Solving the problem of choosing an information system project by methods of tropical mathematics

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Abstract. A multicriteria problem of evaluating the ratings of alternatives based on pairwise comparisons when making a decision on choosing an information system project is considered. To solve the problem, an approach based on the weighted minimax log-Chebyshev approximation and the application of tropical mathematics methods are used. The obtained solution is compared with the known solution by the method of analytical hierarchy process.

Introduction

Multicriteria problems of evaluating alternatives based on pairwise comparisons are significant class of decision-making problems that are common in many areas. In multicriteria problems, the alternatives are compared in accordance with several criteria. The main difficulty of such problems is the absence in the general case of a solution that is the best for all criteria at once. The initial data for the problem are a set of m alternatives and their pairwise comparison by n criteria. The results of comparisons are represented in the form of pairwise comparisons matrices $\boldsymbol{A}_k$, where $1 \leq k \leq n$. The criteria are also compared with each other in pairs, and the results of the comparisons are recorded in a criteria comparisons matrix $\boldsymbol{C}$. The solution of the problem is the vector of absolute ratings, which determines the ranks of alternatives. One of the approaches to solve the problem is based on the log–Chebyshev approximation of pairwise comparisons matrices by consistent matrices (inversely symmetric matrices of unit rank). The log-Chebyshev approximation problem can be represented in terms of tropical mathematics and then solved analytically in a compact vector form.

1. Algebraic definitions

Tropical (idempotent) mathematics studies the theory and applications of algebraic systems with idempotent operations [1, 2, 3]. An operation is called idempotent if, when applied to the same arguments, it results in this argument. For example, the maximum operation is idempotent: $\max(x, x) = x$. Optimization problems formulated in terms of idempotent algebraic systems can be solved by methods of tropical optimization.

The paper uses max–algebra, an algebraic system, which is a set of non-negative real numbers $\mathbb{R}_+ = \{x \in \mathbb{R} | x \leq 0\}$ with addition and multiplication operations. Addition is defined as maximum and denoted by $\oplus$. Multiplication is defined and denoted as usual. Vector and matrix operations are performed according to standard rules with the replacement of arithmetic addition by the operation $\oplus$. The unit matrix is denoted by $\boldsymbol{I}$ and has the usual form. The integer non-negative power of a square matrix $\boldsymbol{A}$ denotes the result of the multiplication of the matrix by itself and is defined for all natural p as $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{p-1}$. The trace of a matrix $\boldsymbol{A} = (a_{ij})$ of order n is calculated by the formula $\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$.

The spectral radius of the matrix $\boldsymbol{A}$ is calculated by the formula

$$\lambda = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}^{1/n}(\boldsymbol{A}^n) = \bigoplus_{i=1}^{n} \operatorname{tr}^{1/i}(\boldsymbol{A}^i).$$

If $\lambda < 1$, then for the matrix $\boldsymbol{A}$ the Kleene operator is defined

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{i=0}^{n-1} \boldsymbol{A}^i.$$

More detailed information on the theory, methods and applications of theoretical mathematics can be found, for example, in [1, 2, 3].

2. Problem of choosing an information system project

Let us consider the problem of choosing an information system project described in [4]. The problem is to choose the most preferable information system project for implementation according to a set of criteria. The paper [4] provides a solution using the analytic hierarchy process of T. Saaty [5].

In the considered problem, a rating scale from 1 to 9 is used. In total, $m = 6$ competing alternative information system projects are considered. Alternatives are compared according to $n = 4$ criteria: increasing the accuracy of clerical operations, the efficiency of information processing, the promotion of organizational learning and the implementation costs.

A pairwise comparisons matrix of criteria is given by:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 1/9 & 1/7 & 1/5 \\ 9 & 1 & 2 & 5 \\ 7 & 1/2 & 1 & 3 \\ 5 & 1/5 & 1/3 & 1 \end{pmatrix}.$$

Pairwise comparisons matrices of alternatives for each criterion are:

