Buckling of reinforced thin shells.

Andrei L. Smirnov

Department of Theoretical and Applied Mechanics
St. Petersburg State University, St. Petersburg, Russia
e–mail: smirnov@bals.usr.pu.ru

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Abstract. *Localized buckling of elastic thin shells reinforced with threads is discussed in this paper. The results of asymptotic analysis for thin isotropic shells reported in [1] are generalized for the case of a shell consisting of a matrix reinforced by fibers. The expressions for critical loadings obtained for the cases when localization of buckling occurs, for example, for the buckling of a convex shell under hydrostatic pressure or under torsion. As examples, buckling of ellipsoidal and cylindrical shells are considered.*

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1 Introduction

We consider a thin shell made of composite material, consisting of the matrix reinforced by fibers situated in planes parallel to the midsurface. On the shell midsurface we introduce the curvilinear coordinates α_1, α_2 coinciding with the curvature lines. The coordinate z is directed along the normal to the midsurface. We assume that the shell if reinforced with N systems of fibers, inclined at angles $\theta^{(k)}, \quad k = 1, 2, \ldots, N$ to the axis α_1.

The shell stress σ_{ij} is a sum of the matrix stress $\sigma_{ij}^{(0)}$ and averaged stress $\sigma_{ij}^{(k)}$, caused by the extensions of the fibers

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sum_{k=0}^{N} \sigma_{ij}^{(k)}. \tag{1}$$

The elastic energy Π of the shell can be expressed as a sum of the stretching energy Π_ε and the bending energy $\Pi_\varkappa$ [1]

$$\Pi = \Pi_\varepsilon + \Pi_\varkappa,$$

where Π_ε and $\Pi_\varkappa$ are given by

$$\begin{aligned} \Pi_\varepsilon &= \frac{1}{2} \int\!\!\int (T_1\varepsilon_1 + T_2\varepsilon_2 + S\omega)\, d\Sigma = \frac{1}{2} \int\!\!\int K_{ij}\varepsilon_i\varepsilon_j d\Sigma, \\ \Pi_\varkappa &= \frac{1}{2} \int\!\!\int (M_1\varkappa_1 + M_2\varkappa_2 + 2H\tau)\, d\Sigma. \end{aligned}$$

Here T_1, T_2, and S are the stress resultants and M_1, M_2, and H are the moment resultants. ε_1, ω, and ε_2 are the stretching-shear strains and $\varkappa_1$, τ, and $\varkappa_2$ are the bending-twisting strains of the midsurface. $d\Sigma = A_1 A_2\, d\alpha_1 d\alpha_2$ is the area element and the integration in Π_ε and $\Pi_\varkappa$ is performed on the entire midsurface. K_{ij} are the coefficients the relations for those have been obtained in citeHas. If the reinforcing fibers are symmetric with respect to the directions α_1 and α_2, i. e. for each fiber system with an angle θ_k corresponds a system with an angle $\theta_l = -\theta_k$, then

$$K_{i3} = D_{i3} = K_{3i} = D_{3i} = 0 \qquad i = 1, 2$$

As a result we obtain *the constructive orthotropic* shell.

2 Bifurcation equations

We simplify the equilibrium equations by using the same assumptions usually made for the Donnell equations for shallow shells [3]. The metric of the midsurface is described as the metric of a plane and we assume that the values of the metric coefficients A_1, A_2 and radii of curvature R_1, and R_2 are constant. Let

$$dx_1 = A_1 d\alpha_1, \qquad dx_2 = A_2 d\alpha_2.$$

Neglecting the small term we derive the following equations for strains

$$\begin{aligned}
\varepsilon_1^l &= \frac{\partial u_1}{\partial x_1} - \frac{w}{R_1}, \quad \omega^l = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}, \quad \varepsilon_2^l = \frac{\partial u_2}{\partial x_2} - \frac{w}{R_2}; \\
\gamma_1 &= -\frac{\partial w}{\partial x_1}, \qquad \gamma_2 = -\frac{\partial w}{\partial x_2}; \\
\varkappa_1 &= \frac{\partial^2 w}{\partial x_1^2}, \quad \tau = \frac{\partial^2 w}{\partial x_1 \partial x_2}, \quad \varkappa_2 = \frac{\partial^2 w}{\partial x_2^2}.
\end{aligned} \tag{2}$$

If the loads q_1, q_2 and q_3 have the same order or $\{q_1, q_2\} \ll q_3$, then with the same error it may be assumed that $q_1 = q_2 = 0$ in equilibrium relations.

