposing that the orders of all terms forming a facet be equal to each other, one can find the relations from which the orders of l for

$$\lambda \sim \lambda^2 \sim p^2 \sim \lambda p^2$$
$$\lambda p^2 \sim p^2 \sim h^4 p^6 \qquad (27)$$
$$\lambda^2 \sim \lambda p^2 \sim \lambda h^4 p^4 \sim h^4 p^6$$

So, if we assume that $l = l_0 h^k$, then for the first and the second relations $k = 0$, and for the third $k = -4$.

Therefore, the entire range of parameter l is divided into 3 domains where the 2D convex hulls are essentially different. Each of these domains defined as Domain I: $k > 0$, Domain II: $0 > k > -4$, Domain III: $k < -4$, as well as the separative cases A: $k = 0$, and B: $\kappa = -4$ must be considered separately. Since the initial equations are valid for $l \ll h^{-4}$, cases III and B have no physical meaning. Case A is special, since in this case the second term in the expansion for p is important [3]. So here, only the solutions in Domains I, II and A are considered. For any l inside a domain the structure of the convex hull and therefore the roots and the eigenvectors are similar. Thus, one can obtain the values of the roots and eigenvectors considering only one value of l for each domain. Using this, we substitute for the case A: $k = 0$, Domains I and II $k = 1$ and $k = -1$ respectively.

Boundary Value Problem: The geometry of the point set and its convex hull for Domain I is shown in Figure 1. For this case the solution may be written as

$$u = \sum_{i=0}^{6} U_i e^{p_i s}, \qquad w = \sum_{i=0}^{6} W_i e^{p_i s}, \qquad (28)$$

where p_i, U_i and W_i are determined from Table 1.

Two types of boundary conditions are considered: freely supported edges and clamped edges. For low frequency vibrations ($l \ll 1$, $k > 0$) of a cylindrical shell with freely supported edges, the boundary conditions have the form

$$u' = w = w'' = 0 \text{ at } s = 0 \text{ and } s = L. \qquad (29)$$

Substituting solution (28) into boundary conditions (29) one gets the characteristic equation from which the first approximation for the frequency parameter l may be found

$$D(l) = \begin{vmatrix} p_1U_1 & p_2U_2 & p_3U_3 & p_4U_4 & p_5U_5 & p_6U_6 \\ W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\ p_1^2U_1 & p_2^2U_2 & p_3^2U_3 & p_4^2U_4 & p_5^2U_5 & p_6^2U_6 \\ p_1U_1e^{p_1L} & p_2U_2e^{p_2L} & p_3U_3e^{p_3L} & p_4U_4e^{p_4L} & p_5U_5e^{p_5L} & p_6U_6e^{p_6L} \\ W_1e^{p_1L} & W_2e^{p_2L} & W_3e^{p_3L} & W_4e^{p_4L} & W_5e^{p_5L} & W_6e^{p_6L} \\ p_1^2U_1e^{p_1L} & p_2^2U_2e^{p_2L} & p_3^2U_3e^{p_3L} & p_4^2U_4e^{p_4L} & p_5^2U_5e^{p_5L} & p_6^2U_6e^{p_6L} \end{vmatrix}$$

(30)

The values of l may me obtained numerically from this equation. But one can also try to simplify this determinant. First, the values of the third and fourth integrals on the left edge and the values of the fifth and the sixth integral on the right edge are neglected. Then, after factorization, the determinant is obtained in the form

$$D(l) = -4e^{\left(\sqrt{2} - ih\sqrt{l}\right)L/h}\left(-1 + e^{2iL\sqrt{l}}\right) l^2 \left(-1 + n^2\right)^2 / h^8 = 0$$

(31)

So, one series for the natural frequency parameter is

$$l = \left(\frac{pk}{L}\right)^2. \qquad (32)$$

This frequency coincides with that for the imperturbed (momentless) system. In this case two additional roots have negative real parts, and two have positive parts. Since there are four additional boundary conditions (two on each edge) this is a case of regular degeneracy and the next corrections for l may be constructed with an iterative method. Note that relation (32) is valid for $l \ll 1$.

Similarly, for high frequency vibrations of a cylindrical shell with freely supported edges ($l \gg 1$, $-4 < k < 0$), the same equation (30) is used, but now p_i, U_i and W_i are determined from Table 3.

