New type of cross section singularity in backward scattering: the Coulomb glory

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For classical scattering by a central potential that exhibits Coulomb behavior (i.e., that is attractive) at small distances, the scattering angle θ tends to π as the orbital angular momentum L decreases. The differential cross section for scattering through angles close to π can be characterized by the power series expansion of the difference $\theta(L)-\pi$ in small L, only odd powers of L being present in this expansion. Expressions are found for the coefficients in the linear (c_1) and cubic (c_3)—in L—terms. It is shown that, for a broad class of screened Coulomb potentials, the coefficient c_1 vanishes at some value of the collision energy E_0. At the energy $E = E_0$ the classical cross section diverges in the case of backward scattering (the Coulomb glory); in wave mechanics the cross section possesses a maximum. The behavior of the cross section for energies close to E_0 is computed. The application of the theory to electron scattering by atoms, in which the Coulomb interaction at small distances is determined by the interaction with the nucleus (charge Z) and $E_0 = 0.0103Z^{4/3}$ keV, is discussed.

§1. INTRODUCTION

The main types of singularities occurring in classical particle scattering by a central field—rainbow, when the scattering angle as a function of the impact parameter ρ or the angular momentum L possesses an extremum, and glory (or glow), when for nonzero ρ and L the scattering angle θ vanishes (forward glory or assumes the value π (backward glory)—are well known. In both cases the effective differential cross section becomes infinite at the classical limit. In the semiclassical analysis there arise two-beam-interference-related cross section peaks described by the Airy function in the case of the rainbow and by a Bessel function in the case of glory.

Here we wish to demonstrate the existence of a new type of cross section singularity—a backward Coulomb glory that is closely tied with the existence of an attractive Coulomb potential at small distances and with the behavior of the scattering angle as ρ or L tends to zero.

Classical scattering through an angle of π for $\rho\to 0$ always occurs in the case of repulsive potentials if the potential energy $U(r)$ is greater than the total energy E for some value of r. In this case no singularities occur in the backward scattering in the classical approximation. If we have an attractive potential, then the classical limiting scattering angle for $\rho\to 0$ depends on the character of the $U(r)$ singularity at $r\to 0$. If $U(r)\sim -C/r^{\varepsilon}$, then, as a result of the scattering, the radius vector **r** of the particle turns through an angle[1)1] $\varphi_0 = 2\pi/(2-\varepsilon)$ (the scattering angle θ can be expressed in terms of the angle φ; e.g., for $\pi < \varphi < 3\pi$ we have $\theta = |\varphi-\pi|$). As $\varepsilon\to 2$, we obtain $\varphi_0\to\infty$, and the "fall-to-the-center" trajectory is converted in the limit into a logarithmic spiral. It is interesting that the geometrically preferred backward (i.e., $\theta = \pi$) scattering is realized for the physically important case of a potential with a Coulomb singularity (the $\varepsilon = 1$ case) (thus, in the case of the interaction of an electron with an atom, such a singularity is due to the attraction to the nucleus). The trajectory in this case remains almost a straight line right into the region of small r, where it turns sharply through an angle $\sim\pi$, rounding the potential center.

For the purely Coulomb potential $U(4) = -Z/r$, we have $\pi-\theta = 2$ arc $\tan(2\rho E/z)$, and the cross section $\sigma(\theta) = (Z/4E)^2 \sin^{-4}(\theta/2)$ for scattering through an angle $\theta\sim\pi$ possesses a smooth minimum (the cross section for backward scattering is four times smaller than for scattering through an angle of $\pi/2$), i.e., as in the case of repulsive potentials, no singularity occurs in the backward scattering at any value of the energy.

