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This manual presents the solutions for the problems provided in the book entitled "Introduction to the Theory of Stability" by D.R. Merkin (Springer-Verlag, 1996) as well as additional supplemental problems on structural stability. It is used by the senior students from the Departments of Mechanics, Faculty of Mathematics and Mechanics at St. Petersburg State University, who take courses in Theory of Stability, Structural Stability and Numerical Methods in Vibration. As such, it may also be recommended for students in Applied Mathematics, Mechanics, Control, Aerospace and Mechanical Engineering at various faculties and universities.

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From the Authors

In 1996 Springer Publishing Company issued the book entitled "The Introduction to the Theory of Stability" written by Prof. D.R. Merkin and translated and edited by Profs. F. Afagh and A. Smirnov. The main advantage of the book is its simple yet simultaneously rigorous presentation of the concepts of the theory, which often are presented in the context of applied problems with detailed examples demonstrating effective methods of solving practical problems.

The above features have made the *Introduction to the Theory of Stability of Motion* the most popular textbook in its field at faculties of mathematics and mechanics as well as engineering faculties in Russian universities and now in the universities of the English speaking countries.

The examples constitute about 25% of the entire volume of the book and cover various areas in science and engineering. Moreover, some of the examples possess an independent value in that they could be used in the analysis of various real structures and mechanisms. The problems are supplied with the answers and some hints.

Using the same numeration as in *Introduction to the Theory of Stability*, the present book contains a detailed solution and discussion of all the problems of the text book. Moreover, the reported errors and misprints of the text book have been corrected in the present volume.

Chapter 8 of this volume does not correspond to the respective chapter in *Introduction to the Theory of Stability.* Instead, a new Chapter 8 entitled "Structural Stability" has been included where some classical problems on stability of equilibrium states in elastic systems have been presented.

The present book is a result of scientific cooperation of the Departments of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics at St. Petersburg State University in Russia and the Department of Mechanical and Aerospace Engineering at Carleton University in Ottawa, Canada.

This work was supported in part by the Russian Foundation for Basic Research under grant # 98-01-01010. A major part of Chapter 9 was prepared by Prof. A. H. Gelig. The help of Mrs. V. Sergeeva and Mr. N. Filippov in typesetting the manuscript and preparing the drawings is highly appreciated. The authors would also like to thank their students for their input and suggestions as well as pointing out the errors that they found out during the preparation of the manuscript.

Chapter 1

Formulation of the Problem

1.1. The perturbed motion of a system is defined by the following equations:

$$\dot{x}_1 = \alpha x_2^3 + \beta x_1 \sqrt[3]{x_1^4 + x_2^4},$$
$$\dot{x}_2 = -\alpha x_1^3 + \beta x_2 \sqrt[3]{x_1^4 + x_2^4}.$$

Determine the stability of the motion of this system. (In the book [11] the "−" sign by β should be replaced by "+".)

S o l u t i o n:

We multiply the first equation by x_1^3 and the second equation by x_2^3, and add the corresponding terms of the resulting equations to get

$$x_1^3 \dot{x}_1 + x_2^3 \dot{x}_2 = \beta \left(x_1^4 + x_2^4\right)^{\frac{4}{3}}$$

or

$$\frac{1}{4}\frac{d}{dt}\left(x_1^4 + x_2^4\right) = \beta \left(x_1^4 + x_2^4\right)^{\frac{4}{3}}.$$

Let $x_1^2 = y_1$ and $x_2^2 = y_2$. Now, stability (or instability) of y_1 and y_2 would mean the stability (or instability) of x_1 and x_2 and visa versa. Let r designate the distance between the point (y_1, y_2) and the reference origin so that $r^2 = y_1^2 + y_2^2$. Now, we have

$$\frac{1}{4}\frac{dr^2}{dt} = \beta r^{\frac{8}{3}}$$

or

$$\frac{1}{2}\frac{dr}{dt} = \beta r^{\frac{5}{3}}.$$

From this it follows that

$$\frac{1}{2}\frac{dr}{r^{\frac{5}{3}}} = \beta dt,$$

which upon integration gives

$$-\frac{3}{2}r^{-\frac{2}{3}} + \frac{3}{2}r_0^{-\frac{2}{3}} = 2\beta\left(t - t_0\right),$$

and

$$r^{\frac{2}{3}} = \frac{r_0^{\frac{2}{3}}}{1 - \frac{4}{3}r_0^{\frac{2}{3}}\beta\left(t - t_0\right)}.$$

Now, if $\beta < 0$, then $r \to 0$ as $t \to \infty$, and the solution is asymptotically stable.

On the other hand, if $\beta > 0$ and $t \to t_0 + \frac{3}{4\beta}r_0^{-\frac{2}{3}}$, we will have $r \to \infty$, and the system is unstable.

For $\beta = 0$ the system is stable (cf. Example 1.1 in [11]).

1.2. The isotropic thin bar with mass m, length l, and horizontal axis of rotation is retained in equilibrium by a spiral spring with stiffness c. The spring is not deformed when the bar is in the upper vertical position. Neglecting all frictional forces, derive the equation that depicts the equilibrium states. Obtain the equation of perturbed motion near the equilibrium state of the bar and the equation of first approximation (see Fig. 1.1).

S o l u t i o n:

In the state of equilibrium of the bar the torque $c\theta$, due to the spring should be equal to the moment $\frac{1}{2}mgl\sin\theta$, caused by the weight of the bar, i. e. ,

$$c\theta = \frac{1}{2}mgl\sin\theta,$$

Figure 1.1: Problem 1.2.

or

$$\sin\theta = k\theta, \tag{1.1}$$

where

$$k = \frac{2c}{mgl}.$$

As it can be seen in Fig. 1.2, for small k, the equation $\sin\theta = k\theta$ has several solutions.

Figure 1.2: Problem 1.2.

Let θ_n be one of the roots of this equation. Denote the change in this angle due to a perturbation as x_n. Then, considering the angular momentum of the rod during this perturbation about the fixed axis O at the support, we have

$$\frac{1}{3}ml^2\frac{d^2}{dt^2}x_n = -c\left(\theta_n + x_n\right) + \frac{1}{2}mgl\sin\left(\theta_n + x_n\right)$$

or, in view of (1.1),

$$\ddot{x}_n + \frac{3g}{2l}k\left(\theta_n + x_n\right) - \frac{3g}{2l}\sin\left(\theta_n + x_n\right) = 0,$$

so that the perturbed motion of bar is described by the equation

$$\ddot{x}_n + \frac{3g}{2l}\left[k\left(\theta_n + x_n\right) - \sin\left(\theta_n + x_n\right)\right] = 0. \tag{1.2}$$

Now, to get the equation of first approximation, let us expand $\sin\left(\theta_n + x_n\right)$ as the following series

$$\sin\left(\theta_n + x_n\right) = \sin\theta_n + x_n\cos\theta_n + \cdots$$

Then, considering only the first two terms of this expansion and substituting it in equation (1.2), we get

$$\ddot{x}_n + \frac{3g}{2l}\left[k\theta_n + kx_n - \sin\theta_n - \cos\theta_n x_n\right] = 0.$$

Finally, noting that θ_n should satisfy (1.1), we obtain the equation of first approximation as

$$\ddot{x}_n + \frac{3g}{2l}\left[kx_n - \cos\theta_n x_n\right] = 0.$$

1.3. The ring M can move freely, without friction, along a circular wire of radius a that is rotating uniformly about a vertical axis. Determine the position of dynamic equilibrium of the ring. Derive the equation of perturbed motion with respect to the equilibrium state and the equation of first approximation. The angular velocity of the uniform rotation of the wire is ω (see Fig. 1.3).

S o l u t i o n:

There are three forces, which act on the ring M. These are:

1) the weight mg of the ring that is directed downward along the vertical axis;

2) the centrifugal force $F_c = ma\omega^2\sin\theta$ that is directed horizontally;

Figure 1.3: Problem 1.3.

3) the reaction from the wire which is directed towards its centre. In a state of equilibrium the resultant of the two first forces should be equal and opposite to the reaction force. Therefore,

$$\tan\theta = \frac{ma\omega^2\sin\theta}{mg}.$$

From this it follows that

$$\cos\theta = \frac{g}{a\omega^2}.$$

Thus, the three angles at which equilibrium prevails are

$$\theta_0 = \arccos\frac{g}{a\omega^2}, \quad \theta_1 = 0, \quad \theta_2 = \pi,$$

where the last two correspond to the evident cases of when the second force is equal to zero.

For the solution $\theta = \theta_0$ we introduce the deviation x for the angle θ_0. Then, to exclude the unknown reaction R from the wire, at point M, we consider Newton's second law in the tangential direction τ:

$$ma\ddot{x} = ma\omega^2\sin(\theta_0 + x)\cos(\theta_0 + x) - mg\sin(\theta_0 + x)$$

or

$$\ddot{x} - \omega^2\sin(\theta_0 + x)\cos(\theta_0 + x) + \frac{g}{a}\sin(\theta_0 + x) = 0. \qquad (1.3)$$

To get the equation of first approximation we can consider that

$$\sin(\theta_0 + x) = \sin\theta_0 + x\cos\theta_0,$$
$$\cos(\theta_0 + x) = \cos\theta_0 - x\sin\theta_0.$$

Substituting these expressions in (1.3) while considering only the first order terms and noting that

$$-\omega^2 \sin\theta_0 \cos\theta_0 + \frac{g}{a}\sin\theta_0 = 0,$$

we obtain the equation of first approximation for the perturbed motion as

$$\ddot{x} - \left(\omega^2 \cos 2\theta_0 - \frac{g}{a}\cos\theta_0\right) x = 0.$$

1.4. The double pendulum depicted in Fig. 1.4 is maintained in the upper vertical position by two spiral springs with stiffness c_1 and c_2. The pendulums have masses m_1 and m_2 and lengths l_1 and l_2. The spiral springs are not deformed when the pendulums are in upper vertical position. Derive the equation for the perturbed motion in the

Figure 1.4: Problem 1.4.

first approximation with respect to the upper vertical position. Neglect the mass of the bars and all frictional forces.

S o l u t i o n:

This system has two independent variables. To write the equation for the perturbed motion we use the Lagrange equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}_k} - \frac{\partial T}{\partial \varphi_k} = -\frac{\partial \Pi}{\partial \varphi_k} \qquad (k = 1, 2)\,. \tag{1.4}$$

The kinetic energy T of the system is $T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$, where v_1 and v_2 are the velocities of mass points M_1 and M_2. Using Fig. 1.4, we can find the coordinates (x_1, y_1) and (x_2, y_2) as follows:

$$\begin{aligned} x_1 &= l_1 \sin\varphi_1, \quad & x_2 &= l_1 \sin\varphi_1 + l_2 \sin\varphi_2, \\ y_1 &= l_1 \cos\varphi_1, \quad & y_2 &= l_1 \cos\varphi_1 + l_2 \cos\varphi_2. \end{aligned}$$

Now, upon differentiation we obtain

$$\begin{aligned} \dot{x}_1 &= l_1 \cos\varphi_1 \dot{\varphi}_1, \quad & \dot{x}_2 &= l_1 \cos\varphi_1 \dot{\varphi}_1 + l_2 \cos\varphi_2 \dot{\varphi}_2, \\ \dot{y}_1 &= -l_1 \sin\varphi_1 \dot{\varphi}_1, \quad & \dot{y}_2 &= -l_1 \sin\varphi_1 \dot{\varphi}_1 - l_2 \sin\varphi_2 \dot{\varphi}_2. \end{aligned}$$

Now, we have $v_1^2 = l_1^2 \dot{\varphi}_1^2$, $v_2^2 = l_1^2 \dot{\varphi}_1^2 + 2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\,\dot{\varphi}_1 \dot{\varphi}_2 + l_2^2 \dot{\varphi}_2^2$. Since the angles φ_1 and φ_2 are small we have $\cos(\varphi_2 - \varphi_1) = 1$ and since we are seeking the equation of first approximation the kinetic energy T can be written as

$$T = \frac{l_1^2}{2}(m_1 + m_2)\,\dot{\varphi}_1^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 + \frac{1}{2} m_2 l_2^2 \dot{\varphi}_2^2.$$

The potential energy Π of the system is due to both the elastic energy of the springs and gravitational potential energy due to weights, i. e. ,

$$\begin{aligned} \Pi = {} & \tfrac{1}{2} c_1 \varphi_1^2 + \tfrac{1}{2} c_2 (\varphi_2 - \varphi_1)^2 - (m_1 + m_2)\, g l_1 (1 - \cos\varphi_1) - \\ & - m_2 g l_2 (1 - \cos\varphi_2)\,, \end{aligned}$$

or more simply,

$$\Pi = \frac{1}{2}[c_1 + c_2 - (m_1 + m_2)\, g l_1]\varphi_1^2 - c_2 \varphi_1 \varphi_2 + \frac{1}{2}[c_2 - m_2 g l_2]\varphi_2^2.$$

Thus, the Lagrange equations (1.4) become

$$(m_1 + m_2)\, l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \ddot{\varphi}_2 + [c_1 + c_2 - (m_1 + m_2)\, g l_1]\varphi_1 - c_2 \varphi_2 = 0,$$

$$m_2 l_1 l_2 \ddot{\varphi}_1 + m_2 l_2^2 \ddot{\varphi}_2 - c_2 \varphi_1 + (c_2 - m_2 g l_2)\,\varphi_2 = 0.$$

These two equations can be presented in the matrix form $A\ddot{\Phi} + C = 0$ where

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad A = \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 + c_2 - (m_1+m_2)gl_1 & -c_2 \\ -c_2 & c_2 - m_2 g l_2 \end{pmatrix}.$$

1.5. The rigid body M with mass m is fixed to the free end of a compressed and twisted cantilever bar that has a uniform bending stiffness (see Fig. 1.5) (see Section 2.12 of [2]) Neglecting the mass of the

Figure 1.5: Problem 1.5.

bar and treating M as a point mass, obtain the equations of perturbed motion near the equilibrium state for the first approximation.

Remarks: Two forces, located in the horizontal plane Oxy, are applied to M under the problem conditions. The radial force F_r is directed from M to O, and the transverse force F_φ is perpendicular to F_r. Both forces are proportional to the distance MO. Neglect any vertical displacement of the rigid body M and all frictional forces.

S o l u t i o n:

The bending force F_r and the twisting force F_φ that are mutually perpendicular are applied to M (see Fig. 1.6). Both forces are proportional to the distance $r = OM$ (see Section 2.12 of [2]), i. e. ,

$$F_r = c_1 r, \quad F_\varphi = c_2 r.$$

Figure 1.6: Problem 1.5.

or in projections on the axes

$$F_{rx} = -c_1 r \sin\alpha = -c_1 x, \quad F_{ry} = -c_1 r \cos\alpha = -c_1 y,$$
$$F_{\varphi x} = c_2 r \cos\alpha = c_2 y, \quad F_{\varphi y} = -c_2 r \sin\alpha = -c_2 x.$$

Invoking Newton's second law and using the magnitudes of F_r and F_φ we obtain the equations of perturbed motion for the first approximation as

$$m\ddot{x} = -c_1 x + c_2 y,$$
$$m\ddot{y} = -c_2 x - c_1 y.$$

1.6. A rigid body with one fixed point moves inertially (the case of Euler–Poinsot). Prove that such a body can rotate uniformly around a fixed axis that coincides in this motion with one of the principal axes of inertia, for instance with z-axis. Considering

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega_0 = \text{const},$$

derive the equation of the perturbed motion in terms of the components of the angular velocity. Let the moments of inertia of the body with respect to its principal axes of inertia x, y, z be designated as A, B, C, respectively.

S o l u t i o n:

Consider the following Euler equations for the given dynamic system:

$$A\dot{\omega}_x + (C - B)\,\omega_y\omega_z = M_x^e,$$

$$\begin{aligned} B\dot{\omega}_y + (A - C)\,\omega_z\omega_x &= M_y^e, \\ C\dot{\omega}_z + (B - A)\,\omega_x\omega_y &= M_z^e, \end{aligned} \tag{1.5}$$

where, according to the conditions of the problem, $M_x^e = M_y^e = M_z^e = 0$. The steady rotation is defined by

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega_0 = \text{const}.$$

In the perturbed motion, let the deviations of the angular velocities ω_x, ω_y and ω_z be designated as x_1, x_2 and x_3, respectively, i. e. ,

$$\omega_x = x_1, \quad \omega_y = x_2, \quad \omega_z = \omega_0 + x_3.$$

Substitute these in equation (1.5), to get

$$\begin{aligned} A\dot{x}_1 + (C - B)\,x_2\,(\omega_0 + x_3) &= 0 \\ B\dot{x}_2 + (A - C)\,(\omega_0 + x_3)\,x_1 &= 0 \\ C\dot{x}_3 + (B - A)\,x_1x_2 &= 0. \end{aligned}$$

1.7. Two boxes with two identical gyroscopes inside are shown in Fig. 1.7. The boxes are connected by gears so that they can rotate

Figure 1.7: Problem 1.7.

in different directions by an equal angle β. The axis of rotation of the external frame that contains the whole apparatus is free. A spiral spring with stiffness c is installed on the axis of rotation of one of the boxes. Neglecting the mass of the external frame and the boxes and all frictional forces, determine the condition of stationary motion under which the angle β and the angular velocity $\dot{\alpha}$ of the frame remain constant. Derive the equation of perturbed motion with respect to the stationary motion.

Solution:

The system consists of two connected identical gyroscopes that each have a fixed point. For each gyroscope, let the mass moment of inertia about each of its two axes x and y be denoted by A while the moment of inertia with respect to the z axes is denoted by C. Then, the kinetic energy T of the system will be

$$T = 2 \cdot \left(\frac{1}{2} A \left(\omega_x^2 + \omega_y^2 \right) + \frac{1}{2} C \omega_z^2 \right). \tag{1.6}$$

If one of the gyroscopes, for example the left one (see Fig. 1.8), is rotated so that

$$\begin{aligned} \omega_x &= -\dot{\beta}, \\ \omega_y &= \dot{\alpha} \cos \beta, \\ \omega_z &= \dot{\varphi} + \dot{\alpha} \sin \beta, \end{aligned}$$

then, for the one on the right we will have $\omega_x = \dot{\beta}$, $\omega_y = -\dot{\alpha} \cos \beta$. Here $\dot{\varphi}$ is the angular velocity of gyroscope. Substituting the expressions for

Figure 1.8: Problem 1.7.

ω_x, ω_y, and ω_z in (1.6), we get

$$T = A\dot{\beta}^2 + A\dot{\alpha}^2 \cos^2 \beta + C \left(\dot{\varphi} + \dot{\alpha} \sin \beta \right)^2. \tag{1.7}$$

The potential energy Π of the system is due to the torsional spring and is

$$\Pi = \frac{1}{2} c\beta^2. \tag{1.8}$$

Since we are considering the steady rotation of the gyroscopes when the induced moment M_φ^{rot} is equal to the resisting moment M_φ^{res}, the angle φ is a cyclic coordinate. Therefore, the generalised force $Q_\varphi = M_\varphi^{rot} - M_\varphi^{res}$ corresponding to the coordinate φ is equal to zero.

Noting that in (1.7) the kinetic energy does not depend on the angle φ, the Lagrange equation with respect to φ becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\varphi} = 0.$$

Now, $\dfrac{\partial T}{\partial \dot\varphi} = 2H$, where H is the angular momentum of each gyroscope ($H = \text{const}$).

Using (1.7), we note that

$$C\left(\dot\varphi + \dot\alpha \sin\beta\right) = H. \tag{1.9}$$

This integral is called cyclic integral. Next, we can write the Lagrange equation with respect to the β coordinate as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\beta} - \frac{\partial T}{\partial \beta} = -\frac{\partial \Pi}{\partial \beta}. \tag{1.10}$$

Using relations (1.7), (1.8) and (1.9), we find

$$\begin{aligned} &\frac{\partial T}{\partial \dot\beta} = 2A\dot\beta, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot\beta} = 2A\ddot\beta, \\ &\frac{\partial T}{\partial \beta} = -2A\dot\alpha^2 \cos\beta \sin\beta + 2H\dot\alpha\cos\beta, \quad \frac{\partial \Pi}{\partial \beta} = c\beta. \end{aligned}$$

Substitution of these expressions into (1.10) results in the differential equation

$$2A\ddot\beta + 2A\dot\alpha^2 \sin\beta\cos\beta - 2H\dot\alpha\cos\beta = -c\beta. \tag{1.11}$$

For the steady motion, we should have

$$\beta = \beta_0 = \text{const}, \quad \ddot\beta = 0, \quad \dot\alpha = \omega = \text{const}. \tag{1.12}$$

Substitute (1.11) into (1.12), to obtain the condition for steady motion as

$$A\omega^2 \cos\beta_0 \sin\beta_0 - H\omega\cos\beta_0 + \frac{1}{2}c\beta_0 = 0. \tag{1.13}$$

To get the equations for perturbed motion, we consider

$$\beta = \beta_0 + x_1, \quad \dot{\alpha} = \omega + x_2. \tag{1.14}$$

Substitute (1.14) into the expressions for kinetic and potential energy (note that $\dot{\beta} = \dot{x}_1$), then

$$\begin{aligned} T &= A\dot{x}_1^2 + A\left(\omega + x_2\right)^2 \cos^2\left(\beta_0 + x_1\right) + \\ &\quad C\left[\dot{\varphi} + \left(\omega + x_2\right)\sin(\beta_0 + x_1)\right]^2, \\ \Pi &= \frac{c}{2}\left(\beta_0 + x_1\right)^2. \end{aligned} \tag{1.15}$$

Now the cyclic integral (1.9) reads as

$$C[\dot{\varphi} + (\omega + x_2)\sin\left(\beta_0 + x_1\right)] = H.$$

Next, considering the Lagrange equation for x_1,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_1} - \frac{\partial T}{\partial x_1} = -\frac{\partial \Pi}{\partial x_1}, \tag{1.16}$$

by virtue of (1.13) we have

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}_1} &= 2A\dot{x}_1, \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} &= 2A\ddot{x}_1, \\ \frac{\partial T}{\partial x_1} &= -2A\left(\omega + x_2\right)^2 \cos\left(\beta_0 + x_1\right)\sin\left(\beta_0 + = x_1\right) + \\ &\quad 2H\left(\omega + x_2\right)\cos\left(\beta_0 + x_1\right), \\ \frac{\partial \Pi}{\partial x_1} &= c\left(\beta_0 + x_1\right). \end{aligned} \tag{1.17}$$

Moreover, the functions $\cos\left(\beta_0 + x_1\right)$, $\sin\left(\beta_0 + x_1\right)$, and $\left(\omega + x_2\right)^2$ each can be expanded into the following series:

$$\begin{aligned} \cos\left(\beta_0 + x_1\right) &= \cos\beta_0 - \sin\beta_0 x_1 + \cdots, \\ \sin\left(\beta_0 + x_1\right) &= \sin\beta_0 + \cos\beta x_1 + \cdots, \\ \left(\omega + x_2\right)^2 &= \omega^2 + 2\omega x_2 + \cdots, \end{aligned} \tag{1.18}$$

where the dots denote the higher-order terms in x_1 and x_2.

Using (1.18) and (1.17) in (1.16), and after a brief manipulation we get

$$A\ddot{x}_1 + A\omega^2 \cos\beta_0 \sin\beta_0 - H\omega\cos\beta_0 + \tfrac{1}{2}\beta_0 +$$
$$\left(A\omega^2\cos 2\beta_0 + H\omega\sin\beta_0 + \tfrac{1}{2}c_1\right)x_1 + \left(A\omega\sin 2\beta_0 - H\cos\beta_0\right)x_2 = X_1.$$

Here X_1 represents all the terms that contain x_1 and x_2 in powers higher than one.

By means of (1.13) we obtain the first equation of perturbed motion as

$$\begin{aligned} A\ddot{x}_1 + \left(A\omega^2\cos 2\beta_0 + H\omega\sin\beta_0 + \tfrac{1}{2}c_1\right)x_1 + \\ + \left(A\omega\sin 2\beta_0 - H\cos\beta_0\right)x_2 = X_1. \end{aligned} \tag{1.19}$$

The α coordinate is also cyclic one, because according to (1.7) the kinetic energy is a function of the velocity $\dot{\alpha}$ only, while the potential energy does not depend on α either. Thus, the differential equation for α coordinate, and hence for x_2, becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}} = \frac{d}{dt}\frac{\partial T}{\partial x_2} = 0. \tag{1.20}$$

By means of (1.15) we have

$$\frac{\partial T}{\partial x_2} = 2A\left(\omega + x_2\right)\cos^2\left(\beta_0 + x_1\right) + 2H\sin\left(\beta_0 + x_1\right).$$

Substituting this into (1.20), we get

$$\begin{gathered} 2A\dot{x}_2\cos^2\left(\beta_0 + x_1\right) - 4A\left(\omega + x_2\right)\cos\left(\beta_0 + x_1\right) = \\ \sin\left(\beta_0 + x_1\right)\dot{x}_1 + 2H\cos\left(\beta_0 + x_1\right)\dot{x}_1 = 0. \end{gathered}$$

Upon dividing this expression by $2\cos\left(\beta_0 + x_1\right)$ and retaining only first order terms in $\dot{x}_1$ and $\dot{x}_2$ we obtain the second equation for perturbed motion as

$$\left(H - 2A\omega\sin\beta_0\right)\dot{x}_1 + A\cos\beta_0\dot{x}_2 = X_2. \tag{1.21}$$

Equations (1.19) and (1.21) define the perturbed motion of the system about the steady state motion.

