Using Tropical Algebra for Evaluating Consumer Preferences for Hotels in Marketing Research

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1. Evaluation of Preferences for Hotels in Marketing Research

We consider the problem of evaluating the ratings (priorities, weights) of criteria used by customers to choose a hotel to stay. The criteria include hotel location, price, type of guests, breakfast included, room equipment, courtesy of staff.

A survey of 202 respondents of both genders aged 17 to 26 is being conducted. Each respondent compares the criteria in pairs to estimate their relative importance and evaluates the ratings and ranks of the criteria directly. The purpose of the research is to obtain the absolute ratings of criteria from the results of pairwise comparisons and then determine the ranks of alternatives.

To solve the problem, we apply an approach based on the application of tropical algebra. The obtained results are compared with the results of application of the principal eigenvector method by T. Saaty and the geometric means method as well as with the results of direct evaluation of criteria.

2. The Problem of Pairwise Comparisons and Its Solution

Let $\boldsymbol{A} = (a_{ij})$ be a pairwise comparison matrix of n alternatives, where the element $a_{ij} > 0$ indicates that alternative i is a_{ij} times more preferred than j. The elements of $\boldsymbol{A}$ satisfy the property $a_{ij} = 1/a_{ji}$, which means that the matrix $\boldsymbol{A}$ is symmetrically reciprocal. Given a pairwise comparison matrix $\boldsymbol{A}$, the problem is to determine the vector $\boldsymbol{x}$ of absolute ratings of alternatives.

A pairwise comparison matrix $\boldsymbol{A}$ is consistent if it has the transitive property $a_{ij} = a_{ik}a_{kj}$ for all i, j, k. For any consistent matrix $\boldsymbol{A}$, there exists a positive vector $\boldsymbol{x} = (x_i)$ that determines the elements of $\boldsymbol{A}$ by the condition $a_{ij} = x_i/x_j$ and thus is a solution to the problem of evaluating absolute ratings of alternatives.

In real-world problems, the pairwise comparison matrices are usually not consistent. In this case, the initial inconsistent matrix is replaced by an approximate consistent matrix, and its vector of absolute ratings is taken as a solution.

The principal eigenvector method by T. Saaty is a wide used heuristic technique that offers a solution in the form of the eigenvector of the pairwise comparison matrix, corresponding to the maximum eigenvalue of the matrix.

The approximation methods solve the problem of minimizing an error of the approximation of a pairwise comparisons matrix $\boldsymbol{A} = (a_{ij})$ by a consistent matrix $\boldsymbol{X} = (x_i/x_j)$. If the error is measured on a logarithmic scale, then a solution can be obtained in analytical form. Specifically, the approximation with the Euclidean norm in logarithmic scale leads to the method of geometric means, where the elements of the vector of absolute ratings are calculated as geometric means of the elements in rows of the matrix $\boldsymbol{A}$. Approximation of the matrix $\boldsymbol{A}$ in the log-Chebyshev metric is reduced to the minimization problem without logarithm

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \max_{1\leq i,j\leq n} a_{ij}x_j/x_i. \tag{1}$$

Both methods of the principal eigenvector and the geometric means provide a unique solution vector (up to a positive factor). The solution based on the log-Chebyshev approximation may be nonunique. If there is a set $\mathcal{S}$ of different solution vectors, it is natural to define some "best" and "worst" solutions that most and least differentiate the alternatives with the maximum and minimum ratings. These best and worst differentiating solutions can be found by solving the following problems:

$$\max_{\boldsymbol{x}\in\mathcal{S}} \max_{1\leq i\leq n} x_i \times \max_{1\leq j\leq n} x_j^{-1}, \qquad \min_{\boldsymbol{x}\in\mathcal{S}} \max_{1\leq i\leq n} x_i \times \max_{1\leq j\leq n} x_j^{-1}. \tag{2}$$

To solve problems (1) and (2), we apply methods of tropical optimization, which allow obtaining analytical solutions in an explicit form [1, 2, 3].

3. Algebraic Solution of the Pairwise Comparison Problem

Tropical (idempotent) mathematics [4, 5, 6] deals with the theory and applications of algebraic systems with idempotent operations. A typical example of the algebraic system under study is the max-algebra, which is the set of non-negative reals $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$, where addition is denoted by the symbol $\oplus$ and defined as $x \oplus y = \max\{x, y\}$, whereas multiplication is denoted and defined as usual.