The simplified system may be used not only for the analysis of shallow shells but also in the analysis of vibration and buckling of arbitrary thin shells. In that case the stress-strain state of a shell may consists of many waves of deformations but in the limit of one deformation wave the shell should be considered as shallow.

Let consider that as a result of loading, there exists in a shell a momentless stress-strain state determined by the initial stress resultants T_1^0, T_2^0, S_0. The stress-strain state is referred to as momentless or membrane-like if the moment resultants $M_1 = M_2 = H = 0$. Next we analyze the stability of such a state.

The bifurcation equations for the equilibrium equations become [4]

$$\begin{aligned}
&\frac{\partial T_1}{\partial x_1} + \frac{\partial S}{\partial x_2} = 0, \\
&\frac{\partial T_2}{\partial x_2} + \frac{\partial S}{\partial x_1} = 0, \\
&T_1^0 \frac{\partial^2 w}{\partial x_1^2} + 2S^0 \frac{\partial^2 w}{\partial x_1 \partial x_2} + T_2^0 \frac{\partial^2 w}{\partial x_2^2} \\
&\qquad - \left(\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 H}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} \right) + \frac{T_1}{R_1} + \frac{T_2}{R_2} = 0.
\end{aligned} \tag{3}$$

These equations together the with strain-displacement relations and the elasticity relations and form a closed system, provided linear approximation for tangential deformations is assumed.

Next we study the one parametric loading by introducing the loading parameter λ as

$$\{T_1^0, T_2^0, S_0\} = -\lambda \{t_1, t_2, t_3\}.$$

The minus sign is necessary in order to seek $\lambda > 0$, since buckling is possible only if there exist directions in which compressive stresses are developed. Such directions exist if and only if at least one of the following inequalities is satisfied

$$T_1^0 < 0 \text{ or } \; T_2^0 < 0 \;\text{ or }\; \left(S^0\right)^2 > T_1^0 T_2^0. \tag{4}$$

We seek the displacements under bifurcation in the form

$$u_1 = u_1^0 \sin z, \quad u_2 = u_2^0 \sin z, \quad w = w^0 \cos z, \quad z = k_1 x_1 + k_2 x_2, \tag{5}$$

where the amplitudes u_1^0, u_2^0, and w^0 and the wave numbers k_1 and k_2 must be determined.

From the first two equations in equilibrium equations we find u_1^0 and u_2^0 as functions of w^0.

Now we can cancel w^0 in the third equation in equilibrium equations since all functions are only of w^0 and find λ as

$$\lambda = f(k_1, k_2) = \frac{B_\varepsilon + B_\varkappa}{B_t}, \tag{6}$$

where

$$B_\varepsilon = \frac{\Delta_k}{\Delta}\left(\frac{k_1^2}{R_2} + \frac{k_2^2}{R_1}\right)^2, \qquad \Delta_k = K_{11}A_{11} + K_{12}A_{12} + K_{13}A_{13},$$
$$B_\varkappa = D_{11}k_1^4 + 4D_{13}k_1^3k_2 + 2\,(D_{12} + 2D_{33})\,k_1^2k_2^2 + 4D_{23}k_1k_2^3 + D_{22}k_2^4,$$
$$\Delta = A_{22}k_1^4 - 2A_{23}k_1^3k_2 + (2A_{12} + A_{33})k_1^2k_2^2 - 2A_{13}k_1k_2^3 + A_{11}k_2^4,$$
$$B_t = t_1k_1^2 + 2t_3k_1k_2 + t_2k_2^2.$$

Here we denote by Δ_k the determinant of the matrix $\{K_{ij}\}$. The variables B_ε, $B_\varkappa$ and B_t are proportional respectively to the bending-twisting shell energy $\Pi_\varkappa$, the tensile-shear shell energy for additional displacements Π_ε, and the work of the initial momentless stress resultants on the additional rotations of the normal.

Since $\Pi_\varkappa$ and Π_ε are positive definite, the matrix $\{K_{ij}\}$ is also positive definite and therefore

$$\Delta_k > 0, \quad K_{ii} > 0, \quad A_{ii} > 0 \quad B_\varkappa > 0,$$
$$\Delta > 0 \quad \text{for} \quad k_1^2 + k_2^2 \neq 0.$$

3 Analysis of the relation for the critical loading

Relation (6) is rather general. It may be used for estimation of the value of a critical loading and expected buckling mode in many problems. We obtain the critical value λ_0 for the parameter λ by minimizing the function $f(k_1, k_2)$ in all real k_1 and k_2, such that $B_t > 0$.