If the deviation of the potential $U(r)$ from the Coulomb potential does not possess a singularity at zero, then $\pi-\theta$ (or $2\pi-\varphi$) is an odd function of ρ (or L), and can be expanded in a series in odd powers of this variable. The fundamentally important result of the present paper is the discovery that the linear—in ρ or L—term in this expansion vanishes at some energy E_0 for a broad class of screened attractive Coulomb potentials. For $E > E_0$ the cross section is regular at all angles. At $E < E_0$ the angle φ of deflection at small ρ values is greater than 2π, after which is decreases, and we have a combination of a rainbow at an angle close to π and a backward glory at small values of the impact parameter. At $E = E_0$ we obtain in the classical cross section a singularity for backward scattering, that is proportional to $(\pi-\theta)^{-4/3}$. A semiclassical treatment leads to the conversion of this singularity into a maximum; if we limit ourselves to the cubic term in the expansion of $\pi-\theta$, then the semiclassical expression for the scattering amplitude is a universal function of the reduced deviation of the angle from π. As the energy is increased (in the region $E > E_0$), the cross section peak decreases and then disappears. As the energy is decreased, the joint action of the glory and the rainbow at first somewhat enhances this peak, and leads to an increase in the total scattering cross section for some scattering-angle range $\pi/2 < \theta_1 < \theta < \pi$ in the rear hemisphere. There also arises in the $E < E_0$ region a six-beam complex interference structure in the scattering cross section.

The combination of the glory and rainbow effects is possible also for nonzero impact parameters. We obtain here as well a sharp enhancement of the backward or forward scattering,[4] but in this case the condition for the function $\pi-\theta$ to

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be odd is not fulfilled, and we can limit ourselves to the consideration of the quadratic term of the power series expansion of $\pi-\theta$ in ρ. The interference structure in this case will be a four-beam one. It is evident from the foregoing that the case considered here is not simply a formal combination of the rainbow and glory phenomena, but is closely tied with the specific nature of the Coulomb scattering, and can be regarded as a new type of quasiclassical singularity in scattering theory.

The question of the possibility of a specific observation of this effect is discussed in §4 and in the Conclusion.

§2. CLASSICAL BACKWARD SCATTERING IN THE CASE OF COULOMB INTERACTION AT SMALL DISTANCES

Let us investigate the behavior of the function $\theta(L)$ in the neighborhood of the point $L = 0$ and the corresponding scattering cross section singularity as a function of the form of the potential, which we represent as

$$U(r)=-Z/r+V(r),$$

where $V(r)$ is the non-Coulomb part of the potential [$rV(r)\to 0$ as $r\to 0$]. The basic expression for the analysis is the well-known expression for the scattering angle or the corresponding formula for the angle φ of deflection of the trajectory[5]:

$$\varphi=\frac{1}{(2m)^{1/2}}\int_{r_0}^{\infty}\frac{L\,dr}{[E-L^2/2mr^2-U(r)]^{1/2}}, \tag{1}$$

where m is the particle mass and r_0 is the distance of closest approach of the particle to the potential center:

$$E-\frac{L^2}{2mr_0^2}-U(r_0)=0. \tag{2}$$

The power series expansion of $r_0(L)$ in small L can easily be obtained by solving Eq. (2) with the aid of the iterative method:

$$r_0=\frac{L^2}{2mZ}+\frac{V(L^2/2mZ)-E}{Z}\frac{L^4}{(2mZ)^2}+\ldots \tag{3}$$

or, assuming that $V(r)$ is finite at $r = 0$,

$$r_0=\frac{L^2}{2mZ}+\frac{V(0)-E}{Z}\frac{L^4}{(2mz)^2}+\ldots. \tag{4}$$

It is convenient to go over to the variable $x = 1/ry_0$ ($y_0 = 1/r_0$) in the integral (1):

$$\varphi(L)=\frac{1}{(2m)^{1/2}}\int_0^1\left\{\frac{E}{L^2y_0^2}-\frac{x^2}{2m}+\frac{Zx}{L^2y_0}-\frac{1}{L^2y_0^2}V\left(\frac{1}{xy_0}\right)\right\}^{-1/2}dx. \tag{5}$$

The investigation of the expression (5) together with (3) yields significantly different results, depending on the nature of the growth of $V(r)$ as $r\to 0$. In the Appendix we briefly consider the case of potentials that increase as $r\to 0$ faster than $r^{-1/2}$. Here we shall discuss in greater detail the simpler and physically more interesting case of potentials for which $V(r)r^{1/2}\to 0$ as $r\to 0$. If $V(r)$ is analytic in the neighborhood of the point $r = 0$, then the function $\theta(L)$ can be expanded in integral powers of L, and, because $\varphi-2\pi$ is an odd function, the expansion contains only odd powers of L, i.e.,