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 1/3 & 1/6 & 1/6 & 1/3 & 1/9 \\ 3 & 1 & 1/3 & 1/3 & 1 & 1/8 \\ 6 & 3 & 1 & 1 & 3 & 1/8 \\ 6 & 3 & 1 & 1 & 3 & 1/8 \\ 3 & 1 & 1/3 & 1/3 & 1 & 1/8 \\ 9 & 8 & 8 & 8 & 8 & 1 \end{pmatrix}; \qquad \boldsymbol{A}_2 = \begin{pmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 1/4 & 1 & 7 & 3 & 1/5 & 1 \\ 1/3 & 1/7 & 1 & 1/5 & 1/5 & 1/6 \\ 1 & 1/3 & 5 & 1 & 1 & 1/3 \\ 1/3 & 5 & 5 & 1 & 1 & 3 \\ 1/4 & 1 & 5 & 3 & 1/3 & 1 \end{pmatrix};$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 1 & 1/5 & 2 & 1/3 & 1/2 & 2 \\ 5 & 1 & 7 & 2 & 3 & 7 \\ 1/2 & 1/7 & 1 & 1/5 & 1/2 & 1 \\ 3 & 1/2 & 5 & 1 & 2 & 5 \\ 2 & 1/3 & 2 & 1/2 & 1 & 3 \\ 1/2 & 1/7 & 1 & 1/5 & 1/3 & 1 \end{pmatrix}; \qquad \boldsymbol{A}_4 = \begin{pmatrix} 1 & 5 & 4 & 2 & 3 & 1/3 \\ 1/5 & 1 & 1/2 & 1/4 & 1/3 & 1/8 \\ 1/4 & 2 & 1 & 1/3 & 1/2 & 1/6 \\ 1/2 & 4 & 3 & 1 & 2 & 1/4 \\ 1/3 & 3 & 2 & 1/2 & 1 & 1/5 \\ 3 & 8 & 6 & 4 & 5 & 1 \end{pmatrix}.$$

We now describe the solution of the problem, which is based on the weighted minimax log-Chebyshev approximation and obtained using methods of tropical optimization proposed in [6, 7].

To solve the problem we use analytical computations in terms of max-algebra which is a tropical semifield with addition defined as maximum.

To determine the weights of the criteria, we first calculate the spectral radius of the matrix $\boldsymbol{C}$ given by

$$\lambda = \operatorname{tr} \boldsymbol{C} \oplus \cdots \oplus \operatorname{tr}^{1/4}(\boldsymbol{C}^4) = (25/9)^{1/3} \approx 1.4057.$$

The calculation of the Kleene star matrix, whose columns generate all the optimal vectors of the weights of the criteria, gives the following result:

$$\boldsymbol{D} = (\lambda^{-1}\boldsymbol{C})^* = \begin{pmatrix} 1 & 1/9\lambda & 2\lambda/25 & \lambda/5 \\ 9\lambda & 1 & 2/\lambda & 5/\lambda \\ 27\lambda/5 & 3/5 & 1 & 3/\lambda \\ 5/\lambda & \lambda/5 & 2/5 & 1 \end{pmatrix}.$$

If the columns in the matrix generate a unique (up to a positive multiplier) vector, this vector is taken as the vector of weights. Otherwise some best and worst differentiating vectors of weights are obtained. As the best (worst) vector of weights, a vector is considered for which the ratio between the maximum and minimum elements is maximal (minimal).

We normalize the columns of the matrix $\boldsymbol{D}$ with respect to the maximum element, which makes the maximum element in each column be equal to 1. In this case, the best solutions corresponds to the columns whose minimum elements are the smallest among all columns, and the worst ones to the vectors whose minimum elements are the largest. The normalized matrix $\boldsymbol{D}$ is equal to

$$\begin{pmatrix} 1/9\lambda & 1/9\lambda & 1/9\lambda & 1/9\lambda \\ 1 & 1 & 1 & 1 \\ 3/5 & 3/5 & \lambda/2 & 3/5 \\ \lambda/5 & \lambda/5 & \lambda/5 & \lambda/5 \end{pmatrix}.$$

The best differentiating vector of weights is the vector

$$\boldsymbol{v} = (1/9\lambda,\ 1,\ 3/5,\ \lambda/5)^{\mathrm{T}},$$

and the worst is vector

$$\boldsymbol{w} = (1/9\lambda,\ 1,\ \lambda/2,\ \lambda/5)^{\mathrm{T}}.$$

Let us take the best differentiating vector of weights $\boldsymbol{v}$ and evaluate a weighted sum

$$\boldsymbol{P} = 1/9\lambda\boldsymbol{A}_1 + \boldsymbol{A}_2 + 3/5\boldsymbol{A}_3 + \lambda/5\boldsymbol{A}_4 =$$
$$= \begin{pmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 3 & 1 & 7 & 3 & 9/5 & 21/5 \\ 2/3\lambda & 2\lambda/5 & 1 & 1/5 & 3/10 & 3/5 \\ 9/5 & 4\lambda/5 & 5 & 1 & 6/5 & 3 \\ 6/5 & 5 & 5 & 1 & 1 & 3 \\ 3\lambda/5 & 8\lambda/5 & 6 & 3 & \lambda & 1 \end{pmatrix}.$$

The spectral radius of the matrix $\boldsymbol{P}$ is defined as

$$\mu = \operatorname{tr}\boldsymbol{P} \oplus \cdots \oplus \operatorname{tr}^{1/6}(\boldsymbol{P}^6) = 45^{1/3} \approx 3.5569.$$