Let

$$k_1 = r\cos\varphi, \qquad k_2 = r\sin\varphi.$$

Taking into account that the functions are homogeneous in k_1 and k_2 we introduce

$$B_\varepsilon = B_\varepsilon^*(\varphi), \quad B_\varkappa = r^4 B_\varkappa^*(\varphi), \quad B_t = r^2 B_t^*(\varphi), \quad \Delta = r^4\Delta^*(\varphi).$$

Minimizing the function (6) in r we obtain

$$\lambda_0 = \min_{\varphi}\{f^*(\varphi)\} = f^*(\varphi_0), \qquad f^*(\varphi) \equiv 2\frac{\sqrt{B_\varepsilon^*(\varphi)B_\varkappa^*(\varphi)}}{B_t^*(\varphi)},$$
$$r_0^4 = \frac{B_\varepsilon^*(\varphi_0)}{B_\varkappa^*(\varphi_0)}.$$

Due to (5), the pits are significantly elongated at angle $-\varphi_0$ to the axis x_2. This relations may be used to study the buckling of convex shells under compression, stretching, torsion, bending or combined loading. In the case of shells of zero gaussian curvature only for the axial compression of cylindrical and conical shells this relation provides the nontrivial result.

In fact, the algorithm described above may be applied only for shells of positive gaussian curvature ($R_1R_2 > 0$). For shells of negative gaussian curvature ($R_1R_2 < 0$), due to (6) we get

$$\lambda_0 = \min_{\varphi}\{f^*(\varphi)\} = 0, \qquad r_0 = 0 \quad \text{for} \quad \tan\varphi_0 = \sqrt{-\frac{R_1}{R_2}}. \tag{7}$$

Similarly, for shells of zero gaussian curvature ($R_1^{-1} = 0$), i.e. cylindrical and conical, we obtain from (6)

$$\lambda_0 = \min_{\varphi}\{f^*(\varphi)\} = 0, \qquad r_0 = 0 \quad \text{for} \quad \varphi_0 = 0. \tag{8}$$

Relations (7) and (8) mean that for shells of zero or negative gaussian curvature the order of the critical loading ($\lambda_0 = 0$) decreases and the buckling mode is not localized ($r_0 = 0$). To obtain the critical loading and buckling modes for such shells one should apply the method of the asymptotic integration that is described below for a circular cylindrical shell as an example. The case of the axially compressed cylindrical shell $t_2 = t_3 = 0$, $t_1 > 0$, is the only one, when the application of relations (6) provides a nontrivial result.

4 Examples

4.1 Orthotropic ellipsoid under external pressure.

As an example we consider an elliptical shell of revolution with the semi-axes (a, a, b). The angle between the axis of symmetry and the normal to the surface is denoted as θ. We select the parameter $R = a$ as a characteristic length. Then for the principle curvatures

$$\rho_2 = R/R_2 = (\sin^2\theta + d\cos^2\theta)^{1/2}, \quad \rho_1 = R/R_1 = \rho_2^3/d^2, \quad d = b/a. \tag{9}$$

Here d is the coefficient of the ellipsoid compression (if $d > 1$ an ellipsoid is prolate, if $d < 1$ it is oblate).

The elliptical shell consists of the matrix made of the uniform material of the thickness h, Young's modulus E and Poisson's ration ν. The shell is reinforced with two similar systems of threads, the angles between the threads and the meridional direction are equal to $\pm\alpha$. The threads occupy the volume $(1-\delta_0)V$, where V is the entire volume of the structure; Young's modulus of the thread material is e times larger than E.

The elliptical shell is under uniform normal (hydrostatic) pressure λ. The relations for the initial stresses are well-known [1]

$$t_1 = \frac{1}{2\rho_2}\text{sign}\lambda, \qquad t_2 = \frac{2\rho_2-\rho_1}{2\rho_2^2}\text{sign}\lambda, \qquad t_3 = 0.$$

For the external pressure sign$\lambda > 0$, and for the internal pressure sign$\lambda < 0$. Note, that for the external pressure the buckling may occur due to (14) for the elliptical shells of arbitrary form and for the internal pressure only for such shells for which $2\rho_2 < \rho_1$, i. e.

$$\rho_2^2 > 2d^2. \tag{10}$$

It follows from (9) that for ρ_2 the following relations hold

$$1 < \rho_2 < d, \text{ for } d > 1, d < \rho_2 < 1, \text{ for } d < 1. \tag{11}$$

Simultaneously inequalities (10) and (11) are satisfied only for $2d^2 < 1$.