$$\theta(L)=\pi+c_1L+c_3L^3+\ldots. \tag{6}$$

The expansion coefficients c_n can be computed directly by differentiating the integrand in (5) [with allowance for the dependence $y_0(L)$]. Below we shall need the expressions for the first two expansion coefficients:

$$c_1=\left(\frac{2}{m}\right)^{1/2}\int_0^{\infty}\frac{V(r)+rV'(r)-E}{r^2(E+Z/r-V(r))^{3/2}}dr=\left(\frac{2}{m}\right)^{1/2}\int_0^{\infty}\frac{U(r)+rU'(r)-E}{r^2(E-U(r))^{3/2}}dr, \tag{7}$$

$$c_3=-\left(\frac{2}{m}\right)^{1/2}\frac{1}{12mZ}\left[(V(0)-E)\left(\frac{2}{Z}\int_0^{\infty}\frac{2(E-V)-rV'}{r^2(E-U)^{3/2}}dr-6\int_0^{\infty}\frac{V+rV'-E}{r^3(E-U)^{5/2}}dr\right)+\int_0^{\infty}\frac{4(V(0)-V)-rV'}{r^3(E-U)^{3/2}}dr\right]. \tag{8}$$

In order to get the initial segment of the expansion of $\theta(L)$ for an arbitrary potential $V(r)$ to have the form (6), we must impose on the behavior of the function $V(r)$ in the neighborhood of the point $r = 0$ certain smoothness requirements that become stricter as the length of the indicated segment increases. These requirements find their expression in convergence conditions on the integrals for the coefficients c_n. The convergence conditions for the integrals (7) and (8) have respectively the forms $r^{1/2}V(r)\to 0$ and $r^{-1/2}[V(r) - V(0)]\to 0$ for $r\to 0$. We can, by integrating by parts, obtain another representation for c_1:

$$c_1=\left(\frac{2}{m}\right)^{1/2}\int_0^{\infty}\left(\frac{1}{r^2[E-U(r)]^{1/2}}-\frac{1}{Z^{1/2}r^{3/2}}\right)dr. \tag{7a}$$

§3. THE CASE OF THE COULOMB GLORY

Of particular interest are the cases in which the coefficient c_1, which is energy and potential-form dependent, vanishes, since we have here a much higher backward scattering intensity, which in this case is determined by the quantity c_3. Indeed, if $c_1\neq 0$, then the classical differential cross section for scattering through an angle $\theta\approx\pi$ is finite:

$$\sigma(\theta)\equiv\frac{\rho}{\sin\theta}\left|\frac{d\rho}{d\theta}\right|\approx(2mEc_1^2)^{-1},$$

whereas the $c_1 = 0$ the cross section is classical mechanics diverges:

$$\sigma(\theta)\approx\frac{|c_3|^{-2/3}}{6mE}(\pi-\theta)^{-4/3}. \tag{9}$$

A necessary condition for $c_1(E)$ to vanish is that the integrand in (7), i.e., the function

$$V(r)+rV'(r)-E=\frac{d}{dr}(rV(r))-E.$$

should change its sign in the integration interval. At any rate the sign change occurs at some values of the energy E if

$$\frac{d}{dr}(rV(r))>0.$$

Since the dominant contribution to the integral is made by the region around $r=0$, it is often sufficient to require that

$$\frac{d}{dr}(rV(r))>0$$

[i.e., that $V(r)>0$] in the vicinity of the point $r=0$.

Let us now discuss the behavior of the function $c_1(E)$ quantitatively. At high energies the asymptotic form of $c_1(E)$ is determined by the behavior of the potential $U(r)$ at small r:

$$c_1(E)=-\frac{2}{Z}\left(\frac{2E}{m}\right)^{1/2}+o(E^{1/2}), \tag{10}$$

i.e., it does not depend on the form of $V(r)$, and turns out to be the same as for scattering in the Coulomb field $-Z/r$.