Chapter 2

The Direct Liapunov Method. Autonomous Systems

2.1. For the given equations of a perturbed motion,

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2, \\ \dot{x}_2 &= -5x_2 - 3x_1^2 \end{aligned} \tag{2.1}$$

determine the Liapunov function, and show that the unperturbed motion $x_1 = x_2 = 0$ is stable in the large.

(There is a misprint in the second equation in the book [11].)

S o l u t i o n:

Multiply the first equation by x_1, and the second one by x_2 and add the corresponding terms of the resulting equations to get

$$x_1\dot{x}_1 + x_2\dot{x}_2 = -\left(x_1^4 + 2x_1^2 x_2 + 5x_2^2\right)$$

or

$$\frac{1}{2}\frac{d}{dt}\left(x_1^2 + x_2^2\right) = -\left(x_1^4 + 2x_1^2 x_2 + 5x_2^2\right).$$

The function $V = x_1^2 + x_2^2$ is a positive definite function for all x_1 and x_2, and its derivative with respect to time,

$$-\left(x_1^4 + 2x_1^2x_2 + 5x_2^2\right)$$

is negative definite for all x_1 and x_2.

The function

$$x_1^4 + 2x_1^2x_2 + 5x_2^2$$

satisfies the Sylvester criterion (see relations (2.9) in [11]) for all x_1 and x_2 because

$$\Delta_1 = 1 > 0, \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 > 0.$$

Thus, according to Liapunov's theorem of stability of motion system (2.1) is stable asymptotically.

2.2. The following functions and their derivatives with respect to time, as determined by virtue of the respective equations of perturbed motion, are given as follows:

$$\begin{array}{lll} 1. & V = x_1^6 + x_2^3, & \dot V = -x_1^6 - x_2^4; \\ 2. & V = 5x_1^4 - 4x_1^2x_2 + x_2^2, & \dot V = -4x_1^4 + 2x_1^2x_2 - x_2^2; \\ 3. & V = x_1^6 + 3x_2^2, & \dot V = -\left(x_1^3 - x_2\right)^2; \\ 4. & V = x_1 - x_2^3, & \dot V = 4x_1^3, \end{array}$$

Can these functions be used to determine stability of motion?

S o l u t i o n:

1. The function $V = x_1^6 + x_2^3$ can not be used because the sign of this function changes (for $x_1 = 0$ and $x_2 > 0$, $V > 0$, while for $x_1 = 0$ and $x_2 < 0$, $V < 0$). Moreover, its derivative $\dot V = -x_1^6 - x_2^4$ is a negative definite function.

2. The function $V = 5x_1^4 - 4x_1^2x_2 + x_2^2$ is positive definite, because the Sylvester criterion is satisfied ((2.9) in [11]):

$$\Delta_1 = 5 > 0,$$

$$\Delta_2 = \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = 1 > 0.$$

Also, the derivative $\dot{V} = -4x_1^4 + 2x_1^2 x_2 - x_2^2$ is a negative definite function, because the Sylvester criterion ((2.10) in [11]) is satisfied:

$$\Delta_1 = -4 < 0,$$

$$\Delta_2 = \begin{vmatrix} -4 & 1 \\ 1 & -1 \end{vmatrix} = 3 > 0.$$

Therefore according to Liapunov's theorem the system is asymptotically stable.

In applying Sylvester's criterion we replace x_1 by x_1^2.

3. The function $V = x_1^6 + x_2^2$ is positive definite, and it's derivative $\dot{V} = -\left(x_1^3 - x_2\right)^2$ is negative semidefinite. Therefore, according to Liapunov's theorem the system is stable.

4. The function $V = x_1 - x_2^3$ is positive for $x_1 > 0$ and $x_2 < 0$ while it's derivative $\dot{V} = 4x_1^3 > 0$ for $x_1 > 0$. Thus according to Chetaev's theorem the system is unstable.

2.3. Show that the equations of the perturbed motion of a rigid body in a uniform rotation (see Problem 1.6) have two integrals:

$$\begin{aligned} Ax_1^2 + Bx_2^2 + C\left(x_3 + \omega_0\right)^2 &= \text{const}, \\ A^2x_1^2 + B^2x_2^2 + C^2\left(x_3 + \omega_0\right)^2 &= \text{const}. \end{aligned}$$

Give the physical meaning of these integrals; compose a bundle of integrals, and prove that the uniform rotation about the large as well as the small axis of the ellipsoid of moment of inertia (in this case, respectively, $C < A < B$ and $C > A > B$) is stable.

Solution:

These two integrals could be obtained in the following manner. Consider the equations derived in Problem 1.6. Multiply the first equation by x_1, the second equation by x_2, and the third one by $(x_3 + \omega_0)$ to get

$$\begin{aligned} Ax_1\dot{x}_1 &= B\omega_0 x_1 x_2 - C\omega_0 x_1 x_2 + Bx_1x_2x_3 - Cx_1x_2x_3, \\ Bx_2\dot{x}_2 &= C\omega_0 x_1 x_2 - A\omega_0 x_1 x_2 + Cx_1x_2x_3 - Ax_1x_2x_3, \\ C\left(x_3 + \omega_0\right)\dot{x}_3 &= Ax_1x_2x_3 - Bx_1x_2x_3 + A\omega_0 x_1 x_2 - B\omega_0 x_1 x_2. \end{aligned}$$

Adding these equations gives

$$Ax_1\dot{x}_1 + Bx_2\dot{x}_2 + C\left(x_3 + \omega_0\right)\dot{x}_3 = 0,$$

so that upon integration one gets

$$Ax_1^2 + Bx_2^2 + C\left(x_3 + \omega_0\right)^2 = \text{const}, \tag{2.2}$$

i.e., the first of the two integrals.

To get the second integral, again refer to the three equations given in Problem 1.6. Multiply the first equation by Ax_1, the second equation by Bx_2, and the third one by $C(x_3 + \omega_0)$. Then, add all the resulting equations to obtain a single equation. Integrate this equation to obtain the second integral as

$$A^2x_1^2 + B^2x_2^2 + C^2(x_3 + \omega_0)^2 = \text{const}. \tag{2.3}$$

We denote the integral in (2.2) by V_1, and the one in (2.3) by V_2.

Now, consider the bundle of integrals

$$V = -V_2 + CV_1 \pm \frac{1}{\omega_0^2}\left(V_1 - C\omega_0^2\right)^2,$$

where the coefficient $\dfrac{1}{\omega_0^2}$ is introduced to retain the dimensional validity of the equation. Upon substituting the expressions for V_1 and V_2 and regrouping of the terms, the bundle becomes:

$$V = A\left(C - A\right)x_1^2 + B\left(C - B\right)x_2^2 \pm \frac{1}{\omega_0^2}\left(Ax_1^2 + Bx_2^2 + Cx_3^2 + 2C\omega_0 x_3\right)^2$$

or,

$$V = A\left(C - A\right)x_1^2 + B\left(C - B\right)x_2^2 \pm 4C^2x_3^2 + \cdots, \tag{2.4}$$

where higher order terms of x_k are denoted by the dots.

First, we consider the " $+$ " sign in (2.4), i.e.,

$$V = A\left(C - A\right)x_1^2 + B\left(C - B\right)x_2^2 + 4C^2x_3^2 + \cdots$$

If $C > A$, $C > B$ and $|x_k|$ is small enough, then V is positive definite and its derivative is equal to zero. Thus, all the corresponding conditions of Liapunov's theorem are satisfied and for $C > A$ and $C > B$ the motion is stable.

Now we consider the " $-$ " sign in (2.4), i.e.,

$$V = A\left(C - A\right)x_1^2 + B\left(C - B\right)x_2^2 - 4C^2x_3^2 + \cdots$$

Here, for $C < A$ and $C < B$ the function V is negative definite, and again according to Liapunov's theorem the motion is stable.

2.4. The rotational motion of a rigid body in a gravitational field about a fixed point O is considered. For a set of principal axes with the origin at O and attached to the rotating body, the equations of motion are

$$\begin{aligned} A\dot{\omega}_x + (C - B)\,\omega_y\omega_z &= \gamma_y m_z - \gamma_z m_y, \\ B\dot{\omega}_y + (A - C)\,\omega_z\omega_x &= \gamma_z m_x - \gamma_x m_z, \\ C\dot{\omega}_z + (B - A)\,\omega_x\omega_y &= \gamma_x m_y - \gamma_y m_x, \end{aligned}$$

where A, B, and C are principal mass moments of inertia of the body with respect to the (x, y, z) set of axes; ω_x, ω_y, and ω_z are components of the angular velocity $\boldsymbol{\omega}$ along the x, y, and z axes; m_x, m_y, and m_z are the static moments of the weight of the rigid body $\boldsymbol{m}$ about the x, y, and z axes; $\boldsymbol{\gamma}$ is the vertical axis of the fixed coordinate system; and γ_x, γ_y, and γ_z are components of the unit vector of $\boldsymbol{\gamma}$ along the x, y, and z axes (direction cosines). Staude and Mlodzeevsky have independently proved that under some conditions a body can rotate with constant angular velocity about an axis $\boldsymbol{\gamma}$. A set of such axes forms a cone. Not all rotations with constant velocity are stable.

Construct the motion integrals and using their bundle, prove stability of rotation with constant angular velocity about that principal axis of the rigid body with respect to which the mass moment of inertia of the body is maximum.

Hint. The integrals of motion are

$$F_1 = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{\gamma}^T \boldsymbol{m} = h, \quad F_2 = \boldsymbol{\gamma}^T \boldsymbol{J}\boldsymbol{\omega} = L, \quad F_3 = \boldsymbol{\gamma}\boldsymbol{\gamma} = 1,$$

where h and L are constants, and

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \quad \boldsymbol{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix}.$$

If the mass moment of inertia is maximal with respect to the z-axis, then stability has to be determined for this axis; in this case $\omega_x = \omega_y = 0$, $m_x = m_y = 0$, $\gamma_x = \gamma_y = 0$. The following bundle of integrals can be considered:

$$V(\boldsymbol{\omega}, \boldsymbol{\gamma}) = F_1 + \lambda F_2 + \frac{1}{2}\mu F_3,$$

where λ and μ are factors to be determined. Show that $\lambda = -|\boldsymbol{\omega}|$, and that for the chosen axis the relation $\mu = A\omega^2 - m_z$ holds. This can help you to prove the stability of uniform rotation about the z-axis.

S o l u t i o n:

$F_1 = h$ is an energy integral, $F_2 = L$ represents an axes transformation, and F_3 expresses a simple well known condition of direction cosines. None of these functions, when considered alone, can lead to a positive definite Liapunov function. Therefore, we can construct the bundle of integrals

$$V(\boldsymbol{\omega}, \boldsymbol{\gamma}) = F_1 + \lambda F_2 + \frac{1}{2}\mu F_3.$$

Substitute the values of F_1, F_2 and F_3 into this bundle and evaluate the first variation of the function V:

$$\delta V = \delta\boldsymbol{\omega}\left(\boldsymbol{J}\boldsymbol{\omega} + \lambda\boldsymbol{J}\boldsymbol{\gamma}\right) + \delta\boldsymbol{\gamma}\left(\boldsymbol{m} + \lambda\boldsymbol{J}\boldsymbol{\omega} + \mu\boldsymbol{\gamma}\right).$$

This variation vanishes if

$$\boldsymbol{J}\left(\boldsymbol{\omega} + \lambda\boldsymbol{\gamma}\right) = 0,$$
$$\boldsymbol{m} + \lambda\boldsymbol{J}\boldsymbol{\omega} + \mu\boldsymbol{\gamma} = 0.$$

From here one concludes that

$$\lambda = -|\boldsymbol{\omega}|, \quad \mu = \boldsymbol{\omega}^2 C - m_z. \tag{2.5}$$

In order to obtain the condition of positive definiteness for V one may consider the second variation of this function in the neighbourhood of a constant set of definite positive we write the second variation of this function in $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$:

$$\delta^2 V = \delta\boldsymbol{\omega}\boldsymbol{J}\delta\boldsymbol{\omega} + 2\lambda\delta\boldsymbol{\gamma}\boldsymbol{J}\delta\boldsymbol{\omega} + \mu\delta\boldsymbol{\gamma}\delta\boldsymbol{\gamma}.$$

We note that $F_3 = \boldsymbol{\gamma}\boldsymbol{\gamma} = 1$ and hence $\boldsymbol{\gamma}\delta\boldsymbol{\gamma} = 0$. From here it follows that in our case $\delta\gamma_z = 0$.

Then we get

$$\delta^2 V = A\delta\omega_x^2 + B\delta\omega_y^2 + C\delta\omega_z^2 + 2\lambda\left(A\delta\gamma_x\delta\omega_x + B\delta\gamma_y\delta\omega_y\right) + \mu\left(\delta\gamma_x^2 + \delta\gamma_y^2\right).$$

For this quadratic form in $\delta\gamma_x^2$, $\delta\gamma_y^2$, $\delta\omega_x^2$, $\delta\omega_y^2$, $\delta\omega_z^2$ we write the matrix of coefficients of the quadratic form as given in (2.7) of [11]:

$$\begin{pmatrix} \mu & 0 & \lambda A & 0 & 0 \\ 0 & \mu & 0 & \lambda B & 0 \\ \lambda A & 0 & A & 0 & 0 \\ 0 & \lambda B & 0 & B & 0 \\ 0 & 0 & 0 & 0 & C \end{pmatrix}.$$

Now, using Sylvester's criterion ((2.9) in [11]) we get:

$$\begin{aligned} &\Delta_1 = \mu > 0, \\ &\Delta_2 = \mu^2 > 0, \\ &\Delta_3 = \mu^2 A - \lambda^2 \mu A^2 = \mu A \left(\mu - \lambda^2 A\right) > 0, \\ &\Delta_4 = AB \left(\mu - \lambda^2 B\right) \left(\mu - \lambda^2 A\right) > 0. \end{aligned}$$

Noting that $m_z > 0$, in view of (2.5), the condition that $\Delta_1 = \mu > 0$ results in

$$\omega^2 > \frac{m_z}{C}. \tag{2.6}$$

From the condition that $\Delta_3 > 0$ it follows that

$$\omega^2 > \frac{m_z}{C - A}. \tag{2.7}$$

If the inequality (2.7) is satisfied, from the condition that $\Delta_4 > 0$ it follows that

$$\omega^2 > \frac{m_z}{C - B}. \tag{2.8}$$

If $C > B > A$, then all conditions of stability for this rotation reduce to the single inequality (2.8).

Chapter 3

Stability of Equilibrium States and Stationary Motions of Conservative Systems

3.1. The end B of a perfectly flexible, weightless, and inextensible cord of length l is fixed (see Fig. 3.1). At the other end a load P is attached.

Block D is fixed and block C can slide on the vertical line that bisects the distance a between points B and D. Block C carries a load Q. Neglecting the dimensions of blocks D and C and all resistant forces, determine the equilibrium positions of the system and investigate the stability of these positions.

S o l u t i o n:

Let the coordinates of points C and P be denoted by z_1 and z_2, respectively. From Fig. 3.1 we have

$$z_1 = \frac{a}{2}\tan\varphi, \quad z_2 = l - 2\frac{a}{2\cos\varphi}.$$

Figure 3.1: Problem 3.1

The potential energy of the system is

$$\Pi = -Qz_1 - Pz_2 = -\frac{Qa}{2}\tan\varphi - P\left(l - \frac{a}{\cos\varphi}\right).$$

In a state of equilibrium we should have $\dfrac{\partial \Pi}{\partial \varphi} = 0$, i.e. by Theorem 3.1:

$$\frac{\partial \Pi}{\partial \varphi} = -\frac{Qa}{2}\frac{1}{\cos^2\varphi} + Pa\frac{\sin\varphi}{\cos^2\varphi} = -\frac{a}{\cos^2\varphi}\left(\frac{Q}{2} - P\sin\varphi\right) = 0.$$

Hence,

$$\sin\varphi = \frac{Q}{2P} < 1, \tag{3.1}$$

so that for $Q \geq 2P$ equilibrium vanishes. Thus, if $Q < 2P$, then equlibrium prevails for

$$\varphi_0 = \arcsin\frac{Q}{2P}.$$

This state is stable because

$$\left(\frac{\partial^2 \Pi}{\partial \varphi^2}\right)_{\varphi=\varphi_0} = \frac{Pa}{\cos\varphi_0} > 0.$$

In evaluating the second derivative, equation (3.1) has been employed.

3.2. Ring A can slide over a smooth wire ring of radius R without friction (see Fig. 3.2). The ring R lies in a vertical plane. Load P is suspended from ring A by a perfectly flexible but inextensible cord. The load Q is suspended from the other end C of the cord, which is

Figure 3.2: Problem 3.2

stretched over the infinitesimal block B. Block B lies on the horizontal diameter of the wire ring R, and its weight is negligible. Determine the equilibrium positions of ring A and investigate their stability.

S o l u t i o n:

It should be noted that the load P is supported by a cord, the upper end of which is attached to ring A. A second cord, which supports the load Q is attached to the same ring. We write the potential energy of the system for the position when ring A is in the upper half of the ring R. The potential energy of P is equal to its weight multiplied by $R\sin\varphi$, i.e., the elevation of ring A. Similarly, the energy of Q is equal to its weight Q multiplied by the length l of the cord AB.

Noting that the angle BAO is equal to $\dfrac{\varphi}{2}$, we get $l = 2R\cos\dfrac{\varphi}{2}$. Considering that the total length of the cord is constant, except for a constant amount, the potential energy of the whole system when the ring A is in the upper half is given as:

$$\Pi = PR\sin\varphi + 2QR\cos\frac{\varphi}{2}$$

or

$$\Pi = PR\left(\sin\varphi + 2\frac{Q}{P}\cos\frac{\varphi}{2}\right).$$

For a state of equilibrium we should have $\dfrac{\partial\Pi}{\partial\varphi} = 0$. Upon differentiation we get

$$\frac{\partial\Pi}{\partial\varphi} = PR\left(\cos\varphi - \frac{Q}{P}\sin\frac{\varphi}{2}\right) = 0. \tag{3.2}$$

Using the identity $\cos\varphi = \cos^2\dfrac{\varphi}{2} - \sin^2\dfrac{\varphi}{2} = 1 - 2\sin^2\dfrac{\varphi}{2}$ we can find an expression for $\sin\dfrac{\varphi}{2}$ in the following manner:

$$1 - 2\sin^2\frac{\varphi_0}{2} - \frac{Q}{P}\sin\frac{\varphi_0}{2} = 0$$

or

$$\sin^2\frac{\varphi_0}{2} + \frac{Q}{2P}\sin\frac{\varphi_0}{2} - \frac{1}{2} = 0.$$

From which we get

$$\sin\frac{\varphi_0}{2} = -\frac{O}{4P} + \sqrt{\frac{Q^2}{16P^2} + \frac{1}{2}}$$

(only the "+" sign in front of the square root should be considered, since $\sin\frac{\varphi_0}{2} > 0$).

From this expression the equilibrium position of the ring A in the upper half is given by

$$\sin\frac{\varphi_0}{2} = \frac{1}{4}\left(\sqrt{\frac{Q^2}{P^2} + 8} - \frac{Q}{P}\right). \tag{3.3}$$

In order to determine the stability of this position, the second derivative of Π should be determined. Using (3.2) we have

$$\frac{\partial^2\Pi}{\partial\varphi^2} = PR\left(-\sin\varphi - \frac{1}{2}\frac{Q}{P}\cos\frac{\varphi}{2}\right) = -PR\left(\sin\varphi + \frac{1}{2}\frac{Q}{P}\cos\frac{\varphi}{2}\right).$$

For the upper half of the ring R we have $0 < \varphi < \pi$, or $0 < \dfrac{\varphi}{2} < \frac{\pi}{2}$. In this interval $\sin\varphi > 0$ and $\cos\dfrac{\varphi}{2} > 0$. Therefore, $\dfrac{\partial^2\Pi}{\partial\varphi^2} < 0$, implying that the equilibrium position (3.3) is unstable.

From (3.3) it is clear that in the interval $0 < \frac{Q}{P} < \infty$ the value $\frac{\varphi_0}{2}$ varies from $\frac{\pi}{4}$ to 0. Therefore, the angle φ_0 lies in the interval $0 < \varphi_0 < \frac{\pi}{2}$.

Next, let us consider the case when the ring A is in the lower half of the ring R, i.e., when

$$\pi < \varphi < 2\pi, \quad \frac{\pi}{2} < \frac{\varphi}{2} < \pi.$$

In this interval

$$\sin\varphi < 0, \quad \sin\frac{\varphi}{2} > 0, \quad \cos\frac{\varphi}{2} < 0. \tag{3.4}$$

The potential energy of the system is given as $\left(l = -2R\cos\frac{\varphi}{2} > 0\right)$

$$\Pi = PR\sin\varphi - 2QR\cos\frac{\varphi}{2} = PR\left(\sin\varphi - 2\frac{Q}{P}\cos\frac{\varphi}{2}\right)$$

(the energy associated with P is negative, while the potential energy due to Q is positive). Now we have

$$\begin{aligned}\tfrac{\partial\Pi}{\partial\varphi} &= PR\left(\cos\varphi + \tfrac{Q}{P}\sin\tfrac{\varphi}{2}\right)\\ &= PR\left(\cos^2\tfrac{\varphi}{2} - \sin^2\tfrac{\varphi}{2} + \tfrac{Q}{P}\sin\tfrac{\varphi}{2}\right)\\ &= PR\left(1 - 2\sin^2\tfrac{\varphi}{2} + \tfrac{Q}{P}\sin\tfrac{\varphi}{2}\right).\end{aligned}$$

By considering $\dfrac{\partial\Pi}{\partial\varphi} = 0$, we can find an expression for $\sin\dfrac{\varphi}{2}$ (note that according to (3.4), we have $\sin\dfrac{\varphi}{2} > 0$):

$$\sin^2\frac{\varphi_0}{2} - \frac{Q}{2P}\sin\frac{\varphi_0}{2} - \frac{1}{2} = 0.$$

Hence,

$$\sin\frac{\varphi_0}{2} = \frac{Q}{4P} + \sqrt{\frac{Q^2}{16P^2} + \frac{1}{2}}$$

or

$$\sin\frac{\varphi_0}{2} = \frac{1}{4}\left(\sqrt{\frac{Q^2}{P^2} + 8} + \frac{Q}{P}\right). \tag{3.5}$$

This expression defines the state of equilibrium for the ring A in the lower half of the ring R. From the first expression for $\dfrac{\partial \Pi}{\partial \varphi}$ we get (note, that $\frac{Q}{P} = -\dfrac{\cos \varphi_0}{\sin \dfrac{\varphi_0}{2}}$)

$$\left.\frac{\partial^2 \Pi}{\partial \varphi^2}\right|_{\varphi=\varphi_0} = PR\left(-\sin\varphi_0 + \frac{1}{2}\frac{Q}{P}\cos\frac{\varphi_0}{2}\right) =$$

$$PR\left(-\sin\varphi_0 - \frac{1}{2}\frac{\cos\varphi_0}{\sin\dfrac{\varphi_0}{2}}\cos\frac{\varphi_0}{2}\right) =$$

$$-PR\frac{\cos\dfrac{\varphi_0}{2}}{\sin\dfrac{\varphi_0}{2}}\left(\left(\sin\frac{\varphi_0}{2}\right)^2 + \frac{1}{2}\right).$$

From (3.4) it follows that $\left.\frac{\partial^2 \Pi}{\partial \varphi^2}\right|_{\varphi=\varphi_0} > 0$ and the equilibrium state in the lower half of the ring A is stable.

Since $\frac{Q}{P} > 0$ then from (3.5) $\frac{\sqrt{2}}{2} < \sin\frac{\varphi_0}{2} < 1$ and the equilibrium position of the ring A in this lower half lies in the interval $\pi < \varphi_0 < \frac{3\pi}{2}$.

3.3. Investigate the stability of the vertical state of the system of pendula depicted in Fig. 3.3 along with all dimensions of the system. The mass of each pendulum and the stiffness of each spring are equal to m and c, respectively. We neglect the mass of the rods and assume that each m is a mass point. In the vertical state of the pendula the springs are not loaded.