Vector and matrix operations are performed according to standard rules with the arithmetic addition replaced by $\oplus$. The zero vector is denoted by $\boldsymbol{0}$ and has the standard form. For a nonzero column vector $\boldsymbol{x} = (x_j)$ the conjugate transpose is the row vector $\boldsymbol{x}^- = (x_j^-)$, where $x_j^- = x_j^{-1}$ if $x_j \neq 0$, and $x_j^- = 0$ otherwise. For the vector $\boldsymbol{1} = (1, \ldots, 1)^T$, the conjugate transpose is $\boldsymbol{1}^- = \boldsymbol{1}^T$.

The conjugate transpose of a matrix $\boldsymbol{A} = (a_{ij})$ is the matrix $\boldsymbol{A}^- = (a_{ij}^-)$, where $a_{ij}^- = a_{ji}^{-1}$ if $a_{ji} \neq 0$, and $a_{ij}^- = 0$ otherwise. The identity matrix is denoted by $\boldsymbol{I}$ and has the usual form. A nonnegative integer power of a square matrix $\boldsymbol{A}$ is defined for all natural p by the conditions $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{p-1}$.

The trace of a matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by $\operatorname{tr}\boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$. The spectral radius of the matrix $\boldsymbol{A}$ is the scalar $\lambda = \operatorname{tr}\boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}^{1/n}(\boldsymbol{A}^n)$. If $\lambda \leq 1$, then the Kleene star matrix is given by $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$.

In terms of max-algebra, the approximation problem at (1) becomes

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}. \qquad (3)$$

All solutions of problem (3) are given in parametric form by

$$\boldsymbol{x}=\boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{u}>\boldsymbol{0}, \qquad \boldsymbol{B}=(\lambda^{-1}\boldsymbol{A})^{*},$$

where λ is the spectral radius of the pairwise comparison matrix $\boldsymbol{A}$.

The obtained solution is unique (up to a positive factor) if all columns $\boldsymbol{b}_j$ in the matrix $\boldsymbol{B}$ are collinear. Otherwise, problems (2) are solved to find the most and least differentiating solutions. The best differentiating solution is given by

$$\boldsymbol{x}_1=\boldsymbol{B}(\boldsymbol{I}\oplus\boldsymbol{B}_{lk}^{-}\boldsymbol{B})\boldsymbol{u}_1, \qquad \boldsymbol{u}_1>\boldsymbol{0},$$

where $\boldsymbol{B}_{lk}$ is obtained from $\boldsymbol{B}=(b_{ij})$ by zeroing all elements except b_{lk} with

$$k=\arg\max_j \boldsymbol{1}^T\boldsymbol{b}_j\boldsymbol{b}_j^{-}\boldsymbol{1}, \qquad l=\arg\max_i b_{ik}^{-1}.$$

The worst differentiating solution is defined as

$$\boldsymbol{x}_2=(\Delta^{-1}\boldsymbol{1}\boldsymbol{1}^T\oplus\lambda^{-1}\boldsymbol{A})^{*}\boldsymbol{u}_2, \qquad \boldsymbol{u}_2>\boldsymbol{0}, \qquad \Delta=\boldsymbol{1}^T\boldsymbol{B}\boldsymbol{1}=\boldsymbol{1}^T(\lambda^{-1}\boldsymbol{A})^{*}\boldsymbol{1}.$$

4. Statistical Analysis of Numerical Results

To evaluate feasibility and accuracy of the algebraic solution based on the log-Chebyshev approximation, a statistical analysis is performed on the numerical results of rating criteria to choose a hotel for all respondents. The data under study include ratings of criteria obtained from the pairwise comparison matrices by applying the methods of the principal eigenvector, geometric means and log-Chebyshev approximation. The vectors of ratings provided by the methods are used to determine corresponding ranks of criteria, which are compared to one another and to the ranks given directly by respondents. As a preliminary result of the analysis, the next table demonstrates the number of complete matches of the ranking for every pair of methods including the direct ranking.

	Direct	Saaty	Geometric	Log-Chebyshev Best	Log-Chebyshev Worst
Direct	202				
Saaty	56	202			
Geometric	56	184	202		
Log-Chebyshev Best	59	124	124	202	
Log-Chebyshev Worst	56	123	125	130	202

Further research consists of correlation analysis, range and scatter plot analysis, cluster and classification analysis for both ratings and ranks obtained. The comparison of results shows that all methods based on pairwise comparison data lead to close results. The method of geometric means and the Saaty method of principal eigenvector give almost the same ranking for the data available from the respondents. The log-Chebyshev approximation provides similar results that are more close to the direct ranking by respondents than the other two methods.

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