On the other hand, at low energies $c_1(E)$ is determined by the asymptotic form of $U(r)$ for large r:

a) if as $r\to\infty$ the potential $U(r)$ decreases sufficiently slowly (i.e., like $\sim -\beta/r\alpha$, $1\leqslant\alpha<2$), then $c_1(E)$ remains finite at $E\to 0$:

$$c_1(0)=\left(\frac{2}{m}\right)^{1/2}\int_0^\infty\frac{U(r)+rU'(r)}{r^2(-U(r))^{3/2}}dr. \tag{11}$$

Let us represent $U(r)$ in the form $U(r)=-Z(r)/r$, where the function $Z(r)$ is such that $r^{-1/2}(Z(r)-Z(0))\to 0$ as $r\to 0$. Then

$$c_1(0)=\left(\frac{2}{m}\right)^{1/2}\int_0^\infty r^{-3/2}\frac{[Z(0)]^{1/2}-[Z(r)]^{1/2}}{[Z(r)]^{1/2}[Z(0)]^{1/2}}dr \tag{11a}$$

b) When the potential decreases faster (i.e., when $\alpha\geqslant 2$), the coefficient $c_1(E)$ increases without restriction as $E\to 0$:

$$c_1(E)\approx\frac{1}{(2m)^{1/2}}\beta^{-1/2}\ln\frac{1}{E},\quad \alpha=2; \tag{12}$$

$$c_1(E)=\left(\frac{2}{\pi m}\right)^{1/2}\frac{2\alpha}{\alpha-2}\Gamma\left(\frac{3}{2}-\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)\beta^{-1/\alpha}E^{1/\alpha-1/2},$$

$$\alpha>2. \tag{13}$$

c) If the potential is truncated at $r=R$ [$U(r)=0, r>R$], then

$$c_1(E)\approx\frac{1}{R}\left(\frac{2}{mE}\right)^{1/2}. \tag{14}$$

Thus, if $U(r)<0$, then in the cases b) and c) the coefficient $c_1(E)$ always has a root. In the case a), for the coefficient to have a root, is is sufficient that $Z(0)>Z(r)$.

Let us now consider specific examples of simple potentials for which the coefficient $c_1(E)$ can be computed explicitly.

For a Coulomb potential truncated at $r=R$, i.e., for

$$U(r)=\begin{cases}-Z/r+Z/R, & r<R\\ 0, & r>R\end{cases} \tag{15}$$

the coefficient

$$c_1(E)=\frac{1}{Z}\left(\frac{2}{mE}\right)^{1/2}\left(\frac{Z}{R}-2E\right) \tag{16}$$

behaves in the limiting cases in accordance with the general formulas (10) and (14), and vanishes at $E=E_0\equiv Z/2R$. Indeed, all the coefficients c_n vanish at the indicated energy, and $\theta(L)=\pi$ at $0<L<(2Em)^{1/2}R$ (giant glory[1-3,4,6,7]).

For the "piecewise Coulomb" potential

$$U(r)=\begin{cases}-Z/r+U_0, & r<R=(Z-Z')/U_0\\ -Z'/r, & r>R\end{cases} \tag{17}$$

($Z>Z'$, $U_0>0$), we obtain

$$c_1(E)=-2\left(\frac{2}{m}\right)^{1/2}\times\left[\frac{E^{1/2}}{Z'}-\frac{1}{ZZ'}(Z-Z')^{1/2}(E(Z-Z')+U_0Z')^{1/2}\right]. \tag{18}$$

in agreement with (10) and (11). The coefficient $c_1(E)$ vanishes at $E=E_0=U_0(Z-Z')/(2Z-Z')$.

§4. THE QUANTUM-MECHANICAL PROBLEM AND THE SCATTERING OF ELECTRONS BY ATOMS

Let us proceed to apply the results obtained in the theory of electron scattering by atoms. Let us choose the effective interaction potential in the form

$$U(r)=-Z/r(1+Ar)^2,\quad A=\alpha b^{-1}Z^{1/3},\quad b={}^1/_2(3\pi/4)^{2/3}, \tag{19}$$

which, for $\alpha\approx 0.5$, guarantees a good approximation to the atomic potential in the Thomas-Fermi statistical theory of the atom (see, for example, Refs. 8 and 9); Z is the nuclear charge; we use atomic units). Introducing the reduced energy $\tilde{E}=E/ZA$, and going over to the variable $t=(1+rA)^{-1}$, we obtain

$$c_1(E)=\left(\frac{2A}{mZ}\right)^{1/2}\int_0^1\frac{(2t^3-\tilde{E})\,dt}{(1-t)^{1/2}(t^3-\tilde{E}t+\tilde{E})^{3/2}}. \tag{20}$$