Solution:

Let us consider the potential energy of the system for small angular displacements φ_k. The deformations of the first, second and third spring are $3h\varphi_1$, $2h(\varphi_2 - \varphi_1)$, and $h(\varphi_3 - \varphi_2)$, respectively. So that the total energy of all springs becomes

$$\Pi_{spr} = \frac{1}{2}c\,(3h\varphi_1)^2 + \frac{1}{2}c\,[2h\,(\varphi_2 - \varphi_1)]^2 + \frac{1}{2}c\,[h\,(\varphi_3 - \varphi_2)]^2\,.$$

The potential energy due to the weight of a mass of an inverted pendulum of length l, when the bar is displaced an angle φ (see Fig. 3.3)

Figure 3.3: Problem 3.3

is

$$\Pi_p = -pl\,(1 - \cos\varphi) \simeq -pl\frac{\varphi^2}{2}.$$

So, the potential energy Π_{p_k} due to the weight of all masses in the system becomes

$$\Pi_{p_k} = -\frac{1}{2}4hp\varphi_1^2 - \frac{1}{2}3hp\varphi_2^2 - \frac{1}{2}2hp\varphi_3^2.$$

Therefore, the overall total potential energy of the system Π is $\Pi_{spr} + \Pi_{p_k}$, i.e.,

$$\begin{aligned}\Pi = {} & \tfrac{1}{2}9ch^2\varphi_1^2 + \tfrac{1}{2}4ch^2\left(\varphi_2 - \varphi_1\right)^2 + \tfrac{1}{2}ch^2\left(\varphi_3 - \varphi_2\right)^2 - \\ & \tfrac{1}{2}4ph\varphi_1^2 - \tfrac{1}{2}3ph\varphi_2^2 - \tfrac{1}{2}2ph\varphi_3^2.\end{aligned}$$

Rearranging the terms we get

$$\begin{aligned}2\Pi = {} & \left(13ch^2 - 4ph\right)\varphi_1^2 + \left(5ch^2 - 3ph\right)\varphi_2^2 + \left(ch^2 - 2ph\right)\varphi_3^2 - \\ & 8ch^2\varphi_1\varphi_2 - 2ch^2\varphi_2\varphi_3.\end{aligned}$$

The necessary and sufficient condition for the potential energy of the system to have a minimum is that Sylvester's criterion must be satisfied (cf. eq. (2.9) in [11]). The matrix of coefficients on the right-hand side of the last equation reads as

$$\begin{pmatrix} 13ch^2 - 4ph & -4ch^2 & 0 \\ -4ch^2 & 5ch^2 - 3ph & -ch^2 \\ 0 & -ch^2 & ch^2 - 2ph \end{pmatrix}.$$

So we have

$$\Delta_1 = h\left(13ch - 4p\right),$$

$$\Delta_2 = h^2 \begin{vmatrix} 13ch - 4p & -4ch \\ -4ch & 5ch - 3p \end{vmatrix} = h^2\left(49c^2h^2 - 59pch + 12p^2\right),$$

$$\Delta_3 = h^3 \begin{vmatrix} 13ch - 4p & -4ch & 0 \\ -4ch & 5ch - 3p & -ch \\ 0 & -ch & ch - 2p \end{vmatrix}$$

$$= h^3\left(36c^3h^3 - 153pc^2h^2 + 130p^2ch - 24p^3\right).$$

From these we get the required conditions for stability as

$$13ch - 4p > 0,$$
$$49c^2h^2 - 59pch + 12p^2 > 0,$$
$$36c^3h^3 - 153pc^2h^2 + 130p^2ch - 24p^3 > 0.$$

3.4. Current i_1 flows along a rectilinear vertical and fixed conductor that attracts a parallel conductor AB (see Fig. 3.4). Current i_2 flows along conductor AB, and l is the length of each conductor. A spring with stiffness c is suspended from conductor AB. If current

Figure 3.4: Problem 3.4

doesn't flow along conductor AB, then the distance between the two conductors is a. Find the equilibrium positions of the system and investigate their stability.

Hint. The interaction force between the two parallel conductors is $F = \dfrac{2i_1 i_2}{d} l$. Here i_1 and i_2 are current flows in the two conductors, d is the distance between the two conductors, and l is the length of each conductor.

S o l u t i o n:

The force acting on the conductor AB is

$$F = \frac{2i_1 i_2 l}{d} - cx = \frac{2i_1 i_2 l}{a-x} - cx.$$

Noting that $a - x > 0$, from here we obtain

$$(a-x)\,F = 2i_1 i_2 l - acx + cx^2$$

or

$$\frac{a-x}{c} F = x^2 - ax + \alpha, \tag{3.6}$$

where

$$\alpha = \frac{2i_1 i_2 l}{c}.$$

The equilibrium of the conductor AB corresponds to $F = 0$. Setting the right-hand side of (3.6) equal to zero, the roots of the resulting equation will give the equilibrium positions of the conductor AB as:

$$x_2 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \alpha}, \quad x_1 = \frac{a}{2} - \sqrt{\frac{a^2}{4} - \alpha}.$$

For positions of equilibrium these roots should be real, and therefore, we must have $\frac{a^2}{4} > \alpha$.

The potential energy Π of the force $F_1 = \frac{a-x}{c} F$ is $\Pi = \int F_1 dx$ or using (3.6),

$$\Pi = \frac{1}{3} x^3 - \frac{1}{2} a x^2 + \alpha x \tag{3.7}$$

A plot of (3.7) is shown in Fig. 3.5.

From this plot it can be observed that the potential energy Π has a minimum at x_2 and a maximum at x_1. Hence, x_2 corresponds to a stable equilibrium, while x_1 represents an unstable state. (The tangents

Figure 3.5: Problem 3.4

at x_2 and x_1 should be parallel to x-axis.) The same conclusions can be arrived at by using a more analytical approach. From (3.7) we have

$$\left.\frac{\partial^2 \Pi}{\partial x^2}\right|_{x=x_2} = (2x-a)_{x=x_2} = a + 2\sqrt{\frac{a^2}{4} - \alpha} - a = 2\sqrt{\frac{a^2}{4} - \alpha} > 0,$$

i.e., at $x = x_2$ the potential energy has a minimum, and this point corresponds to a stable equilibrium state of the conductor AB. Similarly, at $x = x_1$ we have $\dfrac{\partial^2 \Pi}{\partial x^2} < 0$, and therefore, this point corresponds to an unstable equilibrium state. (Note: the answer provided in the book [11] is switched around.)

When $\dfrac{a^2}{4} = \alpha$ there is only one state of equilibrium $x = \dfrac{a}{2}$. This state is unstable, because at this point $d^2\Pi/dx^2 = 0$ and $d^3\Pi/dx^3 \neq 0$, which indicates that Π is not a minimum at the point.

3.5. A solid oscillates freely about the horizontal axis NT (see Fig. 3.6). The axis NT can rotate around the vertical axis Oz with a constant angular velocity ω. Point G is the centre of mass, plane NTG is a plane of symmetry, and axis OG is a principal axis of inertia. KL is parallel to NT, and FD, which passes through point O, is perpendicular to NT and OG. The moments of inertia of the solid about OG, KL, and FD are equal to C, A, and B respectively; h is the length of OG and M is the mass of the solid. Define the possible positions of relative equilibrium of the solid and investigate their stability.

Figure 3.6: Problem 3.5

S o l u t i o n:

The solid can rotate about the axis NT (see Fig. 3.6). Assume that it has an angular velocity $\dot{\varphi}$ so that at any given instant the axis OG makes an angle φ with the vertical z-axis.

The angular velocity $\dot{\varphi}$ is represented by a vector along the axis NT. Moreover, after OG and OD have rotated an angle φ, the components of the angular velocity $\boldsymbol{\omega}$ will be

$$\omega_{OG} = -\omega \cos\varphi, \quad \omega_{OD} = \omega \sin\varphi.$$

The mass moment of inertia of the body about the NT-axis is $A+Mh^2$. Therefore, the kinetic energy of the body is

$$T = \frac{1}{2}\left(A + Mh^2\right)\dot{\varphi}^2 + \frac{1}{2}B\omega^2 \sin^2\varphi + \frac{1}{2}C\omega^2 \cos^2\varphi.$$

The potential energy due to the weight is $\Pi = Mgh\,(1-\cos\varphi)$.

Moreover, from the expression for kinetic energy we have

$$T_2 = \frac{1}{2}\left(A + Mh^2\right)\dot{\varphi}, \quad T_1 = 0, \quad T_0 = \frac{1}{2}\omega^2\left(B\sin^2\varphi + C\cos^2\varphi\right).$$

(It is easy to show that in this example $T_k = R_k$. Cf. relations (3.12) and (3.14) in [11].) The potential energy W of the generalised system becomes (cf. equation (3.20) in [11])

$$W = \Pi - T_0,$$

or

$$W = Mgh\left(1-\cos\varphi\right) - \frac{1}{2}\omega^2\left(B\sin^2\varphi + C\cos^2\varphi\right).$$

Then, for a constant ω we get:

$$\frac{\partial W}{\partial \varphi} = Mgh\sin\varphi - \omega^2\sin\varphi\cos\varphi\left(B-C\right), \tag{3.8}$$

$$\frac{\partial^2 W}{\partial \varphi^2} = Mgh\cos\varphi - \omega^2\left(B-C\right)\left(\cos^2\varphi - \sin^2\varphi\right).$$

The required condition for a stationary motion is to have $\dfrac{\partial W}{\partial \varphi} = 0$. From equation (3.8) three states of equilibrium are deduced: $\varphi = 0$, $\varphi = \pi$, and $\varphi = \arccos\dfrac{Mgh}{\omega^2\left(B-C\right)}$. Let us consider each state separately.

1. $\varphi = 0$. In this case

$$\left.\frac{\partial^2 W}{\partial \varphi^2}\right|_{\varphi=0} = Mgh - \omega^2\left(B-C\right). \tag{3.9}$$

Obviously, for $B < C$ the second variation of W with respect to φ is positive for all ω. Therefore, the state of equilibrium corresponding to $\varphi = 0$ is stable for all ω. From equation (3.9) we can see that for $B > C$ we have $\dfrac{\partial^2 W}{\partial \varphi^2} > 0$ provided $\omega^2 < \dfrac{Mgh}{B-C}$. In this case the state of equilibrium is stable. For $\omega^2 > \dfrac{Mgh}{B-C}$ the state of equilibrium becomes unstable.

2. $\varphi = \pi$. In this case

$$\left.\frac{\partial^2 W}{\partial \varphi^2}\right|_{\varphi=\pi} = -Mgh - \omega^2\left(B-C\right).$$

Therefore, for $B > C$ we have $\dfrac{\partial^2 W}{\partial \varphi^2} < 0$ and the state of equilibrium is unstable.

If $B < C$ the state of equilibrium is stable for $\omega^2 > \dfrac{Mgh}{C-B}$, and it is unstable for $\omega^2 < \dfrac{Mgh}{C-B}$ (in the first case $\dfrac{\partial^2 W}{\partial \varphi^2} > 0$, and in the second case $\dfrac{\partial^2 W}{\partial \varphi^2} < 0$).

3. $\varphi = \arccos \dfrac{Mgh}{(B-C)\,\omega^2}$. In this case a state of equilibrium exists if $Mgh < |B-C|\omega^2$. Then $\sin\varphi \neq 0$, and we have

$$\begin{aligned} \cos\varphi &= \tfrac{Mgh}{\omega^2(B-C)}, \\ \cos^2\varphi &= \tfrac{M^2g^2h^2}{\omega^4(B-C)^2}, \\ \sin^2\varphi &= 1-\cos^2\varphi = 1-\tfrac{M^2g^2h^2}{\omega^4(B-C)^2}. \end{aligned}$$

so that

$$\begin{aligned} \left.\frac{\partial^2 W}{\partial \varphi^2}\right|_{\varphi=\varphi_3} &= \frac{M^2g^2h^2}{\omega^2\,(B-C)} - \omega^2\,(B-C)\left[2\frac{M^2g^2h^2}{\omega^4\,(B-C)^2}-1\right] \\ &= -\frac{M^2g^2h^2}{\omega^2\,(B-C)} + \omega^2\,(B-C)\,. \end{aligned} \tag{3.10}$$

Provided that $Mgh < \omega^2|B-C|$ we will have $M^2g^2h^2 < \omega^4\,(B-C)^2$ or $\dfrac{M^2g^2h^2}{\omega^2\,(B-C)} < \omega^2\,(B-C)$.

Then, from (3.10) we get

$$\left.\frac{\partial^2 W}{\partial \varphi^2}\right|_{\varphi=\varphi_3} > 0,$$

for $B > C$, so that the state of equilibrium is stable.

However, for $B < C$ we have

$$\left.\frac{\partial^2 W}{\partial \varphi^2}\right|_{\varphi=\varphi_3} < 0,$$

and an unstable state of equilibrium.

3.6. In Fig. 3.7 the vertical axis AB is an axis of symmetry of the thin homogeneous round disk with weight P and radius r. AB can

Figure 3.7: Problem 3.6

roll freely around the spherical bearing A. Two mutually perpendicular springs BQ and BD in a horizontal plane hold the axis at point B. Both springs have the same stiffness, i.e., $c_1 = c_2 = c$. They are attached to the axis of the disk at a distance L from the bearing A. The disk is at a distance l from the bearing A. Determine the angular velocity of the disk ω for which the system is stable.

S o l u t i o n:

Let us consider the fixed coordinate system $Axyz$ where the z-axis is pointed upward, and the x and y axes are parallel to springs BQ and BD when the shaft AB is in a vertical position. Moreover, the centroidal coordinate system of the disk is called $(x'y'z')$ which is taken to be parallel to the (xyz)-system, when the axis of the disk is vertical. The mass moment of inertia of the disk with respect to x' and y' is the same and it is $A = \dfrac{mr^2}{4}$. With respect to z'-axis, the mass moment of inertia is $C = \dfrac{mr^2}{2}$, where $m = \dfrac{P}{g}$ is the mass of the disk, and r is its radius.

The mass centre of the disk lies at a distance l from the bearing

A. Now, let us determine the kinetic energy of the disk. It consists of the kinetic energy due to the displacement of the centre of mass of the disk plus the kinetic energy due to the rotation of the disk. For a displaced position of the shaft which is identified by the angles α and β, the coordinates of the mass centre of the disk is denoted by x and y (see Fig. 3.7).

Then, the kinetic energy of the disk is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}A\left(\dot{\alpha}^2\cos^2\beta + \dot{\beta}^2\right) + \frac{1}{2}C\left(\omega + \dot{\alpha}\sin\beta\right)^2.$$

From Fig. 3.7 we can see that

$$\dot{x} = l\dot{\alpha}, \quad \dot{y} = l\dot{\beta},$$

and if we assume that the angles α and β are small ($\cos\alpha = 1$, $\cos\beta = 1$, $\sin\beta = \beta$), we get

$$T = \frac{1}{2}ml^2\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \frac{1}{2}A\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \frac{1}{2}C\left(\omega + \dot{\alpha}\beta\right)^2.$$

Using the expressions for the moments of inertia A and C, we have

$$T = \frac{1}{2}m\left(l^2 + \frac{r^2}{4}\right)\left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \frac{mr^2}{4}\left(\omega + \dot{\alpha}\beta\right)^2. \tag{3.11}$$

Neglecting the terms with order higher than two, the spring deformations are obtained to be $L\alpha$ and $L\beta$, and the potential energy due to the weight P becomes

$$-Pl\left(1 - \cos\alpha + 1 - \cos\beta\right) = -\frac{1}{2}Pl\left(\alpha^2 + \beta^2\right).$$

Then the total potential energy will be

$$\Pi = \frac{1}{2}\left(cL^2 - Pl\right)\left(\alpha^2 + \beta^2\right). \tag{3.12}$$

Using equations (3.11) and (3.12) we get the Lagrange equation with respect to coordinate α as

$$\frac{\partial T}{\partial \dot{\alpha}} = m\left(l^2 + \frac{r^2}{4}\right)\dot{\alpha} + \frac{mr^2}{2}\left(\omega + \dot{\alpha}\beta\right)\beta.$$

Considering that the term $\dot{\alpha}\beta$ is negligible with respect to ω, we may differentiate this expression with respect to time t, and obtain the derivatives of Π with respect to α. In this way we can obtain the equation of motion for α (a similar procedure will result in the equation of motion for β):

$$\begin{aligned} m\left(l^2+\frac{r^2}{4}\right)\ddot{\alpha}+\frac{mr^2}{2}\omega\dot{\beta}+\left(cL^2-Pl\right)\alpha=0,\\ m\left(l^2+\frac{r^2}{4}\right)\ddot{\beta}-\frac{mr^2}{2}\omega\dot{\alpha}+\left(cL^2-Pl\right)\beta=0. \end{aligned} \tag{3.13}$$

Now, assume $\alpha = De^{\lambda t}$ and $\beta = Ee^{\lambda t}$. Substitute these in equation (3.13) and divide by $e^{\lambda t}$ to get

$$\begin{aligned} \left[m\left(l^2+\tfrac{r^2}{4}\right)\lambda^2+\left(cL^2-Pl\right)\right]D+\tfrac{mr^2}{2}\omega\lambda E=0,\\ -\tfrac{mr^2}{2}\omega\lambda D+\left[m\left(l^2+\tfrac{r^2}{4}\right)\lambda^2+\left(cL^2-Pl\right)\right]E=0. \end{aligned}$$

This is a system of homogeneous linear equations with respect to D and E the determinant of which must vanish, i.e.,

$$\begin{vmatrix} m\left(l^2+\frac{r^2}{4}\right)\lambda^2+\left(cL^2-Pl\right) & \frac{mr^2}{2}\omega\lambda \\ -\frac{mr^2}{2}\omega\lambda & m\left(l^2+\frac{r^2}{4}\right)\lambda^2+\left(cL^2-Pl\right) \end{vmatrix}=0.$$

Expanding this determinant we get

$$\begin{aligned} & m^2\left(l^2+\tfrac{r^2}{4}\right)^2\lambda^4+\left[2m\left(l^2+\tfrac{r^2}{4}\right)\left(cL^2-Pl\right)+\tfrac{m^2r^4}{4}\omega^2\right]\lambda^2+ \\ & +\left(cL-Pl\right)^2=0. \end{aligned}$$

This equation can be solved for λ^2:

$$\begin{aligned} \lambda^2=\frac{-1}{2m^2\left(l^2+\frac{r^2}{4}\right)}\Bigg\{ & \left[2m\left(l^2+\frac{r^2}{4}\right)\left(cL^2-Pl\right)+\frac{m^2r^4}{4}\omega^2\right]\pm \\ & \frac{m^2r^4}{4}\omega^2\sqrt{4m\left(l^2+\frac{r^2}{4}\right)\left(cL^2-Pl\right)+\frac{m^2r^2}{4}\omega^2}\Bigg\}. \end{aligned}$$

If $Pl < cL^2$, then all terms are positive and both roots λ_1^2 and λ_2^2 are negative and simple for any angular velocity ω. This means that for $Pl < cL^2$, the vertical state of the shaft is stable in the first approximation for any ω.

Next assume $Pl > cL^2$. Then, the first term under the square root sign becomes negative and for stability to prevail it becomes necessary for the angular velocity ω to satisfy the condition that

$$\omega^2 > \frac{4}{r^4} gl \left(4l^2 + r^2\right) \left(1 - \frac{cL^2}{Pl}\right)$$

or

$$\omega > \frac{2}{r^2} \sqrt{gl \left(4l^2 + r^2\right) \left(1 - \frac{cL^2}{Pl}\right)}.$$

If ω satisfies this condition, then for $cL^2 < Pl$ there is a positive value under the square root sign and both roots λ_1^2 and λ_2^2 are simple and negative. Then, the stability of the disk in the first approximation is proven.

3.7. The mass point depicted in Fig. 3.8 moves over the smooth surface of a torus given by the parametric equations

$$x = \rho \cos \psi, \quad y = \rho \sin \psi, \quad z = b \sin \vartheta,$$

$$\rho = a + b \cos \vartheta,$$

where the z-axis is pointing upward. Find the possible motions of the

Figure 3.8: Problem 3.7.

point when the angle ϑ is a constant, and analyse the stability of these motions.

S o l u t i o n:

Let $\dot{\psi} = \omega = \text{const}$. Therefore $\psi = \omega t$. Then, we have

$$x = (a + b\cos\vartheta)\cos\omega t, y = (a + b\cos\vartheta)\sin\omega t, z = b\sin\vartheta,$$

from which

$$\begin{aligned} \dot{x} &= -b\sin\vartheta\dot{\vartheta}\cos\omega t - \omega(a + b\cos\vartheta)\sin\omega t, \\ \dot{y} &= -b\sin\vartheta\dot{\vartheta}\sin\omega t + \omega(a + b\cos\vartheta)\cos\omega t, \\ \dot{z} &= b\cos\vartheta\dot{\vartheta}. \end{aligned}$$

Now, the kinetic energy of the mass is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[b^2\dot{\vartheta}^2 + \omega^2\left(a + b\cos\vartheta\right)^2\right],$$

and its potential energy is $\Pi = mgz = mbg\sin\vartheta$. From the expression for T we find that $T_2 = \frac{1}{2}mb^2\dot{\vartheta}^2$, and $T_0 = \frac{1}{2}m\omega^2(a + b\cos\vartheta)^2$. Therefore, the potential energy of the generalised system becomes

$$W = \Pi - T_0 = mbg\sin\vartheta - \frac{1}{2}m\omega^2(a + b\cos\vartheta)^2,$$

such that

$$\frac{\partial W}{\partial\vartheta} = mb\left[g\cos\vartheta + \omega^2(a + b\cos\vartheta)\sin\vartheta\right], \tag{3.14}$$

$$\frac{\partial^2 W}{\partial\vartheta^2} = mb\left[-g\sin\vartheta - \omega^2 b\sin^2\vartheta + \omega^2(a + b\cos\vartheta)\cos\vartheta\right] \tag{3.15}$$

We find the relative equilibrium from $\frac{\partial W}{\partial\vartheta} = 0$. Using (3.14), we get

$$g\cos\vartheta + \omega^2(a + b\cos\vartheta)\sin\vartheta = 0.$$

From here

$$a + b\cos\vartheta = -\frac{g}{\omega^2}\cot\vartheta \tag{3.16}$$

or

$$1 + \alpha\cos\vartheta = -\beta\cot\vartheta, \quad \alpha = \frac{b}{a}, \quad \beta = \frac{g}{a\omega^2} \tag{3.17}$$

The two solutions of equation (3.17) can be easily obtained as

$$-\frac{\pi}{2} < \vartheta_1 < 0, \quad \frac{\pi}{2} < \vartheta_2 < \pi.$$

In order to determine which of these is stable, let us consider the right-hand side of (3.15).

First, we proceed as follows:

$$\frac{\partial^2 W}{\partial \vartheta^2} = mb\left[-g\sin\vartheta - \omega^2 b\sin^2\vartheta + \omega^2 a + \omega^2 b\cos^2\vartheta\right]$$

or

$$\tfrac{\partial^2 W}{\partial \vartheta^2} = mb\left[-g\sin\vartheta + a\omega^2 + b\omega^2(\cos^2\vartheta - \sin^2\vartheta)\right] = mb\left[-g\sin\vartheta + a\omega^2(1 + \tfrac{b}{a}\cos 2\vartheta)\right].$$

In the interval $-\frac{\pi}{2} < \vartheta < 0$ we know that $\sin\vartheta < 0$, therefore the first term is positive. Moreover, we have $\frac{b}{a} < 1$, and $|\cos 2\vartheta| < 1$. Thus, the second term is also positive, and hence the motion is stable when $\vartheta = \vartheta_1$.

Next, we note that

$$\frac{\partial^2 W}{\partial \vartheta^2} = mb\left[-g\sin\vartheta - \omega^2 b\sin^2\vartheta + \omega^2(a + b\cos\vartheta)\cos\vartheta\right],$$

so that using equation (3.16), we get

$$\frac{\partial^2 W}{\partial \vartheta^2} = mb\left[-g\sin\vartheta - \omega^2 b\sin^2\vartheta - g\frac{\cos^2\vartheta}{\sin\vartheta}\right].$$

Now, we have $\sin\vartheta > 0$ in the interval $\frac{\pi}{2} < \vartheta_2 < \pi$. Therefore, for $\vartheta = \vartheta_2$

$$\frac{\partial^2 W}{\partial \vartheta^2} < 0,$$

which means that the motion is unstable when $\vartheta = \vartheta_2$.

3.8. The horizontal tube AB shown in Fig. 3.9 can rotate freely about the vertical axis CD. Inside the tube there is a spring with stiffness c. The end of the spring is fixed to the tube wall at A. The solid M is attached to the free end of the spring. The mass of M is m. When the system is at rest, the body M is at distance a from the axis of rotation ($a > 0$ or $a < 0$). During the free rotation of the

Figure 3.9: Problem 3.8.

tube with an angular velocity ω, the system attains a stationary motion in which body M is at relative rest. Assume M is a mass point and neglect any frictional forces and the mass of the spring. If the mass moment of inertia of the tube with respect to the axis of rotation CD is J, determine the parameters of stationary motion and analyse its stability.