Calculation of the Kleene star matrix gives

$$(\mu^{-1}\boldsymbol{P})^* = \begin{pmatrix} 1 & 675/\mu^5 & 378/\mu^4 & 189/\mu^4 & 3/\mu & 7/5 \\ 3/\mu & 1 & 1134/\mu^5 & 567/\mu^5 & 405/\mu^5 & 21/5\mu \\ 2/15 & 2\lambda/5\mu & 1 & 126/\lambda\mu^5 & 90/\lambda\mu^5 & 42/\lambda\mu^4 \\ 81/\mu^4 & 3/5 & 18/\mu^2 & 1 & 243/\mu^5 & 3/\mu \\ 675/\mu^5 & 225/\mu^4 & 14/5 & 7/5 & 1 & 945/\mu^5 \\ 8/15 & 8\lambda/5\mu & 6/\mu & 3/\mu & 72\lambda/5\mu^3 & 1 \end{pmatrix}.$$

Using the generating matrix $(\mu^{-1}\boldsymbol{P})^*$ we calculate the best differentiating vector of ratings:

$$\boldsymbol{x} = (3/\mu,\ 405/\mu^5,\ 90/\lambda\mu^5,\ 243/\mu^5,\ 1,\ 72\lambda/5\mu^3)^{\mathrm{T}} \approx$$
$$\approx (0.8434,\ 0.7114,\ 0.1125,\ 0.4268,\ 1.0000,\ 0.4498)^{\mathrm{T}}.$$

The obtained vector sets the order $A_5 > A_1 > A_2 > A_6 > A_4 > A_3$.

With the worst differentiating vector of weights $\boldsymbol{w}$ we have the weighted sum

$$\boldsymbol{R} = 1/9\lambda\boldsymbol{A}_1 + \boldsymbol{A}_2 + \lambda/2\boldsymbol{A}_3 + \lambda/5\boldsymbol{A}_4 =$$

$$= \begin{pmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 5\lambda/2 & 1 & 7 & 3 & 3\lambda/2 & 7\lambda/2 \\ 2/3\lambda & 2\lambda/5 & 1 & 1/5 & \lambda/4 & \lambda/2 \\ 3\lambda/2 & 4\lambda/5 & 5 & 1 & \lambda & 5\lambda/2 \\ \lambda & 5 & 5 & 1 & 1 & 3 \\ 3\lambda/5 & 8\lambda/5 & 6 & 3 & \lambda & 1 \end{pmatrix}.$$

The matrix $\boldsymbol{R}$ has the spectral radius

$$\mu = \operatorname{tr}\boldsymbol{R} \oplus \cdots \oplus \operatorname{tr}^{1/6}(\boldsymbol{R}^6) = (75\lambda/2)^{1/3} \approx 3.7495.$$

The Kleene star matrix $(\mu^{-1}\boldsymbol{R})^*$ is equal to

$$\begin{pmatrix} 1 & 1125\lambda/2\mu^5 & 315\lambda/\mu^4 & 315\lambda/2\mu^4 & 225\lambda/2\mu^4 & 7/5 \\ 375\lambda^2/4\mu^4 & 1 & 1575\lambda^2/2\mu^5 & 1575\lambda^2/4\mu^5 & 1125\lambda^2/4\mu^5 & 525\lambda^2/4\mu^4 \\ 75\lambda^3/2\mu^5 & 15\lambda^2/\mu^4 & 1 & 21\lambda^2/5\mu^3 & 3\lambda^2/\mu^3 & 105\lambda^3/2\mu^5 \\ 225\lambda^2/4\mu^4 & 3/5 & 15\lambda/\mu^2 & 1 & 675\lambda^2/4\mu^5 & 5\lambda/2\mu \\ 1875\lambda^2/4\mu^5 & 375\lambda/2\mu^4 & 14/5 & 7/5 & 1 & 2625\lambda^2/4\mu^5 \\ 150\lambda^3/\mu^5 & 60\lambda^2/\mu^4 & 6/\mu & 3/\mu & 12\lambda^2/\mu^3 & 1 \end{pmatrix}.$$

The worst differentiating vector of ratings takes the form:

$$\boldsymbol{y} = (4\mu^5/1875\lambda^2,\ 2\mu^4/375\lambda,\ 5/14,\ 5/7,\ 1,\ 4\mu^5/2625\lambda^2)^{\mathrm{T}} \approx$$
$$\approx (0.8001,\ 0.7499,\ 0.3571,\ 0.7143,\ 1,\ 0.5715)^{\mathrm{T}}.$$

The order defined by this vector is the order $A_5 > A_1 > A_2 > A_4 > A_6 > A_3$.

Conclusion

Note that in [4], the order of alternatives obtained using the analytic hierarchy process is given by $A_1 > A_2 > A_5 > A_6 > A_4 > A_3$. If we compare it with the best log-Chebyshev solution, we can see that the difference in solutions is in the first three ranks, the last three ranks completely coincide. In the case of comparison with the results of the worst differentiating solution, the three most preferred and the three least preferred alternatives coincide, but there is a different order within the triples. The difference between solutions makes it somewhat difficult to choose one most preferable alternative, but allows one to recognize a group of three most preferable alternatives.

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