As usual, the values of the edge effect integrals on the other edge are neglected. As a result, after simplifications, $D(l)$ has the following expression

$$D(l) = -\frac{1}{h^8} 4e^{\frac{iL\,l^{1/4}\left((1+i) + h\,l^{1/4}\sqrt{1-n^2}\right)}{h}}$$
$$\left(-1 + e^{\frac{2iL\,l^{1/4}}{h}}\right)\left(-1 + e^{2iL\sqrt{l(1-n^2)}}\right) l^2 \left(n^2 + l\left(1-n^2\right)\right)^2 = 0.$$

(33)

So, one obtains two series for the natural frequency parameter

$$l = \frac{1}{1-n^2}\left(\frac{pk}{L}\right)^2, \qquad (34)$$

and

$$l = \left(\frac{pk}{L}h\right)^4. \qquad (35)$$

Here, there are 4 pure imaginary roots among the additional ones, and this is not a case of regular degeneracy. Expressions (34) and (35) are valid for $l \gg 1$.
For low frequency vibrations of a cylindrical shell with clamped edges ($l \ll 1$, $k > 0$) the boundary conditions

$$u = w = w' = 0 \ \text{at} \ s = 0 \text{ and } s = L. \tag{36}$$

So, the following equation must be solved

$$D(l) = \begin{vmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\ W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \\ p_1 W_1 & p_2 W_2 & p_3 W_3 & p_4 W_4 & p_5 W_5 & p_6 W_6 \\ U_1 e^{p_1 L} & U_2 e^{p_2 L} & U_3 e^{p_3 L} & U_4 e^{p_4 L} & U_5 e^{p_5 L} & U_6 e^{p_6 L} \\ W_1 e^{p_1 L} & W_2 e^{p_2 L} & W_3 e^{p_3 L} & W_4 e^{p_4 L} & W_5 e^{p_5 L} & W_6 e^{p_6 L} \\ p_1 W_1 e^{p_1 L} & p_2 W_2 e^{p_2 L} & p_3 W_3 e^{p_3 L} & p_4 W_4 e^{p_4 L} & p_5 W_5 e^{p_5 L} & p_6 W_6 e^{p_6 L} \end{vmatrix} \tag{37}$$

where p_i, U_i and W_i are determined from Table 1. After some transformations, only the main terms are kept, leading to

$$D(l) = 2e^{\left(\sqrt{2} - ih\sqrt{l}\right)L/h} \, l\left(-1 + e^{2iL\sqrt{l}}\right) + O(h)) = 0 \,. \tag{38}$$

This equation has only the series of roots

$$l = \left(\frac{pk}{L}\right)^2 . \tag{39}$$

Again this is a case of regular degeneracy.

For the higher frequency vibrations (l » 1, – 4 < k < 0) the determinant (37) must be used, but p_i, U_i and W_i should be determined from Table 3.

After some transformations, only the main terms are kept, leading to

$$D(l) = -\frac{1}{h^6} 2e^{-\left(1 + i + h l^{1/4}\sqrt{1-n^2}\, i\right) L l^{1/4}/h}$$
$$l i\left(\left(-1 + e^{2iL\sqrt{l}\sqrt{1-n^2}}\right)\left(1 + e^{2iL l^{1/4}/h}\right) + O(h)\right) = 0. \tag{40}$$

This equation has two series of roots

$$l_1 = \frac{1}{1-n^2}\left(\frac{pk}{L}\right)^2 ,$$
$$l_2 = \left(\frac{p(2k+1)h}{2L}\right)^4 . \tag{41}$$

The second series has no analog for the imperturbed (momentless) system. Again, this is a case of nonregular degeneracy. Note that the above expressions are obtained assuming l »1.

Nonaxisymmetric Vibrations: The same approach may be used to study the nonaxisymmetric vibrations of cylindrical shells. Equation (13) now should be analyzed for $m \neq 0$. The system does not split in this case, and one has to find the roots of the characteristic equation of the

$$\begin{vmatrix} -p^2 + \frac{1-n}{2}m^2 - (1-n^2)l & \frac{1+n}{2}mp & np \\ -\frac{1+n}{2}mp & g(p,m,m) - (1-n^2)l & f(p,m,m) \\ -np & f(p,m,m) & 1 - (1-n^2)l + m^4(p^2 - m^2)^2 \end{vmatrix} = 0, \tag{42}$$