S o l u t i o n:

The potential energy of the system is $\Pi = \frac{1}{2}cx^2$ (for $a > 0$ and for $a < 0$), while its kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2(a+x) + \frac{1}{2}J\omega^2.$$

Hence, $T_2 = \frac{1}{2}m\dot{x}^2$, and $T_0 = \frac{1}{2}m\omega^2(a+x) + \frac{1}{2}J\omega^2$. Thus, the potential energy of the generalised system is

$$\begin{aligned} W &= \Pi - T_0 = \tfrac{1}{2}cx^2 - \tfrac{1}{2}m\omega^2(a+x)^2 - \tfrac{1}{2}J\omega^2, \\ \tfrac{\partial W}{\partial x} &= cx - m\omega^2(a+x), \\ \tfrac{\partial^2 W}{\partial x^2} &= c - m\omega^2. \end{aligned}$$

The state of relative equilibrium is obtained from the equation

$$\frac{\partial W}{\partial x} = cx - m\omega^2(a+x) = 0,$$

from which

$$x_0 = \tfrac{ma\omega^2}{c-m\omega^2},$$

$$\tfrac{\partial^2 W}{\partial x^2} = c - m\omega^2.$$

If $c > m\omega^2$, then $\frac{\partial^2 W}{\partial x^2} > 0$ and the relative equilibrium is stable, if $c < m\omega^2$, then $\frac{\partial^2 W}{\partial x^2} < 0$ and the relative equilibrium is unstable.

(In the book [11], in the following equation

$$\Delta = J\left(1 - \frac{m\omega^2}{c}\right) + m\left(a + x_0\right)^2\left(1 + 4\frac{m\omega^2}{c}\right).$$

should be replaced by $\frac{\partial^2 W}{\partial x^2} = c - m\omega^2$.)

3.9. The rotor depicted in Fig. 3.10 is situated in a horizontal plane and is rigidly mounted at its centre O on a flexible shaft which is supported as shown. The centre of mass of the rotor is C, the mass of the rotor is m, $e = OC$ is the eccentricity of the rotor which has a mass moment of inertia equal to J with respect to the vertical axis. The bending stiffness of the shaft is C and the shaft is driven at a

Figure 3.10: Problem 3.9.

constant angular velocity ω. The shaft axis is bent due to centrifugal forces. One can neglect the mass of the shaft and any frictional forces. In the fixed coordinate system determine the position of point O for the stationary motion and analyse its stability.

S o l u t i o n:

The plane of rotor is horizontal. Point O represents the deflected position of the flexible shaft which has a bending stiffness of c. The mass centre of the rotor, point C, has an eccentricity e. Points O and C are attached to the rotor. Let the x-axis of the coordinate system O_1xy be attached to the plane of the rotor and be parallel to OC. It is required to determine the position of relative equilibrium of point O and analyse its stability. Next, let x and y be the coordinates of point O, then $x+e$ and y are the coordinates of the centre of mass of the rotor; and $\dot{x}$ and $\dot{y}$ are the components of the velocity of point C relative to O_1. The constant velocity vector $\boldsymbol{\omega}$ is normal to O_1xy-plane. The relative velocity of C about point O is

$$\boldsymbol{v}_c^e = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ x+e & y & 0 \end{vmatrix},$$

$$v_{cx}^e = -\omega y, \quad v_{cy}^e = \omega(x+e).$$

The components and the magnitude of the absolute velocity of point C are

$$v_x = \dot{x} - \omega y, \quad v_y = \dot{y} + \omega(x+e),$$
$$v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{y}^2 - 2\omega(\dot{x}y - x\dot{y}) + \omega^2\left[(x+e)^2 + y^2\right].$$

The kinetic energy of the rotor

$$T = \frac{1}{2}J\omega^2 + \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2 - 2\omega(\dot{x}y - x\dot{y}) + \omega^2\left[(x+e)^2 + y^2\right]\},$$

where J is the moment of inertia of the rotor with respect to the axis perpendicular to its plane at point O, and m is the mass of the rotor. The potential energy of the elastic shaft is

$$\Pi = \frac{1}{2}c(x^2 + y^2).$$

Thus, the Lagrange equations become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial \Pi}{\partial x},$$

where

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} - m\omega y; \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = m\ddot{x} - m\omega\dot{y},$$
$$\frac{\partial T}{\partial x} = m\omega\dot{y} + m\omega^2(x+e); \quad \frac{\partial \Pi}{\partial x} = cx.$$

Therefore, the equations of motion become

$$\begin{aligned} m\ddot{x} - 2m\omega\dot{y} - m\omega^2(x+e) &= -cx, \\ m\ddot{y} + 2m\omega\dot{x} - m\omega^2 y &= -cy, \end{aligned} \tag{3.18}$$

where the second equation is obtained in a manner analogous to the first one.

For a state of relative equilibrium we should have $\ddot{x} = \ddot{y} = 0$, $\dot{x} = \dot{y} = 0$. By substituting these into (3.18), the coordinates x_0 and y_0 of point O in relative equilibrium are obtained as

$$-m\omega^2(x_0+e) = -cx_0, \quad -m\omega^2 y_0 = -cy_0.$$

Hence,

$$x_0 = \frac{m\omega^2 e}{c - m\omega^2}, \quad y_0 = 0 \tag{3.19}$$

These equations have simple physical interpretation and can be obtained in a more direct way. In the state of relative equilibrium points O_1, O and C should lie on the same line ($y_0 = 0$), and the elastic force exerted by the flexible shaft, cx_0, should be equal to the centrifugal force $m\omega^2(x_0+e)$. This corresponds to the first equation in (3.19). (There is an error in the book [11] in the expression for x_0.)

To analyse for stability we consider

$$x = x_0 + \varepsilon_1, \quad y = y_0 + \varepsilon_2 = \varepsilon_2.$$

In equations (3.18) we substitute the above expressions for x and y, respectively, to get

$$\begin{gathered} m\ddot{\varepsilon}_1 - 2m\omega\dot{\varepsilon}_2 - m\omega^2(x_0+\varepsilon_1+e) = -c(x_0+\varepsilon_1), \\ m\ddot{\varepsilon}_2 + 2m\omega\dot{\varepsilon}_1 - m\omega^2\varepsilon_1 = -c\varepsilon_1 \end{gathered}$$

such that in view of (3.19), we get

$$\begin{aligned} m\ddot{\varepsilon}_1 - 2m\omega\dot{\varepsilon}_2 + (c - m\omega^2)\varepsilon_1 &= 0, \\ m\ddot{\varepsilon}_2 + 2m\omega\dot{\varepsilon}_1 + (c - m\omega^2)\varepsilon_2 &= 0. \end{aligned} \tag{3.20}$$

As a standard approach, we let $\varepsilon_1 = Ae^{\lambda t}$, $\varepsilon_2 = Be^{\lambda t}$, and substitute these expressions for ε_1 and ε_2 into equations (3.20). In the resulting equation, after rearranging the terms and dividing by $e^{\lambda t}$ we get

$$\begin{gathered} \left[m\lambda^2 + (c - m\omega^2)\right] A - 2m\omega\lambda B = 0, \\ 2m\omega\lambda A + \left[m\lambda^2 + (c - m\omega^2)\right] B = 0. \end{gathered}$$

This is a linear system of homogeneous algebraic equations in A and B whose determinant should vanish, i.e.,

$$\begin{vmatrix} m\lambda^2 + (c - m\omega^2) & -2m\omega\lambda \\ 2m\omega\lambda & m\lambda^2 + (c - m\omega^2) \end{vmatrix} = 0.$$

Expanding this determinant, we have

$$m^2\lambda^4 + 2m(c + m\omega^2)\lambda^2 + (c - m\omega^2)^2 = 0.$$

From where,

$$\lambda^2 = \frac{1}{m^2}\left[-m(c + m\omega^2) \pm \sqrt{m^2(c + m\omega^2)^2 - m^2(c - m\omega^2)^2}\right]$$

or

$$\lambda^2 = -\frac{1}{m}\left(\sqrt{c} \pm \omega\sqrt{m}\right)^2. \tag{3.21}$$

From (3.21) it follows that all four roots of the characteristic equations are simple and pure imaginary. This means that the system is stable for all c and ω and $c \neq m\omega^2$ (there is an error in the answer given in the book [11]). Moreover, for $c = m\omega^2$ also the system is stable, but this can not be concluded from (3.21). For this conclusion one has to consider the matrix

$$A - \lambda E = \begin{pmatrix} m\lambda^2 & -2m\omega\lambda \\ 2m\omega\lambda & m\lambda^2 \end{pmatrix}. \tag{3.22}$$

This matrix is the same as the one in (3.20) for $c = m\omega^2$. Simple reductions of (3.22) will lead to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda(\lambda^2 + 4\omega^2) \end{pmatrix}.$$

From this matrix we can see that the canonical variables should satisfy the following:

$$\dot{z}_1 = 0, \quad \dot{z}_2 = 0, \quad \dot{z}_3 = 2\omega i, \quad \dot{z}_4 = -2\omega i; \quad i = \sqrt{-1},$$

which means that solutions $\varepsilon_1 = \varepsilon_2 = 0$ of equations (3.20) are also stable for $c = m\omega^2$.

We have considered the case when points O_1, O and C were not collinear. Let us now consider the case when these points are collinear — call this line the x-axis. Then the coordinates of points O and C are x and $x+e$, respectively.

The velocity of the centre of mass, point C, is defined as

$$v^2 = \dot{x}^2 + \omega^2(x+e)^2.$$

The kinetic and potential energy of the system are

$$T = \frac{1}{2}J\omega^2 + \frac{1}{2}m\left[\dot{x}^2 + \omega^2(x+e)^2\right], \quad \Pi = \frac{1}{2}cx^2,$$

and the equation of motion becomes

$$m\ddot{x} - m\omega^2(x+e) = -cx. \tag{3.23}$$

The state of relative equilibrium at which $\ddot{x} = 0$ is defined by (3.19):

$$x_0 = \frac{m\omega^2 e}{c - m\omega^2}.$$

We can obtain the equation of the perturbed motion if we let $x = x_0 + \varepsilon$ in (3.23). This will result in

$$m\ddot{\varepsilon} + (c - m\omega^2)\varepsilon = 0.$$

From here, if $c > m\omega^2$, then the unperturbed motion is stable; whereas if $c < m\omega^2$, then it is unstable.

So, the answer is the following: for the stationary motion the centre O has the coordinates

$$\rho_o = \frac{m\omega^2 e}{c - m\omega^2},$$

$$\varphi_o = \omega t.$$

If points O_1, O and C are not colinear, then the relative state equilibrium of point O is stable for all c and ω. If these points lie on the same line, then for $c > m\omega^2$ the relative equilibrium position of point O is stable, and for $c < m\omega^2$ it is unstable.

3.10. For the system given in Problem 1.7 prove that the stationary motion is stable with respect to β, $\dot{\beta}$, and $\dot{\alpha}$.

Hint. For the system under consideration the potential energy of the generalised system, $W = \Pi - R_0$, is

$$W = \frac{1}{2}c\beta^2 + \frac{(n - H\sin\beta)^2}{4A\cos^2\beta},$$

where $n = 2A\dot{\alpha}\cos^2\beta + H\sin\beta$ is the integral corresponding to the cyclic coordinate.

S o l u t i o n:

From the solution of Problem 1.7 we obtain the kinetic and potential energy of the system as

$$\begin{aligned} T &= A\dot{\beta}^2 + A\dot{\alpha}\cos^2\beta + C(\dot{\varphi} + \dot{\alpha}\sin\beta)^2, \\ \Pi &= \frac{1}{2}c\beta^2, \end{aligned} \tag{3.24}$$

where α and φ are cyclic coordinates, because T and Π contain only the velocities $\dot{\alpha}$ and $\dot{\varphi}$. Two cyclic integrals which correspond to these coordinates are $\dfrac{\partial T}{\partial \dot{\alpha}} = n =$ const, and $\dfrac{\partial T}{\partial \dot{\varphi}} = H =$ const. (Here it should be noted that in problems dealing with Routh transform it is not possible for the derivatives $\dfrac{\partial T}{\partial \dot{\alpha}}$ and $\dfrac{\partial T}{\partial \dot{\varphi}}$ to be constant with multipliers. For example, we can not say $\frac{\partial T}{\partial \dot{\varphi}} = 2H$, as we did in Problem 1.7.) Next, from (3.24) we obtain

$$\begin{aligned} \tfrac{\partial T}{\partial \dot{\alpha}} &= 2A\dot{\alpha}\cos^2\beta + H\sin\beta = n, \\ \tfrac{\partial T}{\partial \dot{\varphi}} &= 2C(\dot{\varphi} + \dot{\alpha}\sin\beta) = H. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{\alpha} &= \frac{n - H\sin\beta}{2A\cos^2\beta}, \\ \dot{\varphi} &= \frac{H}{2C} - \dot{\alpha}\sin\beta. \end{aligned} \tag{3.25}$$

Substitute these into the expression for T to get T^* (we neglect the constant quantity $\dfrac{H^2}{4C}$):

$$T^* = A\beta^{-2} + \frac{(n - H\sin\beta)^2}{4A\cos^2\beta}. \tag{3.26}$$

Compose the Routh function R (cf. (3.12) in [11]):

$$R = T^* - n\dot{\alpha} - H\dot{\varphi}.$$

Using equations (3.26) and (3.25) we find

$$R = A\dot{\beta}^2 + \frac{(n - H\sin\beta)^2}{4A\cos^2\beta} - n\frac{n - H\sin\beta}{2A\cos^2\beta} - H\left(\frac{H}{2C} - \frac{n - H\sin\beta}{2A\cos^2\beta}\sin\beta\right),$$

so that after some simple manipulation and neglecting the constant term $\frac{H^2}{2C}$), we get

$$\begin{aligned} R &= A\dot{\beta}^2 - \frac{(n - H\sin\beta)^2}{4A\cos^2\beta} = A\dot{\beta}^2 - R_0, \\ W &= \Pi - R_0 = \frac{1}{2}c\beta^2 + \frac{(n - H\sin\beta)^2}{4A\cos^2\beta}. \end{aligned} \tag{3.27}$$

(In the book [11] there is small error, the coefficient 4 in the dominator is missing.)

We assume that the angle β is small so that $\sin\beta \simeq \beta$, $\cos\beta \simeq 1 - \frac{\beta^2}{2}$. Then, we have

$$\frac{1}{\cos^2\beta} \simeq \frac{1}{\left(1 - \dfrac{\beta^2}{2}\right)^2} \simeq \frac{1}{1 - \beta^2} = (1 - \beta^2)^{-1} = 1 + \beta^2$$

where the terms with order higher than two have been neglected. Then, (3.27) reads

$$W = \frac{(2AC + H^2)\beta^2 - 2nH\beta + n^2}{4A}(1 + \beta^2)$$

Once again retaining only up to the second order terms, we get

$$W = \frac{1}{4A}\left[(2AC + H^2 + n^2)\beta^2 - 2nH\beta + n^2\right].$$

From here, we have

$$\begin{aligned} \tfrac{\partial W}{\partial \beta} &= \tfrac{1}{2A}\left[(2AC + H^2 + n^2)\beta - nH\right], \\ \tfrac{\partial^2 W}{\partial \beta^2} &= \tfrac{1}{2A}(2AC + H^2 + n^2) > 0. \end{aligned}$$

Setting the first equation equal to zero, we find the expression for the angle of relative equilibrium as

$$\beta_0 = \frac{nH}{2AC + H^2 + n^2}. \tag{3.28}$$

From the second equation we find, that this state is stable with respect to the angle β. From the stability of motion with respect to β and from (3.25) the stability with respect to $\dot{\alpha}$ and $\dot{\varphi}$ follows.

Provided equation (3.28) is satisfied, the motion of the system consists of a constant deviation of the axis of gyroscopes by the angle β_0 and the uniform rotation of the whole system with an angular velocity of

$$\dot{\alpha}_0 = \frac{n - H\sin\beta_0}{2A\cos^2\beta_0}.$$

Chapter 4

Stability in First Approximation

4.1. Let the moments of inertia of a rigid body with respect to its principal axes of inertia x, y, and z, be designated as A, B, and C, respectively, such that either $A < C < B$ or $A > C > B$. Prove that the uniform rotation of the rigid body about the z-axis is unstable.

S o l u t i o n:

From the equations of the perturbed motion obtained in Problem 1.6 obtain the following equations in first approximation:

$$\dot{x}_1 = \tfrac{B-C}{A}\omega_0 x_2,$$
$$\dot{x}_2 = \tfrac{C-A}{B}\omega_0 x_1,$$
$$\dot{x}_3 = 0.$$

Let $x_k = D_k e^{\lambda t}$ $(k = 1, 2, 3)$, substitute for x_k into these equations and divide the resulting equations by $e^{\lambda t}$ to obtain

$$\begin{aligned} D_1 A\lambda + D_2(C - B)\omega_0 &= 0, \\ D_2 B\lambda + D_1(A - C)\omega_0 &= 0, \\ D_3 C\lambda &= 0. \end{aligned}$$

This is a system of linear homogeneous equations with respect to D_k the determinant of which must vanish in order to have any nontrivial solutions, i.e., we must have

$$\begin{vmatrix} A\lambda & (C-B)\omega_0 & 0 \\ (A-C)\omega_0 & B\lambda & 0 \\ 0 & 0 & C\lambda \end{vmatrix} = 0,$$

or

$$C\lambda\left[AB\lambda^2 - (C-B)(A-C)\omega_0^2\right] = 0.$$

This equation has one root equal to zero, and for $A < C < B$ or $A > C > B$ it has two real roots:

$$\lambda = \pm\omega_0\sqrt{\frac{(C-B)(A-C)}{AB}}.$$

The existence of one positive root indicates that the uniform rotation of the rigid body about the middle axis of moment ellipsoid, the z-axis, is unstable.

4.2. Prove that the equilibrium of a point mass located on the end of a compressed and twisted bar is unstable (see Problem 1.5).

Solution:

In Problem 1.5 we obtain the following differential equations of the perturbed motion:

$$m\ddot{x} = -c_1x + c_2y,$$
$$m\ddot{y} = -c_2x - c_1y.$$

Letting $x = Ae^{\lambda t}$ and $y = Be^{\lambda t}$, the following homogeneous linear system of algebraic equations in A and B can be obtained:

$$mA\lambda^2 + c_1A - c_2B = 0,$$
$$c_2A + mB\lambda^2 + c_1B = 0.$$

Setting the determinant of this system equal to zero, we have

$$\begin{vmatrix} m\lambda^2 + c_1 & -c_2 \\ c_2 & m\lambda^2 + c_1 \end{vmatrix} = 0,$$

or

$$(m\lambda^2 + c_1)^2 + c_2^2 = 0.$$

Hence,

$$m\lambda^2 + c_1 = \pm i c_2, \quad i = \sqrt{-1}.$$

At least one of the roots of this equation have positive real parts which means that the equilibrium is unstable for $x_1 = x_2 = 0$.

4.3. The motion of a control system is described by the following differential equations:

$$\begin{aligned} \dot{\psi} - \Omega(\gamma_1 + \gamma_2) &= c_2\gamma_2, \\ \dot{\gamma}_1 + \dot{\gamma}_2 + \Omega\psi &= -c_1\gamma_2, \\ \dot{\gamma}_1 + \Omega\psi &= -k(\gamma_1 - \kappa), \end{aligned}$$

where γ_1, γ_2, and ψ are the system coordinates, c_1, c_2, k, and Ω are system parameters, and $\kappa(t)$ is the driving force. Determine the required condition for system parameters such that the motion caused by the driving force κ would be asymptotically stable.

S o l u t i o n:

Multiply the first equation by -1 and rewrite all the equations in the following form:

$$\begin{aligned} \Omega\gamma_1 + (\Omega + c_2)\gamma_2 - \dot{\psi} &= 0, \\ \dot{\gamma}_1 + \dot{\gamma}_2 + c_1\gamma_2 + \Omega\psi &= 0, \\ \dot{\gamma}_1 + k\gamma_1 + \Omega\psi &= k\kappa. \end{aligned}$$

The stability of this nonhomogeneous linear system can be determined by considering its system of first approximation (cf. Example 1.4 in [11]):

$$\begin{aligned} \Omega\gamma_1 + (\Omega + c_2)\gamma_2 - \dot{\psi} &= 0, \\ \dot{\gamma}_1 + \dot{\gamma}_2 + c_1\gamma_2 + \Omega\psi &= 0, \\ \dot{\gamma}_1 + k\gamma_1 + \Omega\psi &= 0. \end{aligned}$$

Let $\gamma_1 = Ae^{\lambda t}$, $\gamma_2 = Be^{\lambda t}$, and $\psi = Ce^{\lambda t}$. Substitute these into the above and divide the resulting equations by $e^{\lambda t}$ to obtain

$$\begin{aligned} \Omega A + (\Omega + c_2)B - \lambda C &= 0, \\ \lambda A + (\lambda + c_1)B + \Omega C &= 0, \\ (\lambda + k)A + \Omega C &= 0. \end{aligned}$$

This homogeneous linear system of algebraic equations in A, B and C must have nontrivial solutions. So the determinant of the system must vanish, i.e.,

$$\begin{vmatrix} \Omega & \Omega + c_2 & -\lambda \\ \lambda & \lambda + c_1 & \Omega \\ \lambda + k & 0 & \Omega \end{vmatrix} = 0,$$

or after expansion

$$\sum_{i=0}^{3} a_i \lambda^{3-i} = \lambda^3 + (k + c_1)\lambda^2 + (\Omega^2 + kc_1)\lambda + (\Omega^2 c_1 + \Omega^2 k + \Omega c_2 k) = 0, \tag{4.1}$$

where Ω, c_1, c_2, and k which are the parameters of the system, all are positive. Therefore, for asymptotic stability to prevail it is only necessary to satisfy inequality (4.30) in [11], i.e.,

$$\Delta_2 = a_1 a_2 - a_0 a_3 > 0.$$

For the problem at hand the corresponding values of a_k are obtained from equation (4.1) to give

$$\Delta_2 = (k + c_1)(\Omega^2 + kc_1) - (\Omega^2 c_1 + \Omega^2 k + \Omega c_2 k) > 0,$$

which can be reduced to

$$kc_1 + c_1^2 > \Omega c_2.$$

4.4. The top view schematic of a uniaxial trailer is shown in Fig. 4.1. Here m is the mass of the trailer; J is the polar inertia moment of the trailer with respect to the vertical axis which is orthogonal to the plane of motion at the hitch point of the tractor to the trailer; G designates the mass centre of the trailer; v is the velocity of the tractor; and the stiffness of the spring is c. If we neglect the nonholonomic reactive force F at the hitch, then equations of motion of the trailer can be reduced to the following equations in the first approximation

$$\begin{aligned} m(b-a)\ddot{x} + cbx + [ma(b-a) - J]\,\ddot{\varphi} &= 0, \\ \dot{x} + b\dot{\varphi} + v\varphi &= 0, \end{aligned}$$

where the second equation describes the nonholonomic constraint at the hitch. Determine the stability conditions of the trailer.

Figure 4.1:

S o l u t i o n:

Determine the differential equations of the motion of the trailer. Considering the moments with respect to the axis which is orthogonal to the plane of motion at the mass centre G of the trailer, we have

$$J\ddot{\varphi} = -cax\cos\varphi + F(b-a).$$

The equation of the motion of the mass center in the direction parallel to the displacement of the spring is

$$m\frac{d^2}{dt^2}(x + a\sin\varphi) = -cax - F\cos\varphi.$$

For small angles φ these equations become

$$\begin{aligned} J\ddot{\varphi} &= -cax + F(b-a), \\ m\ddot{x} + ma\ddot{\varphi} &= -cax - F. \end{aligned}$$

Eliminating F from the above set, we get

$$m(b-a)\ddot{x} + cbx + [ma(b-a) - J]\ddot{\varphi} = 0. \tag{4.2}$$

Since the trailer can not have a motion along its axle the following condition applies:

$$\dot{x}\cos\varphi + b\dot{\varphi} + v\sin\varphi = 0$$

or for small angles,

$$\dot{x} + b\dot{\varphi} + v\varphi = 0. \tag{4.3}$$

In order to determine the stability conditions of the trailer the characteristic equations of differential equations (4.2) and (4.3) are considered:

$$\begin{vmatrix} m(b-a)\lambda^2 + cb & [ma(b-a) - J]\lambda^2 \\ \lambda & b\lambda + v \end{vmatrix} = 0,$$

which expands to

$$\left[m(b-a)^2 + J\right]\lambda^3 + m(b-a)v\lambda^2 + cb^2\lambda + cbv = 0.$$

For asymptotic stability all coefficients must be positive, i.e., we should have

$$b > a, \quad v > 0. \tag{4.4}$$

Besides this condition (4.30) in [11] must also be satisfied,

$$\begin{aligned} \Delta_2 = a_1a_2 - a_0a_3 = mcb^2v(b-a) - cbv\left[m(b-a)^2 + J\right] = \\ = cbv\left[ma(b-a) - J\right] > 0. \end{aligned}$$

From here we obtain the additional condition

$$J < ma(b-a). \tag{4.5}$$

If (4.4) and (4.5) are satisfied, then the motion of the trailer will be asymptotically stable.

