Using Tropical Algebra to Solve a Minimax Optimization Problem in Project Scheduling

Nikolai Krivulin

Abstract. We apply tropical optimization techniques to derive a complete analytical solution to a time-constrained project scheduling problem.

1. Time-Constrained Project Scheduling Problem

Consider a project that involves n activities (jobs, tasks, operations) to be performed in parallel, subject to a set of temporal constraints, including the release time and deadline constraints, as well as start-start and start-finish relationships. The scheduling problem is to find an optimal schedule that minimizes the maximum deviation of the start time of all activities.

For each activity $i = 1, \ldots, n$, we denote the start times by x_i and the finish time by y_i. Given release times g_i that specify the earliest allowed time for activities to start and deadlines f_i that define the latest time for activities to finish, the release time and deadline constraints take the form of the inequalities

$$g_i \leq x_i, \quad y_i \leq f_i, \qquad i = 1, \ldots, n.$$

The start-start constraints indicate the minimum allowed time lag between the start of one activity and the start of another. Given parameters b_{ij}, the start-start constraints are represented as

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i, \qquad i = 1, \ldots, n.$$

The start-finish constraints determine the minimum allowed time lag between the start of one activity and the finish of another. Given parameters a_{ij}, the start-finish constraints are represented as

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq y_i, \qquad i = 1, \ldots, n.$$

Considering the deadlines $y_i \leq f_i$, these constraints can be replaced by

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq f_i, \qquad i = 1, \ldots, n.$$

If some parameters g_i, b_{ij} and c_{ij} are not defined, we set them equal to $-\infty$.

The maximum deviation of the start time of activities is defined as the interval between the earliest and the latest start times of activities and is given by

$$\max_{1 \le i \le n} x_i - \min_{1 \le i \le n} x_i = \max_{1 \le i \le n} x_i + \max_{1 \le i \le n} (-x_i).$$

The scheduling problem with the parameters g_i, h_i, b_{ij} and c_{ij} given for all $i, j = 1, \ldots, n$, is to find start times x_i that solve the minimax problem

$$
\begin{aligned}
&\min_{x_1, \ldots, x_n} \quad \max_{1 \le i \le n} x_i + \max_{1 \le i \le n} (-x_i), \\
&\text{s.t.} \quad \max_{1 \le j \le n} (b_{ij} + x_j) \le x_i, \quad \max_{1 \le j \le n} (c_{ij} + x_j) \le f_i, \qquad (1) \\
&\qquad g_i \le x_i, \quad i = 1, \ldots, n.
\end{aligned}
$$

We observe that problem (1) can be represented as a linear program, and then a numerical solution of the problem can be obtained using one of the computational algorithms available in linear programming. To derive a complete analytical solution to the problem, we formulate it in terms of tropical algebra and apply methods and results of tropical optimization.

2. Elements of Tropical Algebra

Let $\mathbb{X}$ be a set that is equipped with two distinct elements: zero $\mathbb{0}$ and identity $\mathbb{1}$, and closed under associative and commutative binary operations: addition $\oplus$ and and multiplication $\otimes$. Addition is idempotent which means that $x \oplus x = x$ for all $x \in \mathbb{X}$. Multiplication distributes over addition and is invertible which provides each nonzero $x \in \mathbb{X}$ with an inverse $x^{-1} \in \mathbb{X}$ such that $x \otimes x^{-1} = \mathbb{1}$. Finally, the set $\mathbb{X}$ is assumed to be totally ordered by an order relation consistent with that induced by idempotent addition using the rule: $x \le y$ if and only if $x \oplus y = y$.

The algebraic system $(\mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is usually referred to as the idempotent semifield. In what follows, the multiplication sign is omitted to save writing.

As an example, one can consider the real semifield $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, which is usually called the max-plus algebra. In this semifield, the operations are defined as $\oplus = \max$ and $\otimes = +$, and the neutral elements as $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. For any $x \in \mathbb{R}$, the multiplicative inverse is equal to the opposite number $-x$ in the standard representation. The power x^y coincides with the usual product $x \times y$. The order relation $\le$ corresponds to the natural linear order on $\mathbb{R}$.

The scalar operations $\oplus$ and $\otimes$ are extended to vectors and matrices over $\mathbb{X}$ in the usual way. All vectors are assumed column vectors unless transposed. A matrix (vector) with all entries equal to $\mathbb{0}$ is the zero matrix (vector). A vector that has no zero elements is called regular. Any matrix without zero columns is called column-regular. A vector with all elements equal to $\mathbb{1}$ is denoted by $\mathbf{1} = (\mathbb{1}, \ldots, \mathbb{1})^T$. Multiplicative conjugate transposition of a nonzero vector $\boldsymbol{x} = (x_i)$ yields the row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \ne \mathbb{0}$ and $x_i^- = \mathbb{0}$ otherwise.

A square matrix that has all diagonal elements equal to $\mathbb{1}$ and off-diagonal to $\mathbb{0}$, is the identity matrix denoted by $\boldsymbol{I}$. The power notation is defined as follows: $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}\boldsymbol{A}^{p-1}$ for any square matrix $\boldsymbol{A}$ and integer $p > 0$.