For the system of threads described above the shell is orthotropic and the relation for λ has the following form

$$\begin{aligned} &\lambda_0 = \min_{\varphi,\theta}\{f^*(\varphi,\theta)\} = f^*(\varphi_0,\theta_0), \qquad f^*(\varphi,\theta) \equiv 2\frac{\sqrt{B_\varepsilon^*(\varphi,\theta)B_\varkappa^*(\varphi,\theta)}}{B_t^*(\varphi,\theta)}, \\ &r_0^4 = \frac{B_\varepsilon^*(\varphi_0,\theta_0)}{B_\varkappa^*(\varphi_0,\theta_0)}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} B_\varepsilon^* &= \frac{\Delta_k}{R^4\Delta}\left(\rho_2\cos^2\varphi + \rho_1\sin^2\varphi\right)^2, \qquad \Delta_k = K_{11}A_{11} + K_{12}A_{12}, \\ B_\varkappa^* &= D_{11}\cos^4\varphi + 2\left(D_{12}+2D_{33}\right)\cos^2\varphi\sin^2\varphi + D_{22}\sin^4\varphi, \\ B_t^* &= t_1\cos^2\varphi + t_2\sin^2\varphi, \\ \Delta &= A_{22}\cos^4\varphi + (2A_{12}+A_{33})\cos^2\varphi\sin^2\varphi + A_{11}\sin^4\varphi, \\ A_{11} &= K_{22}K_{33} - K_{23}^2, \qquad A_{12} = K_{13}K_{23} - K_{12}K_{33}, \\ A_{22} &= K_{11}K_{33} - K_{13}^2, \qquad A_{33} = K_{11}K_{22} - K_{23}^2. \end{aligned}$$

We start the consideration with the isotropic elliptical shell under external pressure (δ_0=1). In this case

$$\begin{aligned} &A_{11} = A_{22} = K^2\frac{1-\nu}{2}, \quad A_{12} = -\nu A_{11}, \\ &A_{33} = K^2(1-\nu^2), \quad K = \frac{Eh}{1-\nu^2} \\ &\Delta_k = \frac{1-\nu}{2}K^3(1-\nu^2), \quad \Delta = K^2\frac{1-\nu}{2}\left(\cos^2\varphi+\sin^2\varphi\right)^2, \\ &B_\varepsilon^* = K\frac{1-\nu^2}{R^2\left(\cos^2\varphi+\sin^2\varphi\right)^2}\left(\cos^2\varphi\rho_2+\sin^2\varphi\rho_1\right)^2, \\ &B_\varkappa^* = \tfrac{h^2}{12}K\left(\cos^2\varphi+\sin^2\varphi\right)^2 \end{aligned}$$

and relation (6) may be written as

$$\lambda_0 = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} \min_{\varphi,\theta} \frac{(\cos^2\varphi\rho_2 + \sin^2\varphi\rho_1)}{t_1\cos^2\varphi + t_2\sin^2\varphi}$$

Minimising by φ the above expression we obtain

$$\frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\rho_1/t_2, \text{ for } \varphi = \pi/2 \text{ if } t_2\rho_2 > t_1\rho_1$$
$$\frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\rho_2/t_1, \text{ for } \varphi = 0 \text{ if } t_2\rho_2 < t_1\rho_1$$
$$\frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\rho_2/t_1 = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\rho_1/t_2, \text{ if } t_2\rho_2 = t_2\rho_1$$

In the last case the angle φ is undefined. It means that there exist multiple buckling modes. At the same time the value of the buckling loading is unique.

For the case under consideration the condition $t_2\rho_2 = t_1\rho_1$ may be rewritten as

$$\frac{\rho_1}{2\rho_2} = \frac{2\rho_2 - \rho_1}{2\rho_2},$$

or $\rho_1 = \rho_2$, that corresponds to $d = 1$, i. e. spherical shell. For $d > 1$ $t_2\rho_2 > t_1\rho_1$, and for $d < 1$ $t_2\rho_2 < t_1\rho_1$.