4.5. The follower force P is applied to the double pendulum depicted in Fig. 4.2. Spiral springs each having a stiffness c are used at support point O and in joint O_1. The length and mass of both pendulums (mass points) are the same.

Neglecting the mass of the bars, obtain the equations of motion and determine stability conditions of the motion with respect to φ_1, $\dot{\varphi}_1$, φ_2, and $\dot{\varphi}_2$.

S o l u t i o n:

In this problem the force P is nonconservative, so for the generalised forces we get

$$Q_1 = -\frac{\partial \Pi}{\partial \varphi_1} + Q_{1p}, \quad Q_2 = -\frac{\partial \Pi}{\partial \varphi_2} + Q_{2p}, \tag{4.6}$$

Figure 4.2:

where Π is the potential energy due to the springs and gravity, and Q_{1p} and Q_{2p} are generalised forces due to P. We determine the potential energy as

$$\Pi = \tfrac{1}{2}c\varphi_1^2 + \tfrac{c(\varphi_2-\varphi_1)^2}{2} + mgl(1-\cos\varphi_1 + 1 - \cos\varphi_2).$$

Assuming that the angles φ_1 and φ_2 are small, we may consider the following expansions in which only up to the second order terms are retained:

$$1-\cos\varphi_1 = \frac{\varphi_1^2}{2}, \quad 1-\cos\varphi_2 = \frac{\varphi_2^2}{2}.$$

Then the expression for Π can be written as

$$\Pi = \frac{1}{2}c\varphi_1^2 + \frac{c(\varphi_2-\varphi_1)^2}{2} + mgl\varphi_1^2 2 + mgl\frac{(\varphi_2^2+\varphi_1^2)}{2},$$

or

$$\begin{aligned}\Pi &= (c+mgl)\,\varphi_1^2 - c\varphi_1\varphi_2 + (c+mgl)\,\frac{\varphi_2^2}{2} \\ &= (c+mgl)\left(\varphi_1^2 + \frac{\varphi_2^2}{2}\right) - c\varphi_1\varphi_2.\end{aligned} \tag{4.7}$$

In order to evaluate Q_{1p} and Q_{2p} we consider the virtual work done by the force P during a virtual displacement $\delta\boldsymbol{r}$, where $\boldsymbol{r}$ is the radial vector attached to the fixed support at the top that locates the point of application of the load P. Now, we have

$$\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2,$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are vectors representing the length and the direction of the two bars. Therefore, we have

$$\delta\boldsymbol{r} = \delta\boldsymbol{r}_1 + \delta\boldsymbol{r}_2.$$

It is necessary to take note that $\delta\boldsymbol{r}_1$ and $\delta\boldsymbol{r}_2$ are orthogonal to the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. Next, we introduce a coordinate system in the plane of the bars such that the x-axis is vertically downward and the y-axis is horizontal and to the right. If we denote the components of $\delta\boldsymbol{r}$ by δx and δy, then from the last equation we get

$$\begin{aligned}\delta x &= l\left(-\sin\varphi_1\delta\varphi_1 - \sin\varphi_2\delta\varphi_2\right),\\ \delta y &= l\left(\cos\varphi_1\delta\varphi_1 + \cos\varphi_2\delta\varphi_2\right).\end{aligned}$$

The components of the force $\boldsymbol{P}$ are $-P\cos\varphi_2$ and $-P\sin\varphi_2$, and the virtual work is

$$\begin{aligned}\delta W = P_x\delta x + P_y\delta y = -Pl\cos\varphi_2(-\sin\varphi_1\delta\varphi_1 - \\ \sin\varphi_2\delta\varphi_2) - Pl\sin\varphi_2\left(\cos\varphi_1\delta\varphi_1 + \cos\varphi_2\delta\varphi_2\right)\end{aligned}$$

or, upon rearranging of terms,

$$\delta W = -Pl\left(\cos\varphi_2\sin\varphi_1 - \sin\varphi_2\cos\varphi_1\right)\delta\varphi_1.$$

From here we have

$$Q_{1p} = Pl\sin\left(\varphi_2 - \varphi_1\right), \quad Q_{2p} = 0.$$

Using relations (4.6) and (4.7), we find the generalised forces (based on the assumption that angles φ_1 and φ_2 are small, i.e., $\sin\left(\varphi_2 - \varphi_1\right) \simeq \varphi_2 - \varphi_1$)

$$\begin{aligned}Q_1 &= -2\left(c + mgl\right)\varphi_1 + c\varphi_2 + Pl\left(\varphi_2 - \varphi_1\right),\\ Q_2 &= -\left(c + mgl\right)\varphi_2 + c\varphi_1.\end{aligned} \tag{4.8}$$

Now we determine the kinetic energy of the system,

$$\begin{aligned}
&x_1 = l\cos\varphi_1, && x_2 = l\cos\varphi_1 + l\cos\varphi_2,\\
&\dot{x}_1 = -l\dot{\varphi}_1\sin\varphi_1, && \dot{x}_2 = -l\left(\dot{\varphi}_1\sin\varphi_1 + \dot{\varphi}_2\sin\varphi_2\right),\\
&y_1 = l\sin\varphi_1, && y_2 = l\sin\varphi_1 + l\sin\varphi_2,\\
&\dot{y}_1 = l\dot{\varphi}_1\cos\varphi_1, && \dot{y}_2 = l\left(\dot{\varphi}_1\cos\varphi_1 + \dot{\varphi}_2\cos\varphi_2\right),\\
&v_1^2 = l^2\dot{\varphi}_1^2.
\end{aligned}$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l^2\left[\dot{\varphi}_1^2 + 2\cos\left(\varphi_2 - \varphi_1\right)\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2\right].$$

Again based on the assumption that angles φ_1 and φ_2 are small, i.e., $\cos\left(\varphi_2 - \varphi_1\right) = 1$, we get

$$v_2^2 = l^2\left(\dot{\varphi}_1^2 + 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2\right).$$

Thus, we have

$$T = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}ml^2\left(2\dot{\varphi}_1^2 + 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2\right).$$

Now, we can write the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\varphi}_j} - \frac{\partial T}{\partial\varphi_j} = Q_j \quad (j = 1, 2)$$

$$\begin{aligned}
\tfrac{\partial T}{\partial\dot{\varphi}_1} &= ml^2\left(2\dot{\varphi}_1 + \dot{\varphi}_2\right)\\
\tfrac{d}{dt}\tfrac{\partial T}{\partial\dot{\varphi}_1} &= ml^2\left(2\ddot{\varphi}_1 + \ddot{\varphi}_2\right)\\
\tfrac{\partial T}{\partial\varphi_1} &= \tfrac{\partial T}{\partial\varphi_2} = 0.
\end{aligned}$$

Using Q_1 from (4.8), the first equation becomes

$$ml^2\left(2\ddot{\varphi}_1 + \ddot{\varphi}_2\right) + 2H\varphi_1 - c\varphi_2 - Pl\left(\varphi_2 - \varphi_1\right) = 0, \tag{4.9}$$

where

$$H = c + mgl.$$

In a similar fashion we obtain the second equation for φ_2 as:

$$ml^2\left(\ddot{\varphi}_1 + \ddot{\varphi}_2\right) + H\varphi_2 - c\varphi_1 = 0. \tag{4.10}$$

Now, let $\varphi_1 = Ae^{\lambda t}$, $\varphi_2 = Be^{\lambda t}$. Substitute these values into (4.9) and (4.10), divide the resulting equations by $e^{\lambda t}$ to obtain two homogeneous equations in A and B,

$$\begin{aligned}
\left(2ml^2\lambda^2 + 2H + Pl\right)A + \left(ml^2\lambda^2 - c - Pl\right)B &= 0,\\
\left(ml^2\lambda^2 - c\right)A + \left(ml^2\lambda^2 + H\right)B &= 0.
\end{aligned}$$

The determinant of this system must be equal to zero:

$$\begin{vmatrix} 2ml^2\lambda^2 + 2H + Pl & ml^2\lambda^2 - c - Pl \\ ml^2\lambda^2 - c & ml^2\lambda^2 + H \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} 2\lambda^2 + I_1 & \lambda^2 - I_2 \\ \lambda^2 - I_3 & \lambda^2 + I_4 \end{vmatrix} = 0,$$

or

$$\lambda^4 + \lambda^2 (I_1 + I_2 + I_3 + 2I_4) + I_1 I_4 - I_2 I_3 = 0,$$

where

$$I_1 = \frac{2H + Pl}{ml^2}, \quad I_2 = \frac{c + Pl}{ml^2}, \quad I_3 = \frac{c}{ml^2}, \quad I_4 = \frac{H}{ml^2}$$

Note that $I_1 + I_2 + I_3 + 2I_4 > 0$ and $I_1 I_4 - I_2 I_3 = (c^2 + 2cmgl + m^2 g^2 l^2 + Pmgl^2)/(ml^2) > 0$ always, and $(I_1 + I_2 + I_3 + 2I_4)^2 - 4(I_1 I_4 - I_2 I_3) > 0$, so all four roots will be imaginary. Therefore the system is stable in the vertical direction.

4.6. A two-rotor Anschútz gyrocompass[1] with a viscous damper is widely used in some countries. If this type of a gyroscope is mounted in a ship whose northern component of velocity is constant, then the differential equations of motion of the gyroscope are

$$\begin{aligned} \dot{x}_1 - \frac{k^2}{U\cos\varphi} x_2 - \frac{k^2}{U\cos\varphi}(1-\rho)x_3 &= X_1, \\ \dot{x}_2 + U\cos\varphi x_1 &= X_2, \\ \dot{x}_3 + F x_2 + F x_3 &= X_3. \end{aligned}$$

Here x_1, x_2 and x_3 are variations of compass coordinates from its values at dynamic equilibrium; k is the frequency of free vibrations of the sensitive element (gyrosphere); U is the angular velocity of Earth's rotation; φ is latitude of the ship; F is the factor of fluid flow in the viscous damper; $\rho = 1 - \dfrac{c}{Pl}$; c and Pl are the norms of the moments of the damper fluid and the gyrosphere, respectively; and X_1, X_2, X_3 are terms of higher orders in x_1, x_2, x_3 and $\dot{x}_1, \dot{x}_2, \dot{x}_3$.

Determine condition for asymptotic stability.

[1] In honour of the German engineer and industrialist who invented this gyrocompass.

S o l u t i o n:

Determine the equations of the first approximation. Let $X_1 = X_2 = X_3 = 0$. We get the following system of linear differential equations (in the first equation the parameter ρ is replaced by $\rho = 1 - \dfrac{c}{Pl}$):

$$\begin{aligned} \dot{x}_1 - \frac{k^2}{U\cos\varphi}x_2 - \frac{k^2}{U\cos\varphi}\frac{c}{Pl}x_3 = 0,\\ U\cos\varphi x_1 + \dot{x}_2 = 0,\\ Fx_2 + \dot{x}_3 + Fx_3 = 0. \end{aligned} \tag{4.11}$$

We assume $x_1 = Ae^{\lambda t}$, $x_2 = Be^{\lambda t}$, and $x_3 = Ce^{\lambda t}$. Substitute these expressions for x_k into equation (4.11) and divide the resulting equations by $e^{\lambda t}$ to obtain

$$\begin{aligned} \lambda A - \tfrac{k^2}{U\cos\varphi}B - \tfrac{k^2}{U\cos\varphi}\tfrac{c}{Pl}C = 0,\\ U\cos\varphi A + \lambda B = 0,\\ FB + (\lambda + F)\,C = 0. \end{aligned}$$

A, B and C can not be equal to zero simultaneously, hence, the determinant of this system must vanish, i.e.,

$$\begin{vmatrix} \lambda & -\frac{k^2}{U\cos\varphi} & -\frac{k^2}{U\cos\varphi}\frac{c}{Pl} \\ U\cos\varphi & \lambda & 0 \\ 0 & F & \lambda + F \end{vmatrix} = 0,$$

or

$$\lambda^3 + F\lambda^2 + k^2\lambda + k^2F\left(1 - \frac{c}{Pl}\right) = 0.$$

The Hurwitz condition requires that all the coefficients and $\Delta_2 = a_1a_2 - a_0a_3$ be positive. This means that $\dfrac{c}{Pl} < 1$, or $c < Pl$, so that $\Delta_2 = Fk^2 - Fk^2\left(1 - \dfrac{c}{Pl}\right) = Fk^2\dfrac{c}{Pl} > 0$. Thus the only required condition is that we should have $c < Pl$.

4.7. A stable platform is a device which is sometimes used in navigation to determine, simultaneously, the meridian and horizontal plane for a sailing ship. For an anchored ship, the differential equations of its perturbed motion can be reduced to two identical equations:

$$\ddot{x}_1 + 2b_1\dot{x}_1 + (\nu^2 - \Omega^2)x_1 - 2\Omega\dot{x}_2 = X_1,$$

$$\ddot{x}_2 + 2b_2\dot{x}_2 + (\nu^2 - \Omega^2)x_2 + 2\Omega\dot{x}_1 = X_2.$$

Here x_1 is a quantity proportional to the angle of deviation from the meridian plane; x_2 is the variation of the auxiliary variable, which is associated with the constructive angle (see [4]); $b_1 > 0$ and $b_2 > 0$ are coefficients that characterise the dissipative forces; $\nu = \sqrt{g/R} = 0.00124\,1/\text{sec}$ is Schuler frequency[2]; $\Omega = U\sin\varphi$; $U = 7.29 \cdot 10^{-5}\,1/\text{sec}$ is the angular velocity of Earth's rotation; φ is the latitude of the ship; and X_1 and X_2 are terms of higher orders in x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$.

In two other analogous differential equations of perturbed motion x_3 and x_4 determine the angle of deviation from the horizontal plane and the variation of the other auxiliary variable, which is associated with the second constructive angle (see [4]).

Determine the condition for asymptotic stability of the device.

Solution:

Let $X_1 = X_2 = 0$, then we get the system of the first approximation

$$\begin{aligned} &\ddot{x}_1 + 2b_1\dot{x}_1 + \left(\nu^2 - \Omega^2\right)x_1 - 2\Omega\dot{x}_2 = 0,\\ &2\Omega\dot{x}_1 + \ddot{x}_2 + 2b_2\dot{x}_2 + \left(\nu^2 - \Omega^2\right)x_2 = 0. \end{aligned} \tag{4.12}$$

As usual, we take $x_1 = Ae^{\lambda t}$ and $x_2 = Be^{\lambda t}$; substitute these expressions for x_1 and x_2 into (4.12) and divide the resulting equations by $e^{\lambda t}$ to obtain

$$\begin{aligned} &\left[\lambda^2 + 2b_1\lambda + \left(\nu^2 - \Omega^2\right)\right]A - 2\Omega\lambda B = 0,\\ &2\Omega\lambda A + \left[\lambda^2 + 2b_2\lambda + \left(\nu^2 - \Omega^2\right)\right]B = 0. \end{aligned}$$

A and B can not vanish simultaneously, so the determinant of the system must be equal to zero, i.e.,

$$\begin{vmatrix} \lambda^2 + 2b_1\lambda + \left(\nu^2 - \Omega^2\right) & -2\Omega\lambda \\ 2\Omega\lambda & \lambda^2 + 2b_2\lambda + \left(\nu^2 - \Omega^2\right) \end{vmatrix} = 0,$$

or,

$$\begin{aligned} &\lambda^4 + 2\left(b_1 + b_2\right)\lambda^3 + 2\left[\left(\nu^2 - \Omega^2\right) + 2b_1b_2 + 2\Omega^2\right]\lambda^2 + \\ &2\left(b_1 + b_2\right)\left(\nu^2 - \Omega^2\right)\lambda + \left(\nu^2 - \Omega^2\right)^2 = 0. \end{aligned} \tag{4.13}$$

[2] Max Schuler was a German scientist who, in 1912, investigated the period of unperturbed oscillations of the gyroscopic pendulum in a gyrocompass.

For asymptotic stability of the system under consideration which is governed by (4.12), the necessary and sufficient condition is to satisfy Hurwitz's criterion (cf. (4.32) in [11]) as follows:

1. All the coefficients in (4.13) must be positive,
2. $\Delta_3 = a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 > 0$. Δ_3 is obtained as

$$\begin{aligned}\Delta_3 = 8\left(b_1+b_2\right)^2\left[\left(\nu^2-\Omega^2\right)+2b_1b_2+2\Omega^2\right]\left(\nu^2-\Omega^2\right) - \\ 4\left(b_1+b_2\right)^2\left(\nu^2-\Omega^2\right)^2 - 4\left(b_1+b_2\right)^2\left(\nu^2-\Omega^2\right)^2 = \\ 4\left(b_1+b_2\right)^2\left(\nu^2-\Omega^2\right)\left[2\left(\nu^2-\Omega^2\right)+4b_1b_2+4\Omega^2-2\left(\nu^2-\Omega^2\right)\right] = \\ 16\left(b_1+b_2\right)^2\left(\nu^2-\Omega^2\right)\left(b_1b_2+\Omega^2\right).\end{aligned}$$

If $\nu > \Omega$, then Δ_3 and all the coefficients in equation (4.13) are positive and Hurwitz's criterion is satisfied. Hence, it follows that the system (4.2) is asymptotically stable, and therefore, according to Liapunov's theorem of stability in the first approximation (cf. Theorem 4.4 in [11]) the system under consideration, where $X_1 \neq 0$ and $X_2 \neq 0$, is stable.

Chapter 5

Stability of Linear Autonomous Systems

5.1. Given the following equations of a perturbed motion:

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_2 - x_3, \\
\dot{x}_2 &= -x_1 + 3x_2 - x_3 - 2x_4, \\
\dot{x}_3 &= \phantom{-x_1 + {}} 6x_2 - 3x_3 - 3x_4, \\
\dot{x}_4 &= -3x_1 + 3x_2 \phantom{{} - 3x_3} - 3x_4,
\end{aligned}$$

determine the roots of the characteristic equation and the stability of the motion.

S o l u t i o n:

Determine the $A - E\lambda$ matrix for the given problem:

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -2 \\ 0 & 6 & -3-\lambda & -3 \\ -3 & 3 & 0 & -3-\lambda \end{pmatrix}$$

The determinant of this matrix has four roots: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = -1$. Executing the following *elementary matrix operations:* add the third column to the second; multiply the third column by $1-\lambda$ and add the result to the first column; will result in

$$A - \lambda E \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ -2+\lambda & 2-\lambda & -1 & -2 \\ -3+2\lambda+\lambda^2 & 3-\lambda & -3-\lambda & -3 \\ -3 & 3 & 0 & -3-\lambda \end{pmatrix}$$

Interchange the third column with the first one; multiply the first column by -1; using elementary matrix operations obtain zeros for all entries in column one except for the first entry:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-\lambda & -2+\lambda & -2 \\ 0 & 3-\lambda & -3+2\lambda+\lambda^2 & -3 \\ 0 & 3 & -3 & -3-\lambda \end{pmatrix}$$

Next, add the second column to the third column;

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & \lambda(\lambda+1) & -3 \\ 0 & 3 & 0 & -3-\lambda \end{pmatrix}$$

Follow this by multiplying the second column by -1 and subtracting the fourth column from the result to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -2 \\ 0 & \lambda & \lambda(\lambda+1) & -3 \\ 0 & \lambda & 0 & -3-\lambda \end{pmatrix}$$

Subtract the second row from the third row; subtract the second row from the fourth row:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -2 \\ 0 & 0 & \lambda(\lambda+1) & -1 \\ 0 & 0 & 0 & -1-\lambda \end{pmatrix}$$

Multiply the second column by 2; multiply the fourth column by λ: add the fourth column to the second column; change the sign of the fourth column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -\lambda & \lambda(\lambda+1) & 1 \\ 0 & -\lambda(\lambda+1) & 0 & 1+\lambda \end{pmatrix}$$

Divide the second row by 2; subtract the second row from the third one; multiply the second row by $1+\lambda$ and subtract from the fourth:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\lambda & \lambda(\lambda+1) & 0 \\ 0 & -\lambda(\lambda+1) & 0 & 0 \end{pmatrix}$$

Interchange the second and fourth columns; multiply the third row $\lambda+1$ and subtract the result from the fourth row;

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda(\lambda+1) & -\lambda \\ 0 & 0 & -\lambda(\lambda+1)^2 & 0 \end{pmatrix}$$

Interchange columns three and four; change the sign of the third and the fourth column; multiply the third column by $\lambda+1$ and add the result to the fourth column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda(\lambda+1)^2 \end{pmatrix}$$

Thus we get the Smith canonical form of the matrix. The roots are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = -1$, and $\lambda_4 = -1$. The following solutions in normal coordinates corresponds to these roots:

$$z_1 = z_{01}, \quad z_2 = z_{02}, \quad z_3 = z_{03}e^{-t}, \quad z_4 = z_{04}e^{-t}, \tag{5.1}$$

where z_{01}, z_{02}, z_{03}, and z_{04} are the initial values of the corresponding coordinates. Since solution (5.1) is stable with respect to normal coordinates, the solution with respect to x-coordinates is also stable.

5.2. The following equations of a perturbed motion are given:

$$\begin{aligned}
\dot{x}_1 &= x_1 - 2x_2 + x_4,\\
\dot{x}_2 &= -x_1 + 3x_2 - x_3 - 2x_4,\\
\dot{x}_3 &= 3x_2 - 2x_3 - 2x_4,\\
\dot{x}_4 &= -3x_1 + 6x_2 - x_3 - 4x_4.
\end{aligned}$$

Determine the roots of the characteristic equation and the stability of the motion.

S o l u t i o n:

Determine the $A - E\lambda$ matrix for the given equations:

$$A - \lambda E = \begin{pmatrix} 1-\lambda & -2 & 0 & 1 \\ -1 & 3-\lambda & -1 & -2 \\ 0 & 3 & -2-\lambda & -2 \\ -3 & 6 & -1 & -4-\lambda \end{pmatrix}.$$

Multiply the last column by $-(1-\lambda)$ and add it to the first column; then multiply the fourth column by 2 and add it to the second column; next using elementary operations make all the entries in the fourth column, except the first one, to vanish:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1-2\lambda & -1-\lambda & -1 & 0 \\ 2-2\lambda & -1 & -2-\lambda & 0 \\ 1-3\lambda-\lambda^2 & -2-2\lambda & -1 & 0 \end{pmatrix}.$$

Multiply the second and third columns by -1; interchange the first and the fourth columns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\lambda & 1 & 1-2\lambda \\ 0 & 1 & 2+\lambda & 2-2\lambda \\ 0 & 2+2\lambda & 1 & 1-3\lambda-\lambda^2 \end{pmatrix}.$$

Subtract the second row from the fourth row:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\lambda & 1 & 1-2\lambda \\ 0 & 1 & 2+\lambda & 2-2\lambda \\ 0 & 1+\lambda & 0 & -\lambda-\lambda^2 \end{pmatrix}.$$

Subtract the fourth row from the second one:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1-\lambda+\lambda^2 \\ 0 & 1 & 2+\lambda & 2-2\lambda \\ 0 & 1+\lambda & 0 & -\lambda-\lambda^2 \end{pmatrix}.$$

Interchange the second and the third columns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1-\lambda+\lambda^2 \\ 0 & 2+\lambda & 1 & 2-2\lambda \\ 0 & 0 & 1+\lambda & -\lambda-\lambda^2 \end{pmatrix}.$$

Multiply the second column by $1-\lambda+\lambda^2$ and substruct it from the fourth column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2+\lambda & 1 & -\lambda-\lambda^2-\lambda^3 \\ 0 & 0 & 1+\lambda & -\lambda-\lambda^2 \end{pmatrix}.$$

Multiply the second row by the $-(2+\lambda)$ and add it to the third row; Multiply the fourth column by -1:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda+\lambda^2+\lambda^3 \\ 0 & 0 & 1+\lambda & \lambda+\lambda^2 \end{pmatrix}.$$

Multiply the third column by $\lambda+\lambda^2+\lambda^3$ and subtract it from the fourth column; multiply the fourth column by -1:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1+\lambda & \lambda^2+2\lambda^3+\lambda^4 \end{pmatrix}.$$

Multiply the third row by $-(1+\lambda)$ and add to the fourth row.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^2(\lambda+1)^2 \end{pmatrix}.$$

Now we have the normal form of the matrix $A - \lambda E$. We note that the invariant factors are: $E_1 = 1$, $E_2 = 1$, $E_3 = 1$, and $E_4 = \lambda^2(\lambda+1)^2$. Therefore, the $A - \lambda E$ matrix has two elementary divisors: λ^2, $(\lambda+1)^2$, with the corresponding roots: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = -1$.

