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Eigenoscillations in an angular domain and spectral properties of functional equations

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This work studies functional difference equations of the second order with a potential belonging to a special class of meromorphic functions. The equations depend on a spectral parameter. Consideration of this type of equations is motivated by applications in diffraction theory and by construction of eigenfunctions for the Laplace operator in angular domains. In particular, such eigenfunctions describe eigenoscillations of acoustic waves in angular domains with 'semitransparent' boundary conditions. For negative values of the spectral parameter, we study essential and discrete spectrum of the equations and describe properties of the corresponding solutions. The study is based on the reduction of the functional difference equations to integral equations with a symmetric kernel. A sufficient condition is formulated for the potential that ensures existence of the discrete spectrum. The obtained results are applied for studying the behaviour of eigenfunctions for the Laplace operator in adjacent angular domains with the Robin-type boundary conditions on their common boundary. At infinity, the eigenfunctions vanish exponentially as was expected. However, the rate of such decay depends on the observation direction. In particular, in a vicinity of some directions, the regime of decay is switched from one to another and such asymptotic behaviour is described by a Fresnel-type integral.

Key words: Functional difference equations, spectrum, asymptotic behaviour, eigenfunctions of the Laplacian, Malyuzhinets' functional equations

2020 Mathematics Subject Classification: 35A22 (Primary); 35C15, 39B22 (Secondary)

1 Introduction

Recently, functional difference (FD) equations with meromorphic coefficients have become a fairly natural tool of research in diffraction theory [4, 17], in quantum theory [8, 13] in the theory of small oscillations of a fluid [16, 7], and in spectral theory of mathematical physics [10]. Starting with the Malyuzhinets' work [21] on the wave diffraction in angular regions with impedance boundary conditions, see also [23, 25, 6], FD equations of the first order are frequently used in applications and, by analogy with the linear differential equations of the first order, admit an explicit solution in quadratures. However, in the case of the second-order FD equations with variable coefficients, the analysis requires special approaches, in particular, those implying reduction to integral equations. Some additional examples and results, as well as other applications, can be found in [2].

In some applications, FD equations under consideration also depend on a spectral parameter. A natural question of their study is to determine some values of the spectral parameter for which

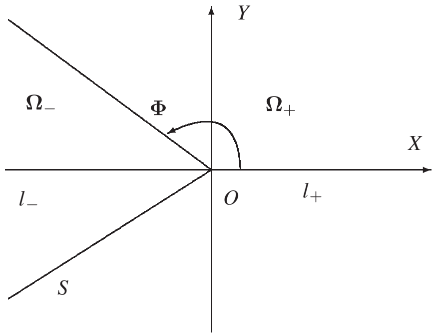

FIGURE 1. The angular domains $\Omega_{\pm}$.

the equations have nontrivial solutions in a certain class. The corresponding spectral problems for the FD equations at hand traditionally arise in the framework of reduction of a spectral boundary-value problem for a partial differential operator. In our case, this operator is a Laplacian in 2D domains with Robin-type boundary conditions, see also [3, 15].

In this work, we consider a particular spectral problem for the Laplace operator in 2D angular domains with Robin-type boundary conditions on their common boundary. On the other hand, this problem enables one reduction to an FD equation of the second order depending on an auxiliary spectral parameter. At the same time, such a reduction leads to an appropriate natural choice of the solutions' class and to description of a set of meromorphic potentials for the equation at hand. Our results dealing with the spectral properties of solutions to the FD equations are then exploited in order to give integral representations for the eigenfunctions of the Laplace operator mentioned above. We also describe behaviour of the corresponding eigenfunctions at large distances making flexible use of the integral representations and their asymptotic evaluation.

Mathematically similar problems arise when studying small oscillations of a fluid continuum in gravitational field. For instance, consider a wedge-like penetrable surface S (with the edge OZ) separating two infinite supplementary domains $\Omega_{\pm}$ (wedges) of the fluid assuming that their closure coincides with R^3, see 2D projection in Figure 1. It is assumed that potential w of the fluid velocity solves the Laplace equation $\triangle_{x,y,z} w = 0$, which is a harmonic function of time t represented in the form $w = \exp(-i\omega t)\cos(\sqrt{-E}z)\, U(x,y;E)$, $E < 0$. The normal component of the velocity ($V_n := \partial_n U$) is continuous across the surface S, whereas U satisfies a condition of the Robin type (with Robin parameter $\gamma > 0$) after separation of the variables t and z. These conditions simulate capacity of the fluid to penetrate across the boundary S. As a result, we can arrive at the problem to determine E and the related solution $U(x,y;E)$, satisfying modified Helmholtz equation, with finite energy (see explicit formulation below).

In the next section, we formulate a spectral problem for a self-adjoint Laplacian in 2D angular domains. By means of the Kontorovich–Lebedev (KL) transform and partial separation of variables, we reduce the corresponding boundary-value problem to a spectral problem for an FD equation of the second order with a meromorphic potential. In the third section, spectral properties of the FD equation at hand are carefully studied. To this end, an FD equation is reduced to a self-adjoint integral equation with spectral parameter. The spectrum of this integral operator is directly associated with that of the FD equation. On this way, we specify essential and discrete spectrum of the FD equations under consideration. The fourth section is devoted to the study of KL integral representation and to derivation of the far-field asymptotics of the eigenfunctions of the Laplacian at hand. In order to derive the asymptotics, it is beneficial to reduce the KL representations for an eigenfunction to Sommerfeld integrals, which are well adapted to asymptotic evaluation. It is shown that the Sommerfeld transformants (functions in the integrand of the Sommerfeld representation) satisfy Malyuzhinets functional equations with trigonometric coefficients. The latter equations enable us to determine poles of the Sommerfeld transformants. The poles of the transformants and the corresponding saddle points play a crucial role in the asymptotic evaluation of the Sommerfeld integrals with the aid of the saddle point technique or its relevant modifications.