where

$$f(p,m,m) = m + m^4\left(-(2-n)mp^2 + m^3\right),$$

and

$$g(p,m,m) = m^4\left(-2(1-n)p^2 + m^2\right) - \frac{1-n}{2}p^2 + m^2 .$$

This equation is represented in the form

$$P(p;h,l) = \sum_i a_i p^{k_i} h^{a_i} l^{b_i} m^{l_i} . \tag{43}$$

The representative points have four coordinates $M_i = \{k_i, \alpha_i, \beta_i, l_i\}$ in the 4D space (p, h, l, m). Similar to the previous axisymmetric case one must construct a convex hull in 4D, the facets of which determine the lines that divide the (l, m)-plane into domains with different structures of the roots of the characteristic equation.

In this paper only the cases for which the order of m is known are considered. This permits to reduce the 4D problem to the 3D one discussed in the previous section.

Let consider the case, when $m = m_0 h^t$, $t = 0$, i.e. $M_i = \{k_i, a_i, b_i\}$. Equation (42) for this case may be written as

$$P(p;h,l) = \sum_i^{24} a_i p^{k_i} h^{a_i} l^{b_i} ,$$

where the 24 representative points $M^*_i = \{a_i, \{k_i, a_i, b_i\}\}$, i = 1,… , 24, with their weights a_i are listed below

$\{-m^2 (1 + m^2),\{0, 0, 1\}\}$,
$\{ - (1 + \nu) (-2 - 3 m^2 + m^2 n),\{0, 0, 2\}\}$,
$\{2 (-1 + n) (1 + n)^2,\{0, 0, 3\}\}$,
$\{ (-1 + m)^2 m^4 (1 + m)^2,\{0, 4, 0\}\}$,
$\{ m^2 (1 + n)(-2 + 3 m^2 - 3 m^4 + m^2 n + m^4 n)$,
$\{0, 4, 1\}\}$,
$\{ -2 m^2 (1 + m^2) (-1 + n) (1 + n)^2,\{0, 4, 2\}\}$,
$\{3 + 2 m^2 + 2 n,\{2, 0, 1\}\}$,
$\{ (-3 + n) (1 + n),\{2, 0, 2\}\}$,
$\{-4 (-1 + m)^2 m^2 (1 + m)^2,\{2, 4, 0\}\}$,
$\{ - (1 + n)$
$\times(-4 + 4 m^2 - 9 m^4 + 4 n + 3 m^4 n - 2 m^2 n^2),\{2, 4, 1\}\}$,
$\{ 4 (1 + m^2 - n) (-1 + n) (1 + n)^2,\{2, 4, 2\}\}$,
$\{1,\{4, 0, 0\}\}$, $\{ -1,\{4, 0, 1\}\}$,
$\{ 2 (2 - 4 m^2 + 3 m^4 - 2 n^2 + m^2 n^2),\{4, 4, 0\}\}$

{ (1 + n) (-4 - 9 m^2 + 4 n+ 3 m^2 n),{4, 4, 1}},
{ -2 (-1 + n) $(1 + n)^2$,{4, 4, 2}},
{m^4 $(-1 + n)^2$ $(1 + n)^2$,{4, 8, 0}},
{ -2 m^2 $(-1 + n)^2$ $(1 + n)^3$,{4, 8, 1}},
{ -4 m^2,{6, 4, 0}},
{ - (-3 + ν) (1 + n),{6, 4, 1}},
{ 4 m^2 (-1 + n) (1 + n),{6, 8, 0}},
{ 4 $(-1 + n)^2$ $(1 + n)^2$,{6, 8, 1}},
{1,{8, 4, 0}},{ -4 (-1 + n) (1 + n),{8, 8, 0}}.

The convex hull for these points is plotted in Figure 5.

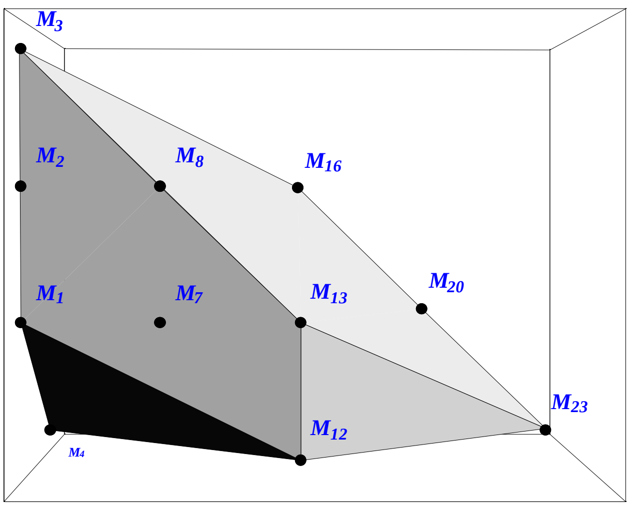

Figure 5. 3D Convex hull for m ~ 1.