The trace of a square matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by

$$\operatorname{tr}(\boldsymbol{A}) = a_{11} \oplus \cdots \oplus a_{nn} = \bigoplus_{i=1}^{n} a_{ii}.$$

A tropical analogue of the matrix determinant is defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr}\boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}\boldsymbol{A}^n = \bigoplus_{m=1}^{n} \operatorname{tr}\boldsymbol{A}^m.$$

If the condition $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ holds, the Kleene star matrix is calculated as

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{m=0}^{n-1} \boldsymbol{A}^m.$$

For any matrix $\boldsymbol{A} = (a_{ij})$ and vector $\boldsymbol{x} = (x_i)$, tropical norms are given by

$$\|\boldsymbol{A}\| = \mathbf{1}^T \boldsymbol{A} \mathbf{1} = \bigoplus_{i,j} a_{ij}, \qquad \|\boldsymbol{x}\| = \mathbf{1}^T \boldsymbol{x} = \boldsymbol{x}^T \mathbf{1} = \bigoplus_{i} x_i,$$

which coincide in max-plus algebra with the maximum entries of $\boldsymbol{A}$ and $\boldsymbol{x}$.

3. Algebraic Solution of Scheduling Problem

We now present a complete analytical solution of the scheduling problem of interest, obtained in the framework of tropical algebra [1]. Other related examples of applications of tropical optimization can be found, e.g., in [2, 3, 4, 5].

Consider the minimax optimization problem at (1) and represent the problem in scalar form in terms of max-plus algebra to write

$$\begin{aligned}
\min_{x_1,\ldots,x_n} \quad & \bigoplus_{1 \leq i \leq n} x_i \bigoplus_{1 \leq j \leq n} x_j^{-1}, \\
\text{s.t.} \quad & \bigoplus_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \quad \bigoplus_{1 \leq j \leq n} c_{ij} x_j \leq f_i, \\
& g_i \leq x_i, \quad i = 1, \ldots, n.
\end{aligned} \tag{2}$$

With the following matrix-vector notation:

$$\boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{C} = (c_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{g} = (g_i), \qquad \boldsymbol{f} = (f_i),$$

problem (2) takes the vector form

$$\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{x}^- \mathbf{1}\mathbf{1}^T \boldsymbol{x}, \\
\text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \\
& \boldsymbol{g} \leq \boldsymbol{x}.
\end{aligned} \tag{3}$$

Application of tropical optimization techniques developed in [2, 3, 4, 5] leads to the next result, which offers a complete analytical solution to problem (3).

Lemma 1. *Let $\boldsymbol{B}$ be a matrix with $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ and $\boldsymbol{C}$ a column-regular matrix. Let $\boldsymbol{g}$ be a vector and $\boldsymbol{f}$ a regular vector such that $\boldsymbol{f}^-\boldsymbol{C}\boldsymbol{B}^*\boldsymbol{g} \leq \mathbb{1}$.*

Then, the minimum of the objective function in problem (3) is equal to

$$\theta = \|\boldsymbol{B}^*\| \oplus \bigoplus_{0 \leq i+j \leq n-2} \|\boldsymbol{f}^-\boldsymbol{C}\boldsymbol{B}^i\|\|\boldsymbol{B}^j\boldsymbol{g}\|,$$

and all regular solutions are given in parametric form by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^-\boldsymbol{C}\boldsymbol{G})^-,$$

where $\boldsymbol{u}$ is a vector of parameters and

$$\boldsymbol{G} = \boldsymbol{B}^* \oplus \bigoplus_{0 \leq i+j \leq n-2} \theta^{-1}\boldsymbol{B}^i\mathbb{1}\mathbb{1}^T\boldsymbol{B}^j.$$

References

[1] N.K. Krivulin and S.A. Gubanov, Algebraic solution of a problem of optimal project scheduling in project management, *Vestnik St. Petersburg University, Mathematics*, **54**(1), 58–68 (2021). `http://doi.org/10.1134/S1063454121010088`

[2] N. Krivulin, A constrained tropical optimization problem: Complete solution and application example, in *Tropical and Idempotent Mathematics and Applications*, ed. by G. L. Litvinov and S. N. Sergeev, AMS, Providence, RI, 2014, 163–177 (Contemp. Math., vol.616). `http://doi.org/10.1090/conm/616/12308`

[3] N. Krivulin, Extremal properties of tropical eigenvalues and solutions to tropical optimization problems, *Linear Algebra Appl.*, **468**, 211–232 (2015). `http://doi.org/10.1016/j.laa.2014.06.044`

[4] N. Krivulin, Tropical optimization problems in time-constrained project scheduling, *Optimization*, **66**(2), 205–224 (2017). `http://doi.org/10.1080/02331934.2016.1264946`

[5] N. Krivulin, Tropical optimization problems with application to project scheduling with minimum makespan, *Ann. Oper. Res.*, **256**(1), 75–92 (2017). `http://doi.org/10.1007/s10479-015-1939-9`

Nikolai Krivulin
Faculty of Mathematics and Mechanics
St. Petersburg State University
St. Petersburg, Russia
e-mail: `nkk@math.spbu.ru`