It follows from the general results that $c_1(E)\approx 2.81\cdot(2A/mZ)^{1/2}\tilde{E}^{-1/6}$ for $E\to 0$ and $c_1(E)\approx -2(2E/m)^{1/2}Z^{-1}$ for $E\to\infty$. Figure 1 shows the result of a numerical computation of the integral (20). The coefficient $c_1(E)$ has only one root at $\tilde{E}_0=0.67$, which corresponds to an electron energy of $E_0=0.67\,\alpha b^{-1}Z^{4/3}$. Thus, for $Z=50$ we find in ordinary units that $E_0=1.9$ keV. In this case, as was discussed above, the large-angle scattering is determined by the magnitude of the coefficient c_3, for which a numerical computation with $\tilde{E}=\tilde{E}_0$ yields the expression

$$c_2(E_0)=-1.93\sqrt{2}\left(\frac{A}{mZ}\right)^{3/2}=-3.28\left(\frac{\alpha}{m}\right)^{3/2}\frac{1}{Z}.$$

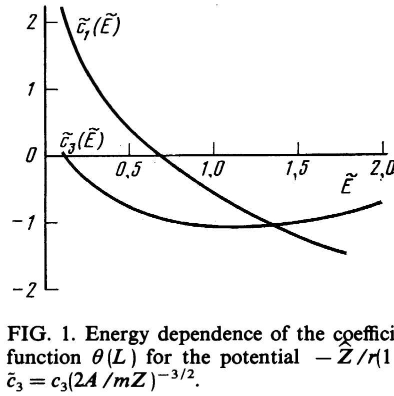

FIG. 1. Energy dependence of the coefficients of the expansion of the function $\theta(L)$ for the potential $-Z/r(1+Ar)^2$: $\tilde{c}_1=c_1(2A/mZ)^{-1/2}$; $\tilde{c}_3=c_3(2A/mZ)^{-3/2}$.

The scattering of an electron by an atom at an energy of several keV is accompanied by intense inelastic processes (excitation and ionization of the atom). The elastic scattering through angles close to π is governed by the interaction of the electron with the atomic nucleus, and its cross section is proportional to Z^2 (see Ref. 10, §139). The total (for all transitions in the atom) differential cross sections for large-angle inelastic scattering is also close to the Rutherford result, but it represents the sum of the cross sections for scattering by the individual atomic electrons, i.e., it is proportional to Z (see Ref. 10, §148). This allows us in the case of backward electron scattering to neglect the inelastic processes. The conditions of applicability of such an approximation get better as well when we choose the energy $E = E_0$, since the elastic backward scattering is especially intense in this case.

It should be noted that, for weakly inelastic processes, when $\Delta E/E \ll 1$, the Coulomb glory will, apparently, also be observed, although at a somewhat different energy.

As was pointed out above, in the $c_1 = 0$ case the classical differential cross section diverges as $\theta \to \pi$ in accordance with the formula (9). Therefore, for small $\pi - \theta$, we should use wave mechanics, which gives a finite result for the cross section.

Let us use for the backward scattering the eikonal approximation, in which the scattering amplitude has the form

$$f(\theta) = -\frac{i}{k}\int_0^\infty e^{2i\delta_L + i\pi L} J_0((\pi-\theta)L)\,L\,dL, \tag{21}$$

where $k = (2mE)^{1/2}$ and $J_0(z)$ is the Bessel function of zeroth order. The scattering phase can be found from the function $\theta(L)$, which is well known from classical mechanics, with the aid of the equation

$$2\frac{d\delta_L}{dL} + \theta(L) = 0, \tag{22}$$

which is valid in the quasiclassical approximation.[10] In particular, limiting ourselves to the first terms of the expansion (6), we obtain

$$2\delta_L + \pi L = 2\delta^{(0)} - \frac{c_1}{2}L^2 - \frac{c_3}{4}L^4, \tag{23}$$

where $\delta^{(0)}$ is an insignificant constant phase. Then the expression for the amplitude assumes the form

$$f(\theta) = |c_3|^{-1/2} e^{2i\delta^{(0)}} F(\Delta\tilde\theta,\ c_1|c_3|^{-1/2}); \tag{24}$$

$$F(\Delta\tilde\theta, c_1|c_3|^{-1/2}) = -\frac{i}{k}\int_0^\infty \exp\left(i\frac{y^4}{4} - i\frac{y^2}{2}c_1|c_3|^{-1/2}\right) J_0(\Delta\tilde\theta y)\,y\,dy.$$