The equation of the perturbed motion (cf.[11]) in canonical variables consists of two Jordan blocks (cf. normal Jordan form (5.40) in [11]):

$$\dot{z}_1 = 0, \quad \dot{z}_2 = z_1, \quad \dot{z}_3 = -z_3, \quad \dot{z}_4 = z_3 - z_4.$$

From these it is easy to get the solution as:

$$z_1 = z_{01}, \quad z_2 = z_{01}t + z_{02}, \quad z_3 = z_{03}e^{-t}, \quad z_4 = (z_{04} + z_{03}t)\, e^{-t}.$$

The canonical variables are unstable ($z_2 \to \infty$ as $t \to \infty$), and therefore the system is unstable.

5.3. The nonhomogeneous linear differential equations

$$\begin{aligned}
\dot{x}_1 &= -5x_1 + 2x_3 + 2t^3 + 5t^2 + 2t,\\
\dot{x}_2 &= 41x_1 + 5x_2 - 19x_3 - 19t^3 - 41t^2 - 10t + 2,\\
\dot{x}_3 &= 5x_1 + 2x_2 - 3x_3 - 3t^3 - 8t^2 - 4t
\end{aligned}$$

have the particular solution

$$\bar{x}_1 = t^2, \quad \bar{x}_2 = 2t, \quad \bar{x}_3 = -t^3.$$

Determine the stability of this solution and construct the solution of the equation of the perturbed motion in terms of canonical variables.

S o l u t i o n:

The stability of the solution

$$x_1 = t^2, \quad x_2 = 2t, \quad x_3 = -t^3 \tag{5.2}$$

could be investigated using the homogeneous parts of the equations, i.e., the equations (cf. Example 1.4 in [11]):

$$\begin{aligned}
\dot{x}_1 &= -5x_1 + 2x_3,\\
\dot{x}_2 &= 41x_1 + 5x_2 - 19x_3,\\
\dot{x}_3 &= 5x_1 + 2x_2 - 3x_3.
\end{aligned}$$

The $A - \lambda E$ matrix for this system of equations is

$$A - \lambda E = \begin{pmatrix} -5-\lambda & 0 & 2 \\ 41 & 5-\lambda & -19 \\ 5 & 2 & -(3+\lambda) \end{pmatrix}. \tag{5.3}$$

Divide the third column by 2; interchange the first and the third columns:

$$\begin{pmatrix} 1 & 0 & -(5+\lambda) \\ -\frac{19}{2} & 5-\lambda & 41 \\ -\frac{3+\lambda}{2} & 2 & 2 \end{pmatrix}.$$

Multiply the first column by $5+\lambda$ and add to the third column; multiply the third column by -2. Now, in the first column, except for the first entry get all zeros:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5-\lambda & 13+19\lambda \\ 0 & 2 & 5+8\lambda+\lambda^2 \end{pmatrix}.$$

Divide the second column by 2; multiply the second column by $-(5 + 8\lambda + \lambda^2)$ and add the result to the third column:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5-\lambda}{2} & \frac{(\lambda+1)^3}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Multiply the third row by $(5-\lambda)/2$ and subtract it from the second row; interchange the second and the third row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\lambda+1)^3 \end{pmatrix}.$$

Matrix (5.3) is in normal diagonal form; it contains three invariant factors

$$E_1 = 1, \quad E_2 = 1, \quad E_3 = (\lambda+1)^3.$$

The root $\lambda_1 = \lambda_2 = \lambda_3 = -1$ is a root with a multiplicity 3 for the invariant factor E_3 as well as the equation. Therefore, the differential equations or the canonical variables are (cf. equation (5.52) in [11])

$$\dot{z}_1 = -z_1, \quad \dot{z}_2 = z_1 - z_2, \quad \dot{z}_3 = z_2 - z_3.$$

The solution is

$$z_1 = z_{01}e^{-t}, \quad z_2 = (z_{02} + z_{01}t)\, e^{-t}, \quad z_3 = \left(z_{03} + z_{02}t + z_{01}\frac{t^2}{2}\right) e^{-t}.$$

This solution is stable, therefore partial solution (5.2) is asymptotically stable.

Chapter 6

The Effect of Force Type on Stability of Motion

6.1. Determine the differential equations which govern the motion in Problem 3.9 and show that they contain gyroscopic forces. For the unstable case when points O_1, O and C are collinear and $m\omega^2 > c$, determine the degree of instability and show that the system may be stabilised by gyroscopic forces.

S o l u t i o n:

1. In Problem (3.9), we consider the equations corresponding to equation (3.18). In these equations, those terms that contain the derivatives in the first power, i.e., $2m\omega\dot{y}$ and $2m\omega\dot{x}$, lead to matrix of coefficients

$$\begin{pmatrix} 0 & -2m\omega \\ 2m\omega & 0 \end{pmatrix}$$

which is a skew-symmetric matrix, indicating that these terms are gyroscopic forces.

2. For the unstable case the degree of instability should be equal to 2. This follows from the observation that

$$c_1 - c_2 = \left(c - m\omega^2\right)^2 > 0.$$

for all c and ω (refer to the solution for Problem 3.9).

6.2. Using the previous problem show the validity of Thomson–Tait–Chetaev Theorems 6.5 and 6.6.

S o l u t i o n:

If we take into account the resistance forces $-b\dot{x}$ and $-b\dot{y}$, then Hurwitz's criterion is satisfied for $m\omega^2 < c$, and the stable system becomes asymptotically stable. For $m\omega^2 > c$ Hurwitz's criterion is not satisfied and the stable system becomes unstable.

6.3. Two unstable potential systems are given:

$$\begin{array}{ll} I)\ \ddot{q}_1 - q_1 + 2q_2 + 3q_3 = 0\,, & II)\ \ddot{q}_1 - q_1 + 2q_2 + 3q_3 = 0\,, \\ \quad \ddot{q}_2 + 2q_1 + q_2 = 0\,, & \quad \ddot{q}_2 + 2q_1 + q_3 = 0\,, \\ \quad \ddot{q}_3 + 3q_1 + q_3 = 0\,; & \quad \ddot{q}_3 + 3q_1 + q_2 + q_3 = 0\,. \end{array}$$

Why are the systems potential? Why are they unstable? Is it possible to stabilise them by gyroscopic forces?

Solutions:

1. Both systems are potential because the coordinate matrices are symmetric.

2. In each case, to determine the stability of the system, Hurwitz's criterion is examined.

For the first system, we have

$$\begin{pmatrix} -1+\lambda^2 & 2 & 3 \\ 2 & 1+\lambda^2 & 0 \\ 3 & 0 & 1+\lambda^2 \end{pmatrix}.$$

For $\lambda = 0$, the determinant of this matrix is evaluated to be -14.

Then, based on Hurwitz's theorem, we can conclude that the first system is unstable and could not be stabilised by adding gyroscopic forces.

For the second system when $\lambda = 0$ we have the determinant

$$\begin{vmatrix} -1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 9 > 0.$$

For all other λ, this determinant becomes

$$\begin{vmatrix} -1+\lambda^2 & 2 & 3 \\ 2 & \lambda^2 & 1 \\ 3 & 1 & 1+\lambda^2 \end{vmatrix} =$$

$$\lambda^2(\lambda^4 - 1) + 6 + 6 - 3\lambda^2 - (\lambda^2 - 1) - 4(1+\lambda^2) = \lambda^4 - 9\lambda^2 + 9.$$

The negative sign in front of λ^2 indicates that the system is unstable, but it could be stabilised by adding gyroscopic forces to the system (cf. Problem 3.9, where, for $c < m\omega^2$, in the absence of the gyroscopic forces $-2m\omega\dot{y}$ and $2m\omega\dot{x}$ the system would be unstable. In fact, the presence of these gyroscopic forces has made the system stable for all c and ω.)

6.4. Kinetic and potential energies of a gyroscopic pendulum at the upper vertical position of its axis of symmetry are, respectively,

$$\begin{aligned} T &= \tfrac{1}{2}\, J_x(\cos^2\!\alpha\, \dot{\beta}^2 + \dot{\alpha}^2) + \tfrac{1}{2}\, J_z(\dot{\varphi} - \dot{\beta}\,\sin\!\alpha)^2, \\ \Pi &= Pl\,\cos\!\beta\,\cos\!\alpha, \end{aligned}$$

where α and β are the angles which define the position of the axis of gyroscope with respect to a vertical axis, φ is the angle of rotation of the gyroscope, J_x and J_z are principle moments of inertia of the gyroscope, P is its weight, and l is the distance from its centre of mass to its point of suspension.

Using the cyclic integral:

$$\frac{\partial T}{\partial \dot{\varphi}} = J_z(\dot{\varphi} - \dot{\beta}\,\sin\!\alpha) = H = \text{const},$$

determine differential equations governing the motion of the gyroscopic pendulum and find that value of the angular momentum H, for which the upper position of the pendulum can be stabilised by gyroscopic forces.

S o l u t i o n:

Assuming small angles α and β, the kinetic and potential energy of the system are:

$$\begin{aligned} T &= \tfrac{1}{2} J_x \left(\dot{\alpha}^2 + \dot{\beta}^2\right) + \tfrac{1}{2} J_z \left(\dot{\varphi} - \dot{\beta}\alpha\right), \\ \Pi &= Pl \left(1 - \tfrac{\alpha^2}{2}\right)\left(1 - \tfrac{\beta^2}{2}\right) = -\tfrac{1}{2} Pl \left(\alpha^2 + \beta^2\right). \end{aligned}$$

where the constant quantity $\dfrac{Pl}{2}$ is ignored. Then, the Lagrange equations read as

$$\begin{aligned} J_x\ddot{\alpha} - H\dot{\beta} - Pl\alpha &= 0, \\ J_x\ddot{\beta} + H\dot{\alpha} - Pl\beta &= 0, \\ H = \tfrac{\partial T}{\partial \dot{\varphi}} &= J_z\left(\dot{\varphi} - \dot{\varphi}\sin\alpha\right) \end{aligned}$$

with the characteristic equation

$$\begin{vmatrix} J_x\lambda^2 - Pl & -H\lambda \\ H\lambda & J_x\lambda^2 - Pl \end{vmatrix} = 0,$$

or

$$J_x^2\lambda^4 + \left(H^2 - 2J_xPl\right)\lambda^2 + P^2l^2 = 0,$$

$$\lambda^2 = \frac{-\left(H^2 - 2J_xPl\right) \pm \sqrt{\left(H^2 - 2J_xPl\right)^2 - 4J_x^2P^2l^2}}{2J_x^2} =$$
$$= \frac{-\left(H^2 - 2J_xPl\right) \pm \sqrt{H^4 - 4H^2J_xPl}}{2J_x^2}.$$

When $H > 2\sqrt{J_xPl}$, the expression under the square root sign will be positive and both values of λ^2 will be real and negative, and therefore the pendulum will be stable in the first approximation.

6.5. The differential equations of a perturbed motion are:

$$A\ddot{\boldsymbol{q}} + HG\,\dot{\boldsymbol{q}} + C\boldsymbol{q} = 0.$$

Here A, G, and C are square $(n \times n)$ matrices of constants. Moreover, $A = A^{\mathrm{T}}$ is a positive definite symmetric matrix, composed from inertia coefficients of the system; $G = -G^{\mathrm{T}}$ is a skew-symmetric matrix of gyroscopic forces; $C = C^{\mathrm{T}}$ is a symmetric matrix of potential forces; $\boldsymbol{q}$ is a column matrix; H is a positive parameter. For $H = 0$ the system is unstable.

Prove the following theorem. If gyroscopic forces satisfying the following conditions:

1) $\det G \neq 0$,

2) the precession system $HG\,\dot{\boldsymbol{q}} + C\boldsymbol{q} = 0$ is stable,

3) the roots of the characteristic equation are simple,

are applied to the unstable potential system, then for rather large values of H, the unstable motion can be stabilised by these gyroscopic forces [4].

(This is a rather difficult problem, and its solution of requires a good level of insight.)

S o l u t i o n:

Recall that for skew-symmetric matrices the determinant of an odd order matrix vanishes, whereas for an even order matrix the determinant is equal to the square of a rational function of its elements. Therefore, the determinant of an odd order skew-symmetric matrix whose elements are real numbers must be nonnegative (cf. Section 5.2 **Matrices and Basic Operations, a) General definitions** in [11]).

First, let us consider the equation

$$A\ddot{\boldsymbol{q}} + HG\dot{\boldsymbol{q}} + C\boldsymbol{q} = 0, \tag{6.1}$$

and show that its characteristic equation

$$|A\lambda^2 + HG\lambda + C| = 0, \tag{6.2}$$

contains only the even powers of the unknown parameter λ. To this end, denote the determinant in (6.2) as $\Delta(\lambda)$. Then, replacing λ by $-\lambda$, we have

$$\Delta(-\lambda) = \left|A\lambda^2 - HG\lambda + C\right|.$$

Since interchanging the columns and rows will not change the determinant, we have:

$$\Delta(-\lambda) = \left|A^T\lambda^2 - HG^T\lambda + C^T\right|.$$

Matrices A and C are symmetric, therefore, $A^T = A$ and $C^T = C$. Matrix G is skew-symmetric, so that $G^T = -G$ (cf. equation (5.16) in [11]). Therefore, we can write

$$\Delta(-\lambda) = \left|A\lambda^2 + HG\lambda + C\right| = \Delta(\lambda).$$

This expression proves that in the determinant given by (6.2) λ appears only in the even powers.

From the condition that $|G| \neq 0$ it follows that n, the order of this matrix, is an even number. Now, let $n = 2s$. Now, based on the condition stated in the problem, the system (6.1) is unstable when $H = 0$. To prove that for large H this system becomes stable, it is necessary and sufficient to show that for large H all roots of equation (6.2) are pure imaginary, and all λ^2 are real negative numbers.

To this end, for a large H, we introduce the small parameter $\mu = H^{-1}$, and let

$$\lambda = \frac{\nu}{H} = \nu\mu.$$

Then equation (6.2) becomes

$$\Delta(\nu,\mu) = \left|\mu^2 A\nu^2 + G\nu + C\right| = 0, \tag{6.3}$$

so that for $\mu = 0$, we get

$$\Delta(\nu,0) = |G\nu + C| = 0. \tag{6.4}$$

Expanding the determinant (6.3) in powers of ν^2, the coefficients of the resulting equation will depend on the small parameter μ. Recalling the theorem that asserts the continuous dependence of the roots of an equation on its coefficients, we observe that for sufficiently small values of μ, i.e., for large H, each of the n roots of the characteristic equation (6.2) that corresponds to differential equation (6.1), is in the neighbourhood of the corresponding roots of equation (6.4). The roots of this latter equation are pure imaginary since this equation contains only the even powers of ν (this can be shown by using the same approach as was used for equation (6.2)). Thus, under the stated conditions system (6.2) is stable.

Denoting these roots by $\nu_k i$, each will correspond to a λ_k. Then, for large H the n roots of equation (6.2) are in the vicinity of the following roots

$$\lambda_k^{(1)} = \pm\frac{\nu_k}{H} i \quad (k = 1, \ldots, n)\,. \tag{6.5}$$

Next, let

$$\lambda = H\gamma = \mu^{-1}\gamma. \tag{6.6}$$

Substitute this expression for λ into equation (6.2), and divide the resulting equation by μ^{-2n}. As the result the characteristic equation becomes

$$\Delta(\gamma,\mu) = \left|A\gamma^2 + G\gamma + \mu^2 C\right| = 0.$$

Upon dividing by γ^n, for $\mu = 0$, we get

$$\Delta(\gamma,0) = |A\gamma + G| = 0. \tag{6.7}$$

In a similar manner we can show that the n roots of equation (6.2) are in the neighbourhood of the pure imaginary roots of (6.7). It should be noted that equation (6.7) is not very different from equation (6.4), and unlike matrix C, matrix A is positive definite. Denoting the roots of (6.7) as $\gamma_k i$, they are related to λ_k by means of (6.6). Thus, the n roots of equation (6.2) will be in the vicinity of the following roots

$$\lambda_k^{(2)} = \pm H\gamma_k i \quad (k = 1, \ldots, n). \tag{6.8}$$

Expressions (6.5) and (6.8) prove that for large H the unstable system (6.1) may be stabilised by gyroscopic forces.

The following two remarks are in order:

1. The roots must be simple because in moving from equation (6.3) to (6.4) if the roots of the characteristic equations (6.3) are not simple then the roots of equation (6.4) can have small real parts.

2. The quantities $\dfrac{\nu_k}{H}$ in (6.5) and $H\gamma_k$ in (6.8) are the frequencies of harmonic vibrations. The parameter H in a gyroscopic system is proportional to the angular velocity of the gyroscope, which is very large (150000-200000 rev./min). The equations obtained for frequencies show that the frequencies $\left(\frac{\nu_k}{H}\right)$ are very small, with very large periods. These represent the system precessions which can be damped slowly in the presence of dissipative forces. The remaining frequencies $(H\gamma_k)$, are very large with small periods. These represent the nutations of the system which are damped very quickly in the presence of dissipative forces. In practical application of theory of gyroscopes, as a rule, the nutations are ignored.

Chapter 7

The Stability of Nonautonomous Systems

7.1. The differential equation of a perturbed motion is

$$\ddot{x} + a\dot{x} + \left(2 - \sqrt{1 - x^2}\,\sin^3 t\right) x = 0,$$

where $a = \text{const}$.

What condition has to be satisfied by a, to ensure asymptotic stability of the system with respect to x and $\dot{x}$?

S o l u t i o n:

This equation is similar to equation (7.23) in [11]. For $\alpha(t, x, \dot{x}) = \text{const}$, this system is stable provided condition (7.43) in [11] is satisfied, where B and b are the maximum and minimum of the function $\beta(t, x, \dot{x})$. In Problem 7.1 this function is

$$\beta = 2 - \sqrt{1 - x^2}\,\sin^3 t.$$

Obviously, $B = 3$ (for $x = 0$ and $t = \pi$), and $b = 1$ (for $x = 0$ and $t = \frac{\pi}{2}$). Thus, in view of (7.43) in [11], the system is asymptotically stable for $a > \sqrt{3} - 1$.

7.2. A perturbed motion is defined by the following set of homogeneous linear differential equations with periodic coefficients

$$\begin{aligned}
\dot{x}_1 &= \quad -x_1 + \sin t \cdot x_2, \\
\dot{x}_2 &= \cos t \cdot x_1 \quad -x_2 - \sin t \cdot x_3 \\
\dot{x}_3 &= \quad \cos t \cdot x_2 \quad -x_3.
\end{aligned}$$

Develop a computer program to integrate these equations over the time interval $[0,\ 2\pi]$ with initial conditions

$$x_{kj} = \begin{cases} 1,\ k = j \\ 0,\ k \neq 0 \end{cases}$$

Obtain the fundamental matrix A. Find the roots of the characteristic equation. Check your results for these roots and analyze the stability of the system.

S o l u t i o n:

This system of linear differential equations with periodic coefficients (the period is equal to 2π) should be integrated numerically using any appropriate computer code. The interval of integration is $[0, 2\pi]$ with the given initial conditions. Then we can get the matrix corresponding to (7.61) and an equation similar to (7.64) in [11]. Solving this equation we obtain the roots of the characteristic equation:

$$\rho_1 = 2.566519 \cdot 10^{-5}, \quad \rho_{2,3} = 0.008405 \pm 0.013532\, i.$$

Since the moduli of these roots are less than one then the system is asymptotically stable.

Using

$$p_1 p_2 p_3 = \exp \int_0^{2\pi} (-3) dt$$

the accuracy or correctness of the results can be checked. The check gives good agreement

$$\rho_1 \rho_2 \rho_3 = 6.512428 \cdot 10^{-9}, \quad \exp -6\pi = 6.512412 \cdot 10^{-9}.$$

7.3. The equations of a perturbed motion are

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + \cos 2t \cdot x_1 x_2^3, \\
\dot{x}_2 &= \left(1 + \sin^2 t\right) x_1^2 x_2^2 - 2x_2^5.
\end{aligned}$$

It is required to investigate the stability of the unperturbed motion $x_1 = x_2 = 0$. (In the book [11], there is an error in the second equation. The last term in this equation must have a coefficient of 2.)

S o l u t i o n:

We consider the following Liapunov function for this system:

$$V = \frac{1}{2}\left(x_1^2 + x_2^2\right).$$

This is a positive definite function and it is an implicit function of time. Its derivative with respect to time is

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2.$$

Substitute the expressions for $\dot{x}_1$ and $\dot{x}_2$ from (7.1) to get

$$\dot{V} = -x_1^4 + \cos 2t x_1^2 x_2^3 + \left(1 + \sin^2 t\right) x_1^2 x_2^3 - 2x_2^6$$

in which after replacing $\cos 2t$ by $\cos^2 t - \sin^2 t$ we can obtain

$$\dot{V} = -x_1^4 + \left(1 + \cos^2 t\right) x_1^2 x_2^3 - 2x_2^6. \tag{7.1}$$

The expression in (7.1) is a quadratic function in terms of x_1^2 and x_2^3. Let us prove that $\dot{V}$ is a negative definite function. To this end we use Sylvester's criterion. The matrix of coefficients for the variables x_1^2 and x_2^3 is

$$A(x,t) = \begin{pmatrix} -1 & \frac{1}{2}\left(1+\cos^2 t\right) \\ \frac{1}{2}\left(1+\cos^2 t\right) & -2 \end{pmatrix}.$$

From which we have

$$\Delta_1 = a_{11} = -1, \quad \Delta_2 = a_{11}a_{22} - a_{12}a_{21} = 2 - \frac{1}{4}\left(1+\cos^2 t\right)^2.$$

Thus,

$$\Delta_1 \le \delta_1 = -1 < 0, \quad \Delta \ge \delta_2 = 1 > 0 \quad (\text{ for } t = \pi n, \quad n = 0, 1, 2, \ldots).$$

These inequalities show that conditions (7.7) in [11] are satisfied, and therefore, $\dot{V}$ is a negative definite function with respect to x_1^2 and x_2^3, and hence with respect to x_1 and x_2. Thus, V is positive definite, whereas its derivative with respect to time is negative definite. Therefore, for the system given in (7.1) all conditions of Liapunov's theorem of asymptotic stability are satisfied.

7.4. Investigate the stability of a perturbed motion which is governed by the following equations:

$$\begin{aligned}\dot{x}_1 &= \frac{\cos^2 t}{\sqrt{1+\sin^2 t}}\,x_1^2 - \frac{x_1 x_2^2}{\sqrt{1+\cos^2 t}}\\ \dot{x}_2 &= \frac{x_1^2 x_2}{\sqrt{1+\cos^2 t}} - x_2^2\end{aligned}$$

S o l u t i o n:

Let us consider the positive definite function $V = \frac{1}{2}\left(x_1^2 + x_2^2\right)$. By virtue of the given expressions, the time derivative of this function is

$$\dot{V} = \frac{\cos^2 t}{\sqrt{1+\sin^2 t}} x_1^3 - x_2^3.$$

The function V is positive definite in the whole x_1, x_2-plane, while its derivative is positive, in the sense of Chetaev, in the domain $x_1 > 0$, $x_2 < 0$. Thus, the equilibrium positions $x_1 = 0$ and $x_2 = 0$ are unstable (Chetaev Theorem).

7.5. The equation of a perturbed motion is

$$\ddot{x} + \left(k - 2\cos^2 0.05\,t\right) x = 0. \tag{7.2}$$

Determine for what values of k parametric resonance occurs.

S o l u t i o n:

This equation could be easily transformed into Mathieu's equation (7.89) in [11]. To this end, we use

$$2\cos^2\alpha = 1 + \cos 2\alpha.$$

Then (7.3) can be written as

$$\frac{d^2x}{dt^2} + (k - 1 - \cos 0.1t)\, x = 0. \tag{7.3}$$

Now, let us introduce the nondimensional time, $\tau = 0.1t$, so that

$$\tfrac{d\tau}{dt} = 0.1,$$

$$\dot{x} = \tfrac{dx}{dt} = \tfrac{dx}{d\tau}\tfrac{d\tau}{dt} = 0.1\tfrac{dx}{d\tau},$$

$$\ddot{x} = 0.01\tfrac{d^2x}{d\tau^2}.$$

Then equation (7.4) can be written as

$$\frac{d^2x}{d\tau^2} + (\delta + \varepsilon\cos\tau)\, x = 0,$$

where

$$\delta = \frac{(k-1)}{0.01}; \quad \varepsilon = -0.01.$$

For small ε parametric resonance occurs at the points $\delta = \dfrac{n^2}{4}$. These points correspond to $k = 1 + \dfrac{n^2}{4}0.01 \quad (n = 0, 1, 2, 3, \ldots)$.