The facets of the convex hull determine the separative points k (l ~ h^k). In the case considered, the 3D convex hull consists of 4 facets: 1. (M_1, M_2, M_3 M_7, M_8, M_{12}, M_{13}); 2. (M_{12}, M_{13}, M_{23}); 3. (M_1, M_4, M_{12}); 4. (M_3, M_8, M_{13}, M_{16}, M_{20}, M_{23}). Imposing that the orders of all terms forming a facet be equal to each other, we find the relations from which the orders of l for separative cases may be determined:

$$l \sim l^2 \sim l^3 \sim lp^2 \sim l^2p^2 \sim p^4 \sim lp^4$$
$$p^4 \sim lp^4 \sim h^4p^8$$
$$h^4 \sim l \sim p^4 \tag{44}$$
$$l^3 \sim l^2p^2 \sim lp^4 \sim l^2h^4p^4 \sim lh^4p^6 \sim h^4p^8$$

So, for the first and the second relations k = 0, for the third k = 4, and for the fourth k = – 4. Note that for m = 1, the representative points M_4 and M_9 are absent since their weights a_i = 0. For this specific case there is no facet 3 and, therefore no separative point k = 4. This case is similar to m = 0.

The most interesting case is $m \sim h^{-1/2}$. For such waves numbers the natural frequency is the lowest [4]. The 3D convex hull for this case is plotted in Figure 6.

Here M_i = {{0,0,3}, {2,0,2}, {4,0,1}, {4,4,2}, {6,4,1}, {8,4,0}, {4,0,0}, {0,-2,1}, {0,-1,2}, {2,-1,1}, {0,0,0}}, where i = {2, 15, 26, 29, 35, 39, 25, 11, 13, 24, 3}.

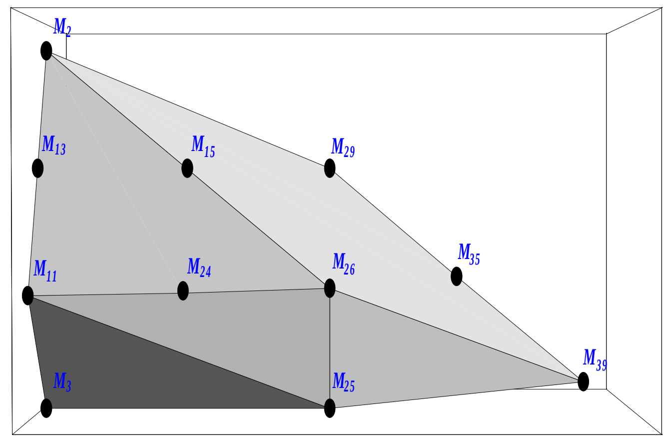

Figure 6. 3D Convex hull for $m \sim h^{-1/2}$.

The asymptotic expansions of the integral have different structures in the four domains separated by the separative points k = 2, k = 0, k = – 1, k = – 4.

The steps of the subsequent analysis are the same as in the case of axisymmetric vibrations: (i) construct all solutions at the separative points and in the domains between the separative points; (ii) find the relative orders of the eigenvectors and substitute the solutions into the imposed boundary conditions; and, (iii) solve the characteristic equation numerically or analytically (if possible) to obtain the natural frequency parameter.

Finally, one considers the general case of non-axisymmetric vibrations of the shell, when the order of m is not given. In this case the coefficients in the characteristic equation (42) depend on three parameters: the small parameter h ($0 < h << 1$), and two positive parameters l ($l > 0$) and m ($m > 0$). The case $m = 0$ (axisymmetric vibrations) has been considered in the section above. The analysis of the roots of the characteristic equation for non-axisymmetric vibrations involves the construction of the 4D convex hull in the space (p, h, l, m).

One assumes that $m = m_0 \; h^t$ and $l = l_0 \; h^k$, where $m_0 \sim 1$ and $l_0 \sim 1$.