For $c_1 = 0$ the reduced amplitude F is a universal function of a single variable: the reduced angle $\Delta\tilde\theta = (\pi - \theta)|c_3|^{-1/4}$. For angles $\Delta\tilde\theta \gg 1$, the corresponding integral can be computed by the stationary phase method, with the use of the asymptotic forms of the Bessel functions, which leads to the classical result (9). In the opposite limiting case we can expand the Bessel function in a series, which yields

$$f(\theta) = \frac{e^{2i\delta^{(0)} + i\pi/4}}{4ik}\left|\frac{4}{c_3}\right|^{1/2} \sum_{n=0}^{\infty} \frac{e^{-i3\pi n/4}}{(n!)^2}\Gamma\left(\frac{n+1}{2}\right)\frac{(\Delta\tilde\theta)^{2n}}{2^n}. \tag{25}$$

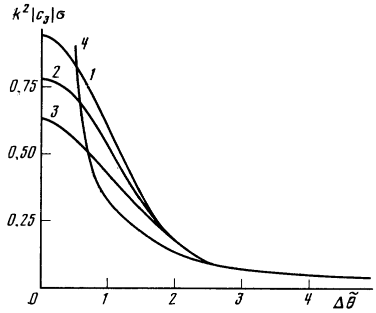

FIG. 2. Differential scattering cross section as a function of the reduced angle $\Delta\tilde\theta$ in the eikonal approximation. The curves 1, 2, and 3 correspond to values of the characteristic parameter $c_1|c_3|^{-1/2}$ equal to -0.25, 0, and 0.25. The curve 4 indicates the cross section in the classical approximation for the case when $c_1 = 0$.

The series (25) converges absolutely at all $\pi - \theta$, and is convenient for computations. A similar, but more unwieldy formula can be obtained in the general $c_1 \neq 0$ case.

Figure 2 shows the results obtained in a calculation of the cross section as a function of the reduced angle $\Delta\tilde\theta$ for different values of the characteristic parameter $c_1/|c_3|^{1/2}$. In accordance with the discussion in §1, the decrease of $c_1|c_3|^{-1/2} < 0$ leads to the flattening out of the peak in the backward scattering. The growth of this parameter leads to the growth of the backward-scattering cross section right up to the value $\pi/k^2|c_3|$ (instead of the value $\pi/4k^2|c_3|$ that we have in the $c_1 = 0$ case) in the limiting case $c_1|c_3|^{-1/2} \to \infty$. But actually the growth of the cross section stops earlier because of the fact that, for real scattering, the restriction to the first terms of the expansion (6) is not valid. Indeed, the backward-scattering amplitude is formed largely by the partial waves with low angular momenta. On the other hand, the value of the integral (24) for $c_1|c_3|^{-1/2} \gg 1$ is determined by the region of large L, which indicates a going beyond the limits of applicability of the formula in question.

Let us now return to the specific problem of electron scattering by an atom. For the typical values $Z = 50$ and $E \approx E_0$, a satisfactory approximation is given by the use of the first four partial waves (with $L = 0$–3). In this case the procedure, leading to (21), whereby the series in terms of the partial waves is replaced by an integral is not justified. Therefore, the cross section was recalculated (Fig. 3) from the exact quantum-mechanical formula for the scattering amplitude, but with the quasiclassical scattering-phase val-

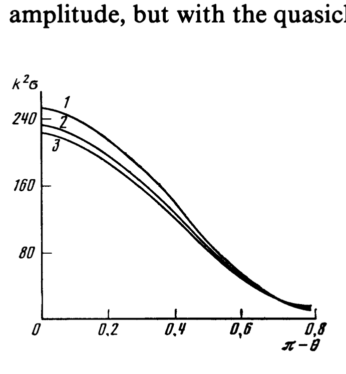

FIG. 3. Cross section for electron scattering by atoms in the case when allowance is made for the first four partial waves. The curves 1, 2, and 3 correspond to reduced-energy values of 0.45, 0.35, and 0.67 respectively.

ues determined from (23). It can be seen from Fig. 3 that the cross section peak is in this case broad. Let us note that, although at energies of several keV the elastic electron scattering is primarily small-angle scattering, the excess of the backward-scattering cross section over the cross section for scattering through angles $\sim 90°$ can, in principle, be observed. As can be seen from Fig. 3, the maximum ratio of the backward-scattering cross section to the cross section for scattering through angles $\pi-\theta \approx 0.8$ is attained at $\widetilde{E} \approx 0.45$, and is approximately equal to 30. In ordinary energy units this corresponds to 1.3 keV. As we move away from this value on either side, the peak flattens out.