Therefore the relation for the critical loading is given as

$$\lambda_0 = \min_{\theta} \begin{cases} \dfrac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} 2\rho_2^2, \text{ for } d \le 1 \\ \dfrac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} \dfrac{2\rho_1\rho_2^2}{2\rho_2 - \rho_1}, \text{ for } d \ge 1 \end{cases}$$

Now minimising by θ we obtain

$$\lambda_0 = \begin{cases} \dfrac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} 2d^2, \text{ for } d \le 1 \\ \dfrac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} \dfrac{2}{2d^2 - 1}, \text{ for } d \ge 1 \end{cases}$$

For $d > 1$ the weakest parallel is on the equator ($\theta_0 = \pi/2$), and the pits are elongated in the direction of the meridian ($\varphi_0 = \pi/2$).

For $d < 1$ the weakest parallel is the pole ($\theta = 0$). Note, that in the last case the value of λ_0 does not depend on angle φ and, therefore, angle φ_0 is undetermined.

Now we consider the orthotropic shell. For such shell

$$A_{11} = \frac{E_2h^2}{1-\nu_1\nu_2}G, \quad A_{22} = \frac{E_1h^2}{1-\nu_1\nu_2}G, \quad A_{33} = \frac{E_1E_2h^2}{1-\nu_1\nu_2},$$
$$A_{12} = -\frac{\nu_1E_2h^2G}{1-\nu_1\nu_2}, \quad \Delta_k = \frac{E_1E_2h^3G}{1-\nu_1\nu_2}.$$

Despite of the isotropic case in the case of orthotropic shell relation (6) cannot be simplified and one should seek numerically for the minimum of function (12). For that we fix the parameters α and δ_0 and find the minimum of the function

$$\lambda_0 = \min_{\varphi,\theta}\{f^*(\varphi,\theta)\} = \\ = \begin{cases} \min\limits_{\substack{\varphi\in[0,\pi/2] \\ \rho_2\in[1,d]}} f^*(\varphi,\theta) & \text{for } d \geq 1 \\ \min\limits_{\substack{\varphi\in[0,\pi/2] \\ \rho_2\in[d,1]}} f^*(\varphi,\theta) & \text{for } \sqrt{2}/2 \leq d \leq 1 \\ \min\limits_{\substack{\varphi\in[\mathrm{Arctan}\sqrt{\frac{d^2}{\rho_2^2-2d^2}},\pi/2] \\ \rho_2\in[d,1]}} f^*(\varphi,\theta) & \text{for } d \leq \sqrt{2}/2 \end{cases} \tag{13}$$

where

$$f^*(\varphi,\theta) = 4\sqrt{\frac{B^*_\varkappa \Delta_k}{\Delta}}\frac{\rho_2^2 d^2\cos^2\varphi + \rho_2^4\sin^2\varphi}{d^2\cos^2\varphi + (2d^2-\rho_2^2)\sin^2\varphi}$$

for the different values of the parameter d. The numerical calculations revealed that the function attains its minimum at $\theta_0 = 0$ for $d < 1$ and at $\theta_0 = \pi/2$ for $d > 1$. This result does not depend on the values of the other parameters.

As it might be expected the increasing of the thread stiffness and their relative volume leads to the increasing of the critical loading. For very small values of the parameter δ_0 (the threads occupy almost all volume) the critical presure comes down drastically since the shell in this case behaves like a system of threads.

The angle φ_0 depends of the values of the parameters d, α and δ_0. For large values of d the pits are elongated in the direction of the meridian, that angle φ_0 converges to $\pi/2$ and for highly elongated orthotropic elliptical shells the buckling modes are similar to those for the isotropic shells.

The increasing of the thread stiffness leads to the smaller angle φ_0.

The dependence of the critical loading and buckling mode on the angle between the system of threads is more complicated. For the angles α larger than $\pi/4$ the critical loading and buckling modes are equal to the the critical loading and modes for the isotropic elliptical shell. For the slightly elongated ellipsoid the critical loading attains its maximum for the angles close to $\pi/8$.

For the oblate orthotropic elliptical shell ($d < 1$) the value of f_0 may be determined in the unique way from the conditions:

$$\varphi_0: \quad \lambda(\varphi_0, 0) = \min_{\varphi}\frac{B^*_\varkappa}{\Delta},$$

from which it follows that

$$\varphi_0 = \pm \text{ Arctan } \left(\frac{E_2}{E_1}\right)^{1/4}$$

For small and large values of d the following approximate formulas may be used to obtain the critical loadings

$$\lambda_0 = \min_\theta \begin{cases} \dfrac{h^2\sqrt{E_1E_2}}{R^2\sqrt{3(1-\nu_1\nu_2)}} 2d^2, \text{ for } d \ll 1 \\ \dfrac{h^2\sqrt{E_1E_2}}{R^2\sqrt{3(1-\nu_1\nu_2)}} \dfrac{2}{2d^2-1}, \text{ for } d \gg 1 \end{cases}$$