Chapter 8

Structural Stability

Buckling analysis is an important consideration in the design of elastic structures in various branches of engineering such as naval architecture, missile and rocket manufacturing, and civil and mechanical structures. In this Chapter the stability of elastic structures under static loading is investigated. Three main methods used in such an investigation are discussed briefly[1]. Using these approaches stability of equilibrium states under conservative and non-conservative loads can be analyzed by considering the corresponding critical loads.

Equilibrium method of stability analysis. Euler approach
In classical problems of linear elasticity where infinitely small deformations are assumed, the equilibrium conditions are assumed to be satisfied by the forces acting on the undeformed elastic system. This assumption which is essential for Kirchhoff's general uniqueness theorem [9, 12]leads to unique solutions for such linear problems. On the other hand, in formulating buckling problems this assumption is dropped and the equilibrium conditions are satisfied by the forces acting on the deformed elastic system. This leads to an essentially nonlinear formulation of such problems in the sense that displacements are not linearly proportional to the externally applied loads, and, in fact, often the deformation of a structure will not be uniquely determined by the applied loading.

[1] In some literature a fourth method, called "imperfection method" is also considered [14].

According to Kirchhoff's theorem there is only one set of solutions of stresses, strains, and displacements for an elastic body in equilibrium, satisfying all basic equations of linear elasticity for a given body force and boundary conditions. In fact, any two sets of solution for the same body force and boundary conditions, at most may differ only by the rigid body displacement of the system, i.e., the difference in any two sets of solution describes the rigid body motion of the system. Therefore, the solution of the such a linearly formulated problem is always stable. The sufficient condition required for satisfying Kirchhoff's theorem is that the potential energy of the elastic system should be a positive definite function.

Buckling equations are obtained by considering variations of the nonlinear equations. To this end, each unknown x in these equations is replaced by $x^0 + \delta x$. Here, the x^0 describe the "initial equilibrium" state. The stability of this initial state which satisfies the nonlinear system of equations is to be investigated. The δx describe adjacent equilibrium states that are infinitesimally close to the initial state. They satisfy the linear homogeneous equations (the buckling equations) and the homogeneous boundary conditions, that are obtained as the result of linearizing the initial nonlinear equations by δx (see Section 1.1 in [11]). Then, considering the non-trivial solutions of the buckling equation the critical load(s) may be determined. In dealing with buckling problems it is convenient to assume that the load varies proportionally to a loading parameter $\lambda > 0$. Then, the variables x_0 describing the initial equilibrium state and the coefficients of the buckling equation depend on λ. In this way, the buckling problem is reduced to an eigenvalue problem. The least (positive) eigenvalue is taken as the first critical value $\lambda = \lambda_*$ leading to the corresponding buckling mode. Such an approach is called *equilibrium or Euler analysis of stability* due to L. Euler who in 1744 used this approach to study the stability of axially compressed bars. His paper [7] is considered to be the first work on structural stability.

Example 8.1

Apply Euler analysis to obtain the critical buckling load for a simply supported bar under axial compression (Fig. 8.1).

Here E is Young's modulus, I is the moment of inertia of the cross-section of the bar with respect to the axis about which the buckling is being considered, l is the bar length, and P is the axial force.

Figure 8.1: Example 8.1

S o l u t i o n:
The equilibrium of the bar is governed by the equation

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}=0, \tag{8.1}$$

where w is the lateral displacement of the bar. The boundary conditions for a simply supported bar are

$$w(0)=0, \quad \left.\frac{d^2w}{dx^2}\right|_{x=0}=0, \quad w(l)=0, \quad \left.\frac{d^2w}{dx^2}\right|_{x=l}=0.$$

We seek those values of P for which the system admits nontrivial equilibrium states. To this end, we consider the solution of (8.1) in the form

$$w(x)=A\sin kx+B\cos kx+Cx+D \tag{8.2}$$

where

$$k^2=\frac{P}{EI} \qquad \text{or} \qquad P=EIk^2.$$

Substituting (8.2) into the boundary conditions we get

$$B=C=D=0, \quad \sin kl=0. \tag{8.3}$$

The lowest non-zero value of kl satisfying (8.3) is π. Therefore,

$$P_{cr}=\frac{\pi^2EI}{l^2}.$$

This approach suffers from a few shortcomings that need to be pointed out. Firstly, this approach does not address the question of stability of a structure directly. It deals with this question in a rather indirect manner by seeking the loading(s) at which there exist infinitesimally close adjacent equilibrium states. Secondly, the Euler approach can not consider the mass distribution in the system. Finally, under some circumstances it may provide the wrong results. Examples of such cases are either

1) when the initial equilibrium state becomes unstable without any infinitesimally close equilibrium states appearing, and the system starts to experience flutter (see second part of solution of Problem 8.1);

2) when the equilibrium state under investigation is stable and the close equilibrium states exist, yet they are unstable (see [14]). Moreover, for many non-conservative systems the results obtained by means of equilibrium approach are not correct (for example, stability analysis of a bar under an axial compressive follower force). Nevertheless, for a large number of conservative systems this approach provides correct results. Unfortunately, to date no reliable criteria have been established that can be used to classify the type of problems or the conditions for which equilibrium method will yield the correct results.

Energy method of stability analysis. Lagrange–Dirichlet approach.

In the stability analyses of an equilibrium state it is convenient to make use of energy principles. These are based on the Lagrange–Dirichlet theorem that states: If for a mechanical system under static conservative forces with ideal holonomic constraints[2] the potential energy at an equilibrium state attains a strict minimum (i.e., is positive definite), then this state is stable. For example, to prove that a system with one degree of freedom has a stable equilibrium state, we should evaluate the potential energy of the system Π, and prove that $\Pi' = 0$ and $\Pi'' > 0$ at this state.

Example 8.2

Investigate the stability of the buckled bar given in Example 8.1. Such a buckled form represents the post-buckling state of the bar.

[2] The work done by *ideal constraints* during any virtual displacements is zero. Holonomic constraints do not depend on velocities and accelerations.

Solution:

When one end of the bar is allowed to be displaced only in the axial direction, then the main post-buckling deformation is of bending form. We assume that the bar is inextensible along its longitudinal axis (elastic axis). Then, the axial displacement $u(s)$ is

$$u\,(s) = -\int_0^s (1-\cos\theta)\,ds,$$

where s is the length along the elastic axis (Fig. 8.1), and $\theta = \dfrac{dw}{ds}$ is the slope of the buckled bar.

The bending moment is

$$M = EI\frac{d\theta}{ds},$$

so that the potential energy of the bar becomes

$$\Pi = \int_0^l \left[\frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2 - P\,(1-\cos\theta)\right]ds$$

For small, yet finite deformations the deformed elastic curve may be approximated by the first mode, i.e., $w_1 = \sin\dfrac{\pi s}{l}$. Then, the solution for the buckled bar in the first approximation can be given as

$$\theta\,(s) = c\theta_1\,(s)\,, \quad \text{where} \quad \theta_1\,(s) = \cos\frac{\pi s}{l}$$

Since θ is small we have

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 - \cdots,$$
$$1-\cos\theta = \tfrac{1}{2}\theta^2 - \tfrac{1}{24}\theta^4 - \cdots$$

Then,

$$\Pi \simeq \frac{EI\pi^2c^2}{4l} - \frac{Pc^2l}{4} + \frac{Pc^4l}{64} = \frac{EI}{4l}\left[c^2 - \frac{P}{P_{cr}}c^2 + \frac{P}{16P_{cr}}c^4\right], \quad P_{cr} = \frac{\pi^2EI}{l^2}$$

and

$$\Pi'_c \simeq \frac{EI}{4l}\left[2c\left(1-\frac{P}{P_{cr}}\right) + \frac{P}{4P_{cr}}c^3\right], \tag{8.4}$$

$$\Pi_c'' \simeq \frac{EI}{4l}\left[2\left(1-\frac{P}{P_{cr}}\right)+\frac{3P}{4P_{cr}}c^2\right]$$

Equation (8.4) has two roots: $c_1 = 0$ which corresponds to the undeformed bar, and $c_2 = \pm 2\sqrt{2}\sqrt{\frac{(P-P_{cr})}{P}}$ which exists only for $P > P_{cr}$, and corresponds to the buckled form represented by the first mode.

For $c = 0$

$$\Pi_c'' = \frac{EI}{2l}\left(1-\frac{P}{P_{cr}}\right)$$

is positive only when $P < P_{cr}$, i.e., the undeformed bar is stable for all P less than the critical load while it is unstable for all $P > P_{cr}$. For $c^2 = \frac{8\,(P-P_{cr})}{P}$

$$\Pi_c'' = \frac{EI}{4l}\left[2\left(1-\frac{P}{P_{cr}}\right)+\frac{3P}{4P_{cr}}\frac{8\,(P-P_{cr})}{P_{cr}}\right] \simeq \frac{4EI}{l}\frac{(P-P_{cr})}{P_{cr}}$$

is positive when $P > P_{cr}$, i.e, the buckled bar is stable for all $P > P_{cr}$. For a more detailed treatment of this problem one may refer to [1, 14, 5, 3].

Kinetic method of stability analysis. Lagrange–Liapunov approach

The most general approach to stability analysis is to consider the free vibration of the elastic system about its equilibrium state and investigate the perturbation of this motion. This method, referred to as the *kinetic method of stability analysis*, was initially proposed by Lagrange for conservative mechanical systems. Later on, A.M. Liapunov developed a rigorous mathematical theory of stability of motion. To this end, he proposed that the equilibrium state (or equilibrium motion) of a mechanical system is considered stable if the deviation from this state is as small as desired for any sufficiently small perturbation. The kinetic analysis can be applied to determine the stability of equilibrium state in any structural problem, but it should be noted that the stability analysis of a perturbed motion is a much more difficult problem than that considered in the Euler approach in which one determines those loading conditions for which the system admits nontrivial equilibrium states.

Therefore, unless absolutely necessary, the kinetic analysis is rarely used in the stability analysis of equilibrium states. It is important to note, however, that for certain stability problems this method is the only viable and reliable approach. Examples of such problems are the stability analysis of motion under dynamic and or non-conservative loads, such as the motion of an elastic body in a gas flux, and analysis of problems due to parametric instability.

A load is considered to be conservative, if the work done by it during a deformation depends only on the two initial and final states of deformation and is independent of its path. In particular, a load that does not change in magnitude and direction is conservative. However, these loads do not comprise the entire class of conservative loads. Hydrostatic pressure forces, the direction of which depend on the deformation state, are also conservative. Note that only the dynamic analysis will yield correct results when non-conservative forces are involved[3].

Example 8.3

Using kinetic analysis investigate the stability of the equilibrium state $w = 0$ for the simply supported bar under axial compression (Fig. 8.1). The bar has a material density of ρ. (Note that, unlike in equilibrium method, using the kinetic approach one needs to know the mass distribution of the system.)

S o l u t i o n:

The small free vibrations of the bar near the equilibrium state $w = 0$ are given by

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{8.5}$$

where t is time. The general solution of (8.5) has the form

$$w\left(x,t\right) = A\sin\left(\omega t + \alpha\right)W\left(x\right),$$

where A and α may be found from the initial conditions. The equilibrium state is stable if the frequency ω is real. Otherwise we have two complex conjugate frequencies that correspond to two solutions one of

[3] According to [11] (see Section 6.2) a force $\boldsymbol{R} = -\boldsymbol{P}\boldsymbol{q}$ whose components are linear functions of the generalized coordinates $\boldsymbol{q}$ with a skew-symmetric matrix of coefficients $\boldsymbol{P} = (p_{kj})$ is called a non-conservative force.

which increases unbounded with time. To determine $W(x)$ we consider the equation

$$EI\frac{d^4W}{dx^4} + P\frac{d^2W}{dx^2} - \rho\omega^2 W = 0, \tag{8.6}$$

with the boundary conditions

$$W(0) = 0, \quad \left.\frac{d^2W}{dx^2}\right|_{x=0} = 0, \quad W(l) = 0, \quad \left.\frac{d^2W}{dx^2}\right|_{x=l} = 0.$$

The characteristic equation of (8.6) is

$$k^4 + \lambda k^2 - \Omega^2 = 0, \qquad \lambda = \frac{P}{EI}, \qquad \Omega^2 = \omega^2\frac{\rho}{EI}$$

and its solution has the general form

$$W(x) = A\sinh k_2 x + B\cosh k_2 x + C\sin k_1 x + D\cos k_1 x, \tag{8.7}$$

where $k_1^2 = \frac{1}{2}\left(\sqrt{\lambda^2 + 4\Omega^2} + \lambda\right)$, and $k_2^2 = \frac{1}{2}\left(\sqrt{\lambda^2 + 4\Omega^2} - \lambda\right)$. Substituting (8.7) into the boundary conditions we get $B = D = 0$, and

$$\begin{vmatrix} \sinh k_2 l, & \sin k_1 l \\ k_2^2 \sinh k_2 l, & -k_1^2 \sin k_1 l \end{vmatrix} = 0.$$

The lowest non-zero solution of this equation is $k_1 l = \pi$. We note that Ω is real if $\Omega^2 = k_1^2\left(k_1^2 - \lambda\right) \geq 0$, i.e., if $\lambda \leq k_1^2 = \dfrac{\pi^2}{l^2}$. Then the critical load corresponds to the largest λ for which the last inequality holds, i.e., $P_{cr} = \dfrac{\pi^2 EI}{l^2}$. The equilibrium state is stable if $P < P_{cr}$.

Problems

8.1. The horizontal pipe AB carries a fluid as shown in Fig. 8.2. The pipe has a length L, modulus of elasticity of E, and the moment of inertia I. The velocity of the flow is V with a mass of m per second flowing through the pipe. Determine the stability of the tube if:

1) both ends of the pipe are simply supported;
2) one end of the pipe is fixed with the other end being free.

S o l u t i o n:

The equilibrium equation of the pipe is

$$EI\frac{d^4y}{dx^4} + mV\frac{d^2y}{dx^2} = 0 \tag{8.8}$$

Figure 8.2: Problem 8.1.

or

$$\frac{d^4y}{dx^4} + K^2 \frac{d^2y}{dx^2} = 0,$$

where $K^2 = \frac{mV}{EI}$.

The solution of this equation has the form

$$y = A \sin Kx + B \cos Kx + Cx + D.$$

1) If both ends of the pipe are simply supported then the boundary conditions are:

$$y(0) = 0, \quad \frac{d^2y}{dx^2}|_{x=0} = 0, \quad y(L) = 0, \quad \frac{d^2y}{dx^2}|_{x=L} = 0.$$

From these conditions it follows that

$$B = 0, \quad C = 0, \quad D = 0$$

and $\sin KL = 0$. This means that the critical value of the flow parameter is

$$(mV)|_{cr} = \frac{\pi^2 EI}{L^2}.$$

2) If one end of the pipe is fixed with the other end being free, then the boundary conditions become:

$$y(0) = 0, \quad \frac{dy}{dx}|_{x=0} = 0, \quad \frac{d^2y}{dx^2}|_{x=L} = 0 \quad \frac{d^3y}{dx^3}|_{x=L} = 0. \tag{8.9}$$

In this case, no non-trivial solution to equation (8.8) is available that can satisfy the boundary conditions (8.9). Hence, in this case we should consider the equation of perturbed motion:

$$\frac{m}{V}\frac{\partial^2 y}{\partial t^2} + mV\frac{\partial^2 y}{\partial x^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

or

$$\frac{\partial^4 y}{\partial x^4} + \frac{mV}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{m}{EIV}\frac{\partial^2 y}{\partial t^2} = 0.$$

Using separation of variables $y\left(x,t\right) = X\left(x\right)T\left(t\right),$ we get

$$\frac{1}{X}\left(\frac{d^4X}{dx^4} + K^2\frac{d^2X}{dx^2}\right) = -\frac{m}{EIV}\frac{1}{T}\frac{d^2T}{dt^2} = \Lambda^2.$$

From these two equalities we conclude that

$$\frac{d^2T}{dt^2} + \frac{\Lambda EIV}{m}T = 0, \quad T = A\sin\left(\omega t + \alpha\right), \quad \omega^2 = \frac{\Lambda EIV}{m}$$

and

$$\frac{d^4X}{dx^4} + K^2\frac{d^2X}{dx^2} - \Lambda X = 0. \tag{8.10}$$

The characteristic equation of (8.10) is

$$s^4 + K^2 s^2 - \Lambda = 0$$

with the roots

$$s_1^2 = \frac{\sqrt{K^4 + 4\Lambda} - K^2}{2}, \quad s_2^2 = -\tilde{s}_2^2 = \frac{\sqrt{K^4 + 4\Lambda} + K^2}{2}, \tag{8.11}$$

so that the solution of equation (8.10) can be given as

$$X\left(x\right) = A\sin\tilde{s}_2 x + B\cos\tilde{s}_2 x + C\sinh s_1 x + D\cosh s_1 x.$$

From boundary conditions (8.9) we have

$$X\left(0\right) = 0, \quad \frac{dX\left(0\right)}{dx} = 0, \quad \frac{d^3X\left(L\right)}{dx^3} = 0 \quad \frac{d^2X\left(L\right)}{dx^2} = 0.$$

Using these we get

$$B = -D, \quad A\tilde{s}_2 = -Cs_1$$

and

$$Cs_1\left(\tilde{s}_2\sin\left(\tilde{s}_2L\right)+s_1\sinh\left(s_1L\right)\right)+D\left(\tilde{s}_2^2\cos\left(\tilde{s}_2L\right)+s_1^2\cosh\left(s_1L\right)\right)=0$$

$$Cs_1\left(\tilde{s}_2^2\cos\left(\tilde{s}_2L\right)+s_1^2\cosh\left(s_1L\right)\right)+D\left(s_1^3\sinh\left(s_1L\right)-\tilde{s}_2^3\sin\left(\tilde{s}_2L\right)\right)=0.$$

The last two equations constitute a system of linear homogeneous equations in C and D, the determinant of which must vanish in order to provide a non-trivial solution, i.e., characteristic equation of the perturbation becomes:

$$s_1^4+\tilde{s}_2^4+2s_1^2\tilde{s}_2^2\cos\left(\tilde{s}_2L\right)\cosh\left(s_1L\right)-s_1\tilde{s}_2\left(s_1^2-\tilde{s}_2^2\right)\sinh\left(s_1L\right)\sin\left(\tilde{s}_2L\right)=0.$$

Now from relations (8.11) we get

$$s_1^4+\tilde{s}_2^4=K^4+2\Lambda,\quad s_1^2\tilde{s}_2^2=\Lambda,\quad \tilde{s}_2^2-s_1^2=K^2.$$

Using these expressions the characteristic equation reduces to

$$F\left(K,\Lambda\right)=K^4+2\Lambda+2\Lambda\cos\tilde{s}_2L\cosh s_1L+\sqrt{\Lambda}K^2\sinh s_1L\sin\tilde{s}_2L=0,$$

or in dimensionless form to

$$F\left(\bar{K},\bar{\Lambda}\right)=\bar{K}^2+2\bar{\Lambda}+2\bar{\Lambda}\cos\bar{s_2}\cosh\bar{s_1}+\sqrt{\bar{\Lambda}}\bar{K}\sinh\bar{s_1}\sin\bar{s_2}=0.$$

Here

$$\bar{s_2}=\tilde{s}_2L,\quad \bar{s_1}=s_1L,$$

with the nondimensional frequency parameter $\bar{\Lambda}$, and the nondimensional parameter $\bar{K}$ that characterizes the flow, defined as

$$\bar{\Lambda}=\Lambda L^4=\omega^2\frac{mL^4}{EIV},\quad \bar{K}=K^2L^2=\frac{mVL^2}{EI}.$$

The dependence of $\bar{K}$ on the frequency parameter $\bar{\Lambda}$ is shown in Fig. 8.3.

At the limit point N ($\bar{K}_{cr}\simeq 20.19$) the first and the second frequencies of the system coalesce and for $\bar{K}>\bar{K}_{cr}$ the system becomes unstable. Hence, the critical parameter of the flow is

$$\left(mV\right)|_{cr}\simeq\frac{20.19EI}{L^2}.$$

8.2. Use the kinetic approach to investigate the stability of the equilibrium state $w=0$ of a massless bar when subjected to the axial

Figure 8.3: The dependence of $\bar{K}$ on the frequency parameter $\bar{\Lambda}$

Figure 8.4: Problem 8.2.

follower force. The bar is clamped at the bottom but it carries a mass m at its top (Fig. 8.4).

S o l u t i o n:

The governing differential equation of the beam is

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0 \tag{8.12}$$

with the boundary conditions

$$w = w' = 0 \quad \text{at} \quad x = 0; \quad \text{and} \quad w'' = 0, \quad EIw''' = m\frac{\partial^2 w}{\partial t^2} \quad \text{at} \quad x = l.$$

We consider the solution in the form

$$w(x,t) = f(x)\sin(\lambda t + \varepsilon), \tag{8.13}$$

where λ is the frequency, and $f(x)$ and ε are unknowns. Substituting (8.13) into (8.12) we get

$$f^{IV} + k^2 f'' = 0, \quad k^2 = \frac{P}{EI}.$$

Hence, the solution has the form

$$f(x) = A + Bx + C\cos kx + D\sin kx$$

Substituting this into the boundary conditions we get the characteristic equation from which we get

$$\lambda^2 = \frac{k^3 EI}{ml^3}\frac{1}{\sin kl - kl\cos kl}.$$

Then the displacement of the bar is given by

$$w(x,t) = C(\tan kl - kx + \sin kx - \tan kl\cos kx)\sin(\lambda t + \varepsilon).$$

When λ is real the beam oscillates about the equilibrium state $w = 0$, otherwise it will diverge from this state. Therefore, the system is stable if

$$\frac{k^3 EI}{ml^3}\frac{1}{\sin kl - kl\cos kl} \geq 0.$$

Letting $z = kl$, the following inequality will be satisfied

$$\sin z \geq z\cos z$$

when $0 \leq z \leq 4.493$. The critical load corresponds to the largest value of z for which the above inequality holds, i.e.,

$$P_{cr} = \frac{20.19EI}{l^2}.$$

For $P \leq P_{cr}$ the system is stable.

8.3. The equilibrium equation for a cylindrical shell spinning with a constant angular velocity around its axis of symmetry is:

$$L_0 U + 2\omega\Omega L_c U + \Omega^2 L_\Omega U + \omega^2 U = 0,$$

where

$$L_c = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix} \quad L_\Omega = \begin{pmatrix} -m^2 & 0 & 0 \\ 0 & -m^2 & 2m \\ 0 & 2m & -m^2 \end{pmatrix}.$$

Here m is the circumferential wave number, Ω is the angular velocity of the spin , ω is the natural frequency of shell vibrations, and $U = (u, v, w)$ is the displacement vector with (u, v, w) being the displacement components in the local coordinate system along the axial, circumferential and normal directions, respectively. L_0 is a linear differential operator describing the non-spinning shell. Investigate the stability of the shell.

S o l u t i o n:

We will use the static analysis, i.e., we will try to determine if there exist any angular velocities (critical speeds) for which the frequency ω would vanish. Consider the non-spinning shell, the equilibrium of which is described by

$$L_0 U_0 + \omega_0^2 U_0 = 0,$$

where U_0 and ω_0 denote, respectively, the mode shape and the natural frequency of the non-spinning shell. Assuming that the displacements of the spinning shell are approximately equal to those of the non-spinning shell, we have

$$2\omega\Omega L_c U + \Omega^2 L_\Omega U + \omega^2 U - \omega_0^2 U = 0.$$

The characteristic equation of this is

$$|2\omega\Omega L_c + \Omega^2 L_\Omega + \omega^2 I - \omega_0^2 I| = 0,$$

where I is the identity matrix. This equation has six roots:

$$\begin{gathered} \omega = -\sqrt{\Omega^2 m^2 + \omega_0^2}, \\ \omega = \sqrt{\Omega^2 m^2 + \omega_0^2}, \\ \omega = -\Omega - \sqrt{\Omega^2 (1-m)^2 + \omega_0^2}, \\ \omega = -\Omega + \sqrt{\Omega^2 (1-m)^2 + \omega_0^2}, \\ \omega = \Omega - \sqrt{\Omega^2 (1+m)^2 + \omega_0^2}, \\ \omega = \Omega + \sqrt{\Omega^2 (1+m)^2 + \omega_0^2}. \end{gathered}$$

When $\Omega \neq 0$, and $\omega \neq 0$, only the fourth root will vanish provided $m = 1$. Thus, the critical speed is equal to ω_0. A similar result can be obtained for spinning shafts.