For one and the same interaction potential the magnitude of the coefficient c_3 decreases with increasing incoming-particle mass m like $m^{-3/2}$ (the value of E_0 does not depend on m). In other words, the peak in the backward scattering is formed by higher partial waves, and is narrower.

§5. CONCLUSION

A screened spherically-symmetric attractive Coulomb potential is realized first and foremost in electron scattering by atoms. The energy region where the Coulomb glory occurs in scattering of negative singly-charged particles by atoms corresponds roughly to a quarter of the important atomic constant: the potential of the electron cloud in the atomic nucleus. For the Thomas-Fermi potential this quantity is equal to $1.588b^{-1}Z^{4/3}$; and for the potential $-Z/r(1+Ar)^2$ it is equal to $2ZA$.

Unfortunately, in the energy region where the Coulomb glory should occur the quasiclassicality condition is fulfilled only to a slight degree: the electron wavelength is only several times smaller than the dimensions of an atom with $Z = 50$. This means that the Coulomb glory phenomenon manifests itself only for the first 2–4 partial waves, and we can expect only a relatively broad peak in the range of $\pi - \theta$ values from ~ 20 to 40°.

In this respect, more suitable objects for the observation of the effect would be heavy negatively-charged particles, in particular, μ mesons. Experiments on the observation of the backward scattering of kilovolt muons by heavy atoms are at present quite practicable. The role of the inelastic processes for the heavy particles will also be less significant, since they move significantly more slowly than the electrons, and the collision will be adiabatic. But even for the electrons, even though the inelastic processes do weaken the effect and a more exact estimate of their contribution is necessary, the detection of the peak in the backward scattering remains within the feasibility limits of practicable experiment.

APPENDIX

In the case of potentials that grow as $r \to 0$ faster than $r^{-1/2}$, an investigation of (5) together with (3) shows that the expansion of the function $\theta(L)$ contains fractional powers of L:

$$\theta(L) \approx \pi + c_\gamma L^{2(1-\gamma)}, \tag{A1}$$

$$c_\gamma = \frac{2\alpha(0)}{Z\pi^{1/2}}(2mZ)^{\gamma-1}\Gamma\left(\gamma - \frac{1}{2}\right)\Gamma(2-\gamma)\sin\pi\gamma,$$

where the exponent γ characterizes the behavior of $V(r)$ as $r \to 0$:

$$V(r) = \frac{\alpha(r)}{r^\gamma}\left(\frac{1}{2} < \gamma < 1,\ \alpha(0) \neq 0,\ \alpha(0) \neq \infty\right). \tag{A2}$$

The case $\gamma = \frac{1}{2}$ is an intermediate case, and corresponds to the logarithmic terms in the expansion:

$$\theta(L) \approx \pi - 4m(2mZ)^{-1/2}\alpha(0)L\ln L. \tag{A3}$$

As an example of a potential with a more complicated singularity at the origin, we cite the function

$$V(r) = \alpha(r)/r\ln r, \tag{A4}$$

in which case

$$\theta(L) \approx \pi - \frac{\pi\alpha(0)}{2Z}\Big/\ln^2\frac{L}{(2mZ)^{1/2}}. \tag{A5}$$

In all the considered cases the deviation $\theta(L)-\pi$ decreases as $L \to 0$ more slowly than L, the coefficient in the corresponding term being independent of the energy, and being entirely determined by the character of the growth of the potential $V(r)$ as $r \to 0$.

[1)] Notice that, with the aid of the solution to the inverse scattering problem, we can construct potentials for which the scattering angle is constant and equal to a fixed value in some interval $0 < \rho < R$ (Refs. 2 and 3).

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