4.2 Orthotropic elliptical shell under internal pressure.

We start the consideration with the case of the isotropic shell (δ_0=1). Since for the shell under internal normal pressure $t_1 < 0$, and $t_2 > 0$, then the inequality $t_2\rho_2 > t_1\rho_1$ holds for any values of the parameter d and relation (6) has the form

$$\lambda_0 = \min_\theta \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} \rho_1/t_2, \text{ for } \varphi = \pi/2$$

or

$$\lambda_0 = -\min_\theta \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}} \frac{2\rho_1\rho_2^2}{\rho_1 - 2\rho_2}, \text{ for } d \le \frac{\sqrt{2}}{2}$$

We seek the minimum of the function $\dfrac{2\rho_1\rho_2^2}{\rho_1 - 2\rho_2} = \dfrac{2\rho_2^4}{r_2^2 - 2d^2}$ under condition $2d^2 < r_2^2 < 1$. For $d < 1/2$ the function attains its minimum at $\rho_2 = 2d$, i. e. on the parallel $\theta = \arcsin\frac{3d^2}{1-d^2}$ and this minimum is equal to $16d^2$. For $1/2 < d < \frac{\sqrt{2}}{2}$ the minimum attains at $\rho_2 = 1$, i. e. on the equator and it is equal to $\frac{2}{1-2d^2}$.

Hence, the pits are elongated in the direction of the meridian and they moves from the equator to the pole as d decreases.

Consider the orthotropic elliptical shell described in the previous section. then the relation for the critical loading may be written as

$$\lambda_0 = \min_{\varphi,\theta}\{f^*(\varphi,\theta)\} = \min_{\substack{\varphi\in(\text{Arctan}\sqrt{\frac{d^2}{\rho_2^2-2d^2}},\pi/2] \\ \rho_2\in(\sqrt{2}d,1]}} f^*(\varphi,\theta) \tag{14}$$

where

$$f^*(\varphi,\theta) = 4\sqrt{\frac{B_\varkappa^*\Delta_k}{\Delta}} \frac{\rho_2^2 d^2\cos^2\varphi + \rho_2^4\sin^2\varphi}{-d^2\cos^2\varphi - (2d^2-\rho_2^2)\sin^2\varphi}$$

As before, we seek the minimum for all positive λ_0. We remind that the buckling of the elliptical shell under the internal pressure may occur only if $d < \sqrt{2}/2$.

For the shell reinforced with the threads the critical loading is higher than for the isotropic shell and the weakest parallel is closer to the equator for $1/2 < d < \sqrt{2}/2$. At the same time the orientation of the pit axis φ_0 changes significantly.

The critical loading decreases as the angle α increases. For $\alpha > \pi/4$ the buckling modes of isotropic and orthotropic shell practically coincide.

4.3 Cylindrical shell under axial compression

Finally we consider circular cylindrical shell under axial compressive force. In this case the dimensionless initial stress-couples are $t_2 = t_3 = 0$ and $t_1 > 0$. Substituting the coefficients in the elasticity relations [2] into the espression for the energies we obtain for the axisymmetric buckling mode ($k_2 = 0$)

$$B_\varepsilon = \frac{GhE_1E_2k_1^4}{R^2\left(GE_1k_1^4\right)} \qquad B_\varkappa = \frac{h^3\left(E_1k_1^4\right)}{12\left(1-\nu_1\nu_2\right)} \qquad B_t = k_1^2t_1 \tag{15}$$

and

$$T_1^0 = \lambda t_1 = 2\sqrt{\frac{hE_1E_2h^3}{R^2 12\left(1-\nu_1\nu_2\right)}} = \frac{h^2}{R}\sqrt{\frac{E_1E_2}{3\left(1-\nu_1\nu_2\right)}} \tag{16}$$

For the axisymmetric mode we get

$$T_1^0 = C\sqrt{\frac{E_1E_2}{3\left(1-\nu_1\nu_2\right)}}\cdot\frac{h^2}{R}$$

where

$$C = \sqrt{\frac{\sqrt{E_1E_2} + 2G\left(1-\nu_1\nu_2\right) + E_1\nu_2}{\sqrt{E_1E_2} + \left(E_1E_2/2G\right) - E_1\nu_2}}$$

Both formulas coincide with those given in [5].

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