8.4. The critical axial compressive load for a cylindrical shell of medium height, i.e., when $\sqrt{h/R} < L/R < \sqrt{R/h}$, where h is the shell thickness, L is the shell height and R is the shell radius, can be determined by using the equations of shallow shells. Using non-dimensional variables the governing differential equations of a cylindrical shell with an initial imperfection $\tilde{w}$ are ([6]):

$$\begin{aligned} &\Delta^2 w - \frac{\partial^2 \Phi}{\partial x^2} - L\left(\tilde{w} + w, \Phi\right) = 0, \\ &\Delta^2 \Phi + \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} L\left(w, w\right) + L\left(\tilde{w}, w\right) = 0, \end{aligned} \tag{8.14}$$

where w is the deflection function, Φ is the force function and,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad L\left(u, v\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}.$$

Here x and y represent the coordinates in the axial and circumferential directions, respectively. Moreover, the non-dimensional variables are related to their corresponding physical variables (*) as follows:

$$w = \frac{w^* c}{h}, \quad \tilde{w} = \frac{\tilde{w}^* c}{h}, \quad (x, y) = \frac{(x^*, y^*)\sqrt{c}}{\sqrt{hR}}, \quad \Phi = \frac{\Phi^* c^2}{E h^3},$$

where E is Young's modulus, ν is Poisson's ratio, and $c^2 = 12\left(1 - \nu^2\right)$. The resultant force T, representing the load parameter, is

$$T = \frac{\partial^2 \Phi}{\partial y^2}, \quad T = \frac{T^* R c}{E h^2}.$$

1) Determine the stability of a simply supported shell. (Ignore the boundary conditions in the circumferential direction.)

2) For a cylindrical shell with an axisymmetric imperfection $\tilde{w} = \xi \cos x$, investigate its bifurcation into a non-axisymmetric form with equal wavelengths in the axial and circumferential directions.

S o l u t i o n:

1) We represent the stress function as $\Phi = \dfrac{T y^2}{2} + \Phi_a$, where Φ_a is some additional stress function. Next, linearizing system (8.14) gives

the governing differential equations for a shell without the initial imperfections, i.e., for a shell with $\tilde{w} = 0$,

$$\Delta^2 w - \frac{\partial^2 \Phi_a}{\partial x^2} - T\frac{\partial^2 w}{\partial x^2} = 0, \qquad \Delta^2 \Phi_a + \frac{\partial^2 w}{\partial x^2} = 0,$$

or

$$\Delta^4 w - \Delta^2 \frac{\partial^2 \Phi_a}{\partial x^2} - T\Delta^2 \frac{\partial^2 w}{\partial x^2} = 0, \quad \Delta^2 \Phi_a = -\frac{\partial^2 w}{\partial x^2};$$

and finally as

$$\Delta^4 w - T\Delta^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^4 w}{\partial^4 x} = 0. \tag{8.15}$$

First, we assume that the buckling mode of the shell is axisymmetric. Then equation (8.15) becomes

$$\frac{d^8 w}{dx^8} - T\frac{d^6 w}{dx^6} + \frac{d^4 w}{dx^4} = 0, \tag{8.16}$$

and we seek the solution of this equation as

$$w = W \sin \frac{\pi n x^*}{L} = W \sin \lambda x, \quad \text{with} \quad \lambda = \frac{\pi n \sqrt{hR}}{L\sqrt{c}},$$

where n is wave number in the axial direction, and

$$w\left(0\right) = w\left(L\right) = \frac{d^2 w\left(0\right)}{dx^2} = \frac{d^2 w\left(L\right)}{dx^2}.$$

Substituting the assumed solution into equation (8.16) we get

$$W\left[\lambda^8 + T\lambda^6 + \lambda^4\right] = 0, \quad \text{or} \quad T = -\left(\lambda^2 + \lambda^{-2}\right),$$

so that the critical load ($\min |T|$) is equal to -2 for $\lambda = 1$. In terms of dimensional variables $T_{cr} = \dfrac{Eh^2}{\sqrt{3(1-\nu^2)}R}$.

Next, we assume that the buckling mode is non-axisymmetric, i.e., $w = W \sin(px + \alpha) \sin(qy + \beta)$, where α and β can be equal to either 0 or π. Then, after substitution into equation (8.15), we obtain

$$\begin{array}{c} W\left[\left(p^2+q^2\right)^4 + T\left(p^2+q^2\right)^2 p^2 + p^4\right] = 0, \quad \text{or} \\ -T = f\left(p,q\right) = \frac{\left(p^2+q^2\right)^2}{p^2} + \frac{p^2}{(p^2+q^2)^2} \end{array}.$$

In our investigation of buckling, we are interested only in the lowest eigenvalue. Therefore,

$$T_{cr} = -2 \quad \text{when} \quad p^2 + q^2 = p. \tag{8.17}$$

Thus, according to the classical shell theory there exist an infinite number of buckling modes that are characterised by the wave length parameters p and q in the axial and circumferential directions, respectively. These parameters must satisfy the relation $p^2 + q^2 = p$. For example, a pair (p, q)=$(1, 0)$ determines an axisymmetric mode.

Experimental results have indicated that for buckling of shells the wavelength parameters in the axial and circumferential directions are close to each other [13]. The "squares" form with $p \approx q$ is the most sensitive to the imperfections ([10, 8]).

2) Here we consider a cylindrical shell with an initial axisymmetric imperfection $\tilde{w} = \xi \cos x$ that buckles into a non-axisymmetric form with equal wavelengths in the axial and circumferential directions. In this case, the pre-buckling axisymmetric deformation w_0 of the shell may be obtained from the equations

$$\frac{d^4 w_0}{dx^4} - T\frac{d^2 (w_0 + \tilde{w})}{dx^2} - \frac{d^2 \Phi_0}{d^2 x^2} = 0, \quad \frac{d^4 \Phi_0}{dx^4} + \frac{d^2 w_0}{dx^2} = 0,$$

or

$$\frac{d^4 w_0}{dx^4} - T\frac{d^2 w_0}{dx^2} + w_0 = -T\frac{d^2 \tilde{w}}{dx^2}, \quad w_0 = \frac{-T\xi}{T+2}\cos x.$$

For the non-axisymmetric component of the deflection function, w_1, and the load function, Φ_1, we get from (8.14)

$$\Delta^2 w_1 - T\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2}\left(\frac{2\xi}{2+T}\right)\cos x - \frac{T\xi}{2+T}\frac{\partial^2 w_1}{\partial y^2} = 0, \tag{8.18}$$

$$\Delta^2 \Phi_1 + \frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_1}{\partial y^2}\left(\frac{2\xi}{2+T}\right)\cos x = 0. \tag{8.19}$$

According to (8.17), if the non-axisymmetric form has equal wavelengths in the axial and circumferential directions, then $p = q = 1/2$, and we should seek the buckling mode in the form $w = W_1 \cos\frac{x}{2}\cos\frac{y}{2}$. In [10, 8] it is shown that the lowest buckling load corresponds to this mode. Equation (8.19) gives

$$\Phi_1 = F_1 \cos\frac{x}{2}\cos\frac{y}{2} + F_3 \cos\frac{3x}{2}\cos\frac{y}{2} + \cdots, \quad F_1 \simeq W_1\left(1 - \frac{\xi}{2+T}\right).$$

Substitution into equation (8.18) results in

$$\frac{1}{4}W_1\left[1+T+1-\frac{2\xi}{2+T}+\frac{T\xi}{2\,(2+T)}\right]=0,$$

or, considering that T is close to the classical load -2, for $\xi \ll 1$

$$2+T-\frac{3\xi}{2+T}=0, \quad \text{or} \quad T=-2+\sqrt{3\xi}.$$

Hence, if the amplitude of the initial axisymmetric imperfection is equal to ξ, then the absolute value of the critical load (bifurcation load) decreases by $\sqrt{3\xi}$, or, in dimensional variables,

$$T_{cr}=\frac{Eh^2}{\sqrt{3(1-\nu^2)}R}\left(1-\frac{3^{3/4}(1-\nu^2)^{1/4}{\xi_*}^{1/2}}{2h^{1/2}}\right),$$

where $\xi=\dfrac{\xi^* c}{h}$.

Chapter 9

Frequency Method of Stability Analysis

9.1. The governing differential equations of a gyroscope are

$$\frac{d\vartheta}{dt} = -\vartheta - \sigma,$$
$$\frac{d\sigma}{dt} = \vartheta + \sigma - \varphi(\sigma),$$

where ϑ is the roll angle of the plant, σ is a parameter which is proportional to the angle of rotation of the inner gimbals of the gyroscope, $\varphi(\sigma)$ is the function that describes the change of the control moment, i. e., and satisfies the following conditions:

$$\varphi(0) = 0, \quad \varphi(\sigma)\sigma > 0 \quad \text{for} \quad \sigma \neq 0, \quad \int_0^\infty \varphi(\sigma)d\sigma = \infty.$$

Investigate the stability of the system.

S o l u t i o n:

Find the transfer function (from the input $-\varphi$ to the output σ):

$$p\vartheta = -\vartheta - \sigma,$$
$$p\sigma = \vartheta + \sigma - \varphi.$$

To eliminate ϑ, from the first equation we have

$$(p+1)\,\vartheta = -\sigma, \quad \vartheta = -\frac{\sigma}{p+1},$$

which upon substitution into the second equation gives

$$p\sigma = -\frac{\sigma}{p+1} + \sigma - \varphi,$$

or,

$$\left(p^2 + p\right)\sigma = -\sigma + p\sigma + \sigma - (p+1)\,\varphi.$$

Therefore,

$$\sigma = -\frac{p+1}{p^2}\varphi, \quad W(p) = \frac{p+1}{p^2}.$$

We have the critical case with two zero poles. Using Theorem 9.3 of the book [11], we get

$$\alpha = \lim_{p\to 0} p^2 \frac{p+1}{p^2} = 1 > 0,$$

$$\rho = \lim_{p\to 0} \frac{d}{dp}\left[p^2 \frac{p+1}{p^2}\right] = 1 > 0,$$

$$W\,(i\omega) = \frac{i\omega + 1}{-\omega^2}, \quad \Im W\,(i\omega) = -\frac{1}{\omega}, \quad \pi\,(\omega) = \omega\left(-\frac{1}{\omega}\right) = -1 < 0$$

Thus the given system is absolutely stable.

9.2. The behaviour of a gyroscopic system that controls the orientation of a spaceship in the pitch plane is described by the following equations:

$$\begin{aligned} &a\dot{u} + Hv = 0, \\ &b\dot{v} - Hu + \varepsilon v = \varphi(\sigma), \\ &\dot{\sigma} = u. \end{aligned}$$

Here H is the angular momentum of the gyroscope about its axis of rotation, σ is the pitch angle, $v = \beta$, is the precession angle of the gyroscope, ε is the coefficient of viscous friction, a and b are the principal moments of inertia, $\varphi(\sigma)$ is the nonlinear characteristic of the control moment that satisfies the following conditions (see Fig. 9.1 and (9.13) of the book [11]):

$$\varphi(0) = 0, \quad 0 < \frac{\varphi(\sigma)}{\sigma} < k \leq +\infty, \quad \sigma \neq 0.$$

Determine the conditions for absolute stability.

S o l u t i o n:

Find the transfer function (from the input $-\varphi$ to the output σ):

$$\begin{aligned} &apu + Hv = 0, \\ &bpv - Hu + \varepsilon v = \varphi, \\ &p\sigma = u. \end{aligned}$$

To eliminate u and v, from the first and the third equations we have

$$u = p\sigma, \quad v = -\frac{ap}{H}u = -\frac{ap^2}{H}\sigma,$$

which after substituting into the second equation we get

$$-\frac{abp^3}{H}\sigma - Hp\sigma - \frac{a\varepsilon p^2}{H}\sigma = \varphi$$

$$\sigma = -\frac{H}{abp^3 + a\varepsilon p^2 + H^2 p}\varphi,$$

$$W\,(p) = \frac{H}{abp^3 + a\varepsilon p^2 + H^2 p}.$$

We have the critical case with a single zero pole. Using Theorem 9.2, we get

$$\rho = \lim_{p \to 0} pW(p) = \frac{1}{H} > 0$$

$$W(i\omega) = \frac{H}{-a\varepsilon\omega^2 + i\omega\left(H^2 - ab\omega^2\right)} = H\frac{-a\varepsilon\omega^2 + i\omega\left(ab\omega^2 - H^2\right)}{a^2\varepsilon^2\omega^4 + \omega^2\left(H^2 - ab\omega^2\right)^2}.$$

Now, we check frequency condition (9.14) in [11]:

$$\frac{1}{k} + \Re\left[\left(1 + i\omega\vartheta\right)W\,(i\omega)\right] \geq 0.$$

Letting $\frac{1}{Hk} = \mu$, this condition becomes

$$\begin{array}{l}\mu\left[a^2\varepsilon^2\omega^4 + \omega^2\left(H^2 - ab\omega^2\right)^2\right] + \\ \Re\left[(1 + i\omega\vartheta)\left(-a\varepsilon\omega^2 + i\omega\left(ab\omega^2 - H^2\right)\right)\right] \geq 0\end{array},$$

or

$$\mu a^2\varepsilon^2\omega^4 + \mu a^2b^2\omega^6 - 2\mu H^2ab\omega^4 + \mu H^4\omega^2 - a\varepsilon\omega^2 + \vartheta H^2\omega^2 - \vartheta ab\omega^4 \geq 0.$$

Dividing by ω^2, and denoting $\omega^2 = t$, we have

$$\mu a^2b^2t^2 + \left(\mu a^2\varepsilon^2 - 2\mu H^2ab - \vartheta ab\right)t + \mu H^4 + \vartheta H^2 - a\varepsilon \geq 0.$$

Next, we obtain the determinant

$$\begin{array}{l}D = \left(\mu a^2\varepsilon^2 - 2\mu H^2ab - \vartheta ab\right)^2 - 4\mu a^2b^2\left(\mu H^4 + \vartheta H^2 - a\varepsilon\right) = \\ \quad \mu^2a^4\varepsilon^4 + 4\mu^2H^4a^2b^2 + \vartheta^2a^2b^2 - 4\mu^2H^2a^3b\varepsilon^2 - 2\mu\vartheta a^3b\varepsilon^2 \\ \quad +4\mu H^2\vartheta a^2b^2 - 4\mu^2H^4a^2b^2 - 4H^2\mu\vartheta a^2b^2 + 4\mu a^3b^2\varepsilon = \\ \quad a^2b^2\vartheta^2 - 2\mu a^3b\varepsilon^2\vartheta + \mu a^3\varepsilon(\mu a\varepsilon^3 + 4b^2 - 4\mu H^2b\varepsilon) < 0.\end{array}$$

It is necessary to find a ϑ such that $D < 0$. Here D is a quadratic polynomial in terms of ϑ, so that $D \to \infty$ as $|\vartheta| \to \infty$. If $\Delta > 0$, we need to find a real ϑ such that $D < 0$:

$$\Delta = 4\mu^2a^6b^2\varepsilon^4 - 4a^2b^2\left(\mu^2a^4\varepsilon^4 + 4\mu a^3b^2\varepsilon - 4\mu^2H^2a^3b\varepsilon^2\right) > 0,$$

or

$$4\mu^2H^2a^3b\varepsilon^2 - 4\mu a^3b^2\varepsilon > 0,$$

or

$$\mu H^2\varepsilon > b, \qquad \text{or} \qquad \frac{1}{kH}H^2\varepsilon > b.$$

Thus, for a nonlinear system, we obtained the following sufficient condition for absolute stability

$$H\varepsilon > kb$$

which satisfies condition (9.13) in [11], i. e., $0 < \dfrac{\varphi(\sigma)}{\sigma} < k$ for $\sigma \neq 0$, $\varphi(0) = 0$.

It's easy to check, that the condition

$$H\varepsilon \geq kb \tag{9.1}$$

is necessary for absolute stability under condition (9.13) in [11]. In fact, if we consider $\varphi(\sigma) = \lambda\sigma$, then we have a linear system whose characteristic polynomial,

$$abp^3 + a\varepsilon p^2 + H^2 p + H\lambda$$

satisfies Hurwitz's condition for $a\varepsilon H^2 > abH\lambda$, i. e., $H\varepsilon > b\lambda$. If condition (9.1) does not hold, then there exists a $\lambda \in (0, k)$ such that $H\varepsilon < b\lambda$, and the linear system is not asymptotically stable.

9.3. Consider the control system of a steam turbine with a hydraulic amplifier. The feedback is by means of a slider with friction. Under some simplifying assumptions the control system is described by following equations:

$$\begin{aligned} \dot{\zeta} &= \eta_2, \\ \eta &= \zeta + \varphi(\eta_2), \\ \tau_1\dot{\eta}_1 + \eta_1 &= -\eta, \\ \tau_2\dot{\eta}_2 + \eta_2 &= \eta_1. \end{aligned}$$

The second equation describes the behaviour of the intermediate amplifier; ζ, η_1, η_2, and η are variable parameters that describe the state of the system; τ_1 and τ_2 are the relative time constants; $\varphi(\eta_2)$ is the characteristic of the frictional force that satisfies the conditions

$$\varphi(0) = 0, \qquad \varphi(\eta_2)\eta_2 > 0 \quad \text{for} \quad \eta_2 \neq 0.$$

Determine under what conditions absolute stability prevails.

Solution:

Find the transfer function (from the input $-\varphi$ to the output η_2):

$$p\zeta = \eta_2, \quad (\tau_1 p + 1)\,\eta_1 = -\zeta - \varphi(\eta_2), \quad (\tau_2 p + 1)\,\eta_2 = \eta_1.$$

Eliminate ζ and η_1 to get

$$(\tau_1 p + 1)\,(\tau_2 p + 1)\,\eta_2 = -\frac{\eta_2}{p} - \varphi.$$

$$\eta_2 = -\frac{p}{\tau_1\tau_2 p^3 + (\tau_1 + \tau_2)\,p^2 + p + 1}\varphi.$$

$$W(p) = \frac{p}{\alpha p^3 + \beta p^2 + p + 1},$$

where $\alpha = \tau_1\tau_2$, $\beta = \tau_1 + \tau_2$. In Example 9.4 in [11] it is shown that this transfer function satisfies the frequency condition (9.14) of Theorem 9.1 for $\tau_1 + \tau_2 > \tau_1\tau_2$.

9.4. Consider a control system of a steam turbine with two amplifiers connected in series and the steam boiler. The piston of the system actuator is subjected to a nonlinear friction. The equations of the system are

$$\begin{aligned} \dot{\zeta} &= -\pi, \\ \psi_\eta \dot{\eta} &= \zeta - \eta, \\ \dot{\xi} &= \varphi(\sigma), \quad \sigma = \eta - \xi, \\ \psi_\pi \dot{\pi} + \pi &= \xi. \end{aligned}$$

In these equations, ϑ, η, ξ, and π are relative coordinates of the machine, the preliminary amplifier, the actuator and steam pressure, respectively, ψ_η and ψ_π are the relative positive time constants of the preliminary amplifier and the steam boiler. The nonlinear friction satisfies the conditions

$$\varphi(0) = 0, \quad \text{for} \quad \sigma \neq 0 \quad \varphi(\sigma)\sigma > 0, \quad \int_0^\infty \varphi(\sigma) d\sigma = +\infty.$$

Determine the domain of absolute stability.

S o l u t i o n:

Find the transfer function (from the input $-\varphi$ to the output σ):

$$p\zeta = -\pi, \quad \psi_\eta p \eta = \zeta - \eta, \quad p\xi = \varphi,$$

$$(\psi_\pi p + 1)\,\pi = \xi, \qquad \sigma = \eta - \xi.$$

Eliminate all variables except σ and φ:

$$\xi = \frac{\varphi}{p}, \quad \pi = \frac{\xi}{\psi_\pi p + 1} = \frac{\varphi}{p\,(\psi_\pi p + 1)},$$

$$\eta = \frac{\zeta}{\psi_\eta p + 1} = -\frac{\pi}{p\,(\psi_\eta \varphi + 1)} = -\frac{\varphi}{p^2\,(\psi_\pi p + 1)\,(\psi_\eta p + 1)},$$

$$\sigma = \eta - \xi = -\left[\frac{1}{p^2\left(\nu p^2 + \mu p + 1\right)} + \frac{1}{p}\right]\varphi,$$

where $\nu = \psi_\eta \psi_\pi$, and $\mu = \psi_\eta + \psi_\pi$.

Therefore

$$W(p) = \frac{\nu p^3 + \mu p^2 + p + 1}{p^2\left(\nu p^2 + \mu p + 1\right)}.$$

We have the critical case with two zero poles. Using Theorem 9.3, we have

$$\alpha = \lim_{p \to 0} p^2 W(p) = 1 > 0.$$

$$\rho = \lim_{p \to 0} \frac{d}{dp}\left(p^2 W(p)\right) = \frac{1-\mu}{(1)^2} = 1 - \mu > 0.$$

Now, the following conditions should be satisfied:

$$\mu < 1 \quad (\text{ or } \psi_\pi + \psi_\nu < 1); \tag{9.2}$$

$$W(i\omega) = -\frac{1 - \mu\omega^2 + i\omega(1 - \nu\omega^2)}{\omega^2(1 - \nu\omega^2 + i\omega\mu)};$$

$$\begin{aligned} \Im W(i\omega) &= -\tfrac{1}{\omega^2}\tfrac{(1-\mu\omega^2)(-\mu\omega)+\omega(1-\nu\omega^2)^2}{(1-\nu\omega^2)^2+\omega^2\mu^2} \\ &= \tfrac{1}{\omega}\tfrac{\mu(1-\mu\omega^2)-(1-\nu\omega^2)^2}{(1-\nu\omega^2)^2+\omega^2\mu^2}; \end{aligned}$$

$$\pi(\omega) = \omega \Im W(i\omega) = \frac{\mu(1-\mu\omega^2) - (1-\nu\omega^2)^2}{(1-\nu\omega^2)^2 + \omega^2\mu^2};$$

$$\pi(\omega) < 0, \quad \omega^2 = t,$$

$$\mu - \mu^2 t - (1 - \nu t)^2 < 0,$$

$$\nu^2 t^2 + \left(\mu^2 - 2\nu\right)t + 1 - \mu > 0. \tag{9.3}$$

Since $\mu^2 - 2\nu = \left(\psi_\pi + \psi_\eta\right)^2 - 2\psi_\pi\psi_\eta = \psi_\pi^2 + \psi_\eta^2 > 0$, it follows from condition (9.2) that condition (9.3) is satisfied for all $t \geq 0$. Also due to (9.2) we have

$$\lim_{\omega \to 0} \pi(\omega) = \frac{\mu - (1)^2}{(1)^2} = \mu - 1 < 0.$$

Thus, according to Theorem 9.2 in [11], (9.1) is a sufficient condition for absolute stability of the system.

9.5. Consider a gyrostabiliser with forced rotation of its gimbals. Assume friction in the precession axis. The behaviour of this gyrostabiliser is described by the following equations:

$$\begin{aligned}\dot{\sigma}_1 &= -\nu\sigma_1 + \mu v + \sigma_2,\\ \dot{\sigma}_2 &= -\sigma_1 - \varphi(\sigma_2),\\ \dot{v} &= \sigma_2.\end{aligned}$$

Here, σ_1, σ_2, and v are relative coordinates of the gyrostabiliser, ν and μ are constant positive parameters, and the nonlinear function $\varphi(\sigma_2)$ satisfies the conditions

$$\varphi(0) = 0, \quad \text{for} \quad \sigma_2 \neq 0 \quad \varphi(\sigma_2)\sigma_2 > 0,$$

Determine under what conditions the gyrostabiliser is absolutely stable.

S o l u t i o n:

Find the transfer function (from the input $-\varphi$ to the output σ_2):

$$\begin{aligned}(p+\nu)\sigma_1 &= \mu v + \sigma_2,\\ p\sigma_2 &= -\sigma_1 - \varphi(\sigma_2),\\ pv &= \sigma_2.\end{aligned}$$

Eliminate v and σ_1:

$$v = \frac{\sigma_2}{p}, \quad \sigma_1 = \frac{\mu\sigma_2}{(p+\nu)p} + \frac{\sigma_2}{p+\nu} = \frac{p+\mu}{p(p+\nu)}\sigma_2,$$

$$p\sigma_2 = -\frac{p+\mu}{p(p+\nu)}\sigma_2 - \varphi(\sigma_2),$$

$$\sigma_2 = -\frac{p(p+\nu)}{p^3 + p^2\nu + p + \mu}\varphi,$$

$$W(p) = \frac{p(p+\nu)}{p^3 + p^2\nu + p + \mu}.$$

Let $\nu > \mu$, then we have a non-critical case. Using Theorem 9.1 when $k = \infty$, $\vartheta = 0$, we have

$$\Re W(i\omega) = \Re\frac{\nu\omega i - \omega^2}{\mu - \nu\omega^2 + i\omega(1-\omega^2)} = \frac{\nu\omega^4 - \mu\omega^2 + \nu\omega^2(1-\omega^2)}{(\mu-\nu\omega^2)^2 + \omega^2(1-\omega^2)^2} \geq 0,$$

where $(\nu - \mu)\omega^2 \geq 0$ is satisfied for $\nu > \mu$. Thus, the condition for absolute stability is $\nu > \mu$.

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