The steps of the algorithm are the same as for the 3D case, but to construct the 4D convex hull the code *Qhull* has been used. Since only the cases when parameter h is small are of interest, after constructing the 4D convex hull one should select only the facets on the "lower" part of the convex hull, i.e. the facets that are visible from the point (p, h, l, m) = (0, -∞, 0, 0). Each facet is determined by 4 or more than 4 vertices. Assuming that the orders of the terms corresponding to the vertices of each facet are equal to each other, one finds the orders of l and m, i.e. the separative points in (k, t) plane.

After finding the separative points k_i, t_I,, one can construct the separative lines in the plane (k, t). In order to do so, the horizontal lines $t = t_i$ through the separative points (k_i, t_i) are represented. In the case of non-axisymmetric vibrations, the separative points are (k_i, t_i)={(0,0), (0,1), (4,0), (-4,2)}, so the horizontal lines are t_i = 0, 1, 2. These lines split the entire plane into zones $0 < t < 1$, $1 < t < 2$, $2 < t$. For any fixed k inside one zone the structures of the corresponding 3D convex hulls are similar. So, one may choose an arbitrary point inside each domain and obtain the relations between k and t which determine the separative lines.

For the case under consideration, the domain $0 < t < 1$ is analyzed. Setting $t = 1/2$ arbitrarily, one can find the facets of the 3D convex hull, which are determined by the facets $\{\{M_{11}, M_3, M_{25}\}, \{M_{11}, M_{25}, M_{24}, M_{26}\}, \{M_{11}, M_{13}, M_{15}, M_{24}, M_{26}, M_2\}, \{M_{25}, M_{39}, M_{26}\}, \{M_2, M_{26}, M_{39}, M_{15}, M_{35}, M_{29}\}\}$ (see Figure 6). Note that for any t in the domain (0, 1) the 3D convex hull has such a form.

This leads to the following relations:

(1) $h^{(-4a)} l \sim h^{-4(-1+2a)} \sim p^4$ Þ $l \sim h^{(4-8a+4a)}$ Þ $b=4-4a$ Þ $a = 1-b/4$.

(2) $h^{(-4a)} l \sim p^4 \sim l p^4$ Þ $l \sim 1$ Þ $b = 0$.

(3) $h^{(-4a)} l \sim l p^4 \sim l^3$ Þ $a = -b/2$.

(4) $p^4 \sim h^4 p^8 \sim l p^4$ Þ $b = 0$.

(5) $l^3 \sim l p^4 \sim h^4 p^8 \sim h^4 l^2 p^4$ Þ $b = -4$.

For the domain analyzed, these segments are plotted in Figure 7.

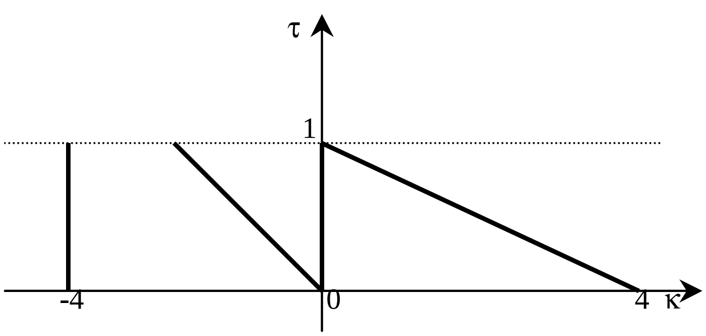

Figure 7. t versus k.

For any point (k, t) inside one domain the structures of the corresponding 2D convex hulls are similar.

The other 2 domains are analyzed in the same manner. The final graph representing the domains and separative lines is similar to that obtained in [2]. Since the initial equations describing vibrations of shells are valid if the frequency is not too high and the number of wave in circumferential direction is not too large [2], the analysis is limited to the cases $-4 < k$ and $-2 < t < 0$.

Conclusions: The geometrical approach appears to be fruitful, since the construction of the convex hull of points set permits to build formal asymptotic solutions in different domains of the space of the parameters. The constructed solutions were used for studying the free vibration spectra of the shells. The most important cases of the relations between the parameters were analyzed, but developing an algorithm for an arbitrary number of small parameters is desirable. To pursue this, the geometrical method can be generalized for an arbitrary number of dimensions. In such a form the algorithm may successfully compete with the standard numerical methods of solution, especially when the relative shell thickness is small.

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