It should be remarked that in order to study eigenfunctions and their asymptotics, we make use of various techniques recently developed for similar problems of wave scattering by wedges and cones [19, 6]. On the other hand, analogous results have been recently published (online) that deal with the asymptotic behaviour of the eigenfunctions in a 3D cone-like domain [20]. However, contrary to the current paper, where the behaviour of the eigenfunctions near the so-called singular directions is described by a Fresnel-type integral, in the mentioned work, a parabolic-cylinder-type function is used for evaluation of the asymptotics in the transition zone. Although the applied methods are, in general, similar in both works, the results and technical details in 2D and 3D cases are qualitatively different.

2 Formulation of the problem and reduction to an FD equation

2.1 Formulation

Consider two adjacent angular domains Ω_+ and Ω_- (Figure 1) with their common boundary S which consists of two half-lines $S_+ = \{r > 0, \quad \varphi = \Phi\}$ and $S_- = \{r \geqslant 0, \quad \varphi = -\Phi\}$ with the same origin O, $(\pi/2 < \Phi < \pi)$

$$X = r\cos\varphi, \quad Y = r\sin\varphi.$$

These domains are symmetric with respect to (w.r.t.) their bisectors l_+ and l_- correspondingly.

We study solutions of the homogeneous problem with spectral parameter E

$$A_\gamma\, U = E\, U \tag{2.1}$$

for a formally symmetric operator A_γ in (2.1) defined in the ‘classical’ terms of differential equations and boundary conditions,

$$\begin{aligned} -\triangle u^+(r,\omega) - Eu^+(r,\omega) = 0, \quad (r,\varphi) \in \Omega_+, \\ -\triangle u^-(r,\omega) - Eu^-(r,\omega) = 0, \quad (r,\varphi) \in \Omega_-, \end{aligned} \tag{2.2}$$

$\triangle = \nabla\cdot\nabla = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$. The boundary conditions on $S = S_+ \cup S_-$ read

$$\begin{aligned} \left.\frac{\partial u^+}{\partial n}\right|_S &= \gamma(u^+|_S - u^-|_S), \\ \left.\frac{\partial u^+}{\partial n}\right|_S &= \left.\frac{\partial u^-}{\partial n}\right|_S, \end{aligned} \tag{2.3}$$

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where $\gamma > 0$ is Robin parameter, the normal n is directed from Ω_+ to Ω_- and $\frac{\partial}{\partial n}\Big|_{S_\pm} = \pm\frac{1}{r}\frac{\partial}{\partial\varphi}\Big|_{\varphi=\pm\Phi}$, $\pi/2 < \Phi < \pi$. It is worth commenting on that, in view of the elliptic regularity of solutions and smoothness of the surface S as $r \neq 0$, the equations (2.2) and boundary conditions (2.3) can be understood in classical sense. In physics of acoustic waves, the conditions (2.3) are traditionally called conditions of semitransparency [17]. These conditions can be also interpreted as singular δ'−interaction with the support on S [3]. The Schrödinger operator A_γ with singular potential corresponding to the classical problem at hand can be formally written as $-\triangle - \gamma\delta'_S$. It should be noticed, however, that our approach, with appropriate modifications, enables one to consider operator $-\triangle - \gamma\delta_S$, i.e. depending on singular interaction with δ−potential. In this case, the boundary conditions on S corresponding to δ−potential are exploited for the simulation of interaction of an electromagnetic wave with a thin dielectric layer (impedance sheet), see Chapter 10 in [2].

We consider negative spectrum $E < 0$ of A_γ and construct classical solutions of (2.2), (2.3) which additionally satisfy the so called Meixner's condition ($r \to 0$) at the vertex O,

$$\begin{aligned} &|u^\pm(r,\varphi)| < c_\pm, \\ &|\nabla u^\pm(r,\varphi)| < c_\pm r^{\delta_0^\pm - 1}, \end{aligned} \tag{2.4}$$

($\delta_0^\pm > 0$) uniformly[1] w.r.t. (with respect to) the direction φ.

A traditional rigorous way to define a selfadjoint opertor A_γ attributed to the problem (2.1) implies making use of the corresponding sesquilinear form. The symmetric sesquilinear form ($\gamma > 0$)

$$a_\gamma[u,v] = (\nabla u_+, \nabla v_+)_{L_2(\Omega_+)} + (\nabla u_-, \nabla v_-)_{L_2(\Omega_-)} - 2\gamma(u_+|_S - u_-|_S, v_+|_C - v_-|_S)_{L_2(S)},$$

where $\mathrm{Dom}[a_\gamma] = H^1(\Omega_+) \oplus H^1(\Omega_-)$ and $(u,v)_{L_2(B)} = \int_B u(x)\bar{v}(x)dx$ is closed and bounded from below so that it uniquely specifies a selfadjoint operator A_γ (for some additional details see [3]).

It can be shown [3, 22], however, although we are not dealing with this herein, that, provided $E < -4\gamma^2$, the spectrum of the operator A_γ is discrete and finite. So, in order to find the corresponding eigenfunctions and to prove their exponential decay, we require the following conditions: let there exist such $\delta_\pm > 0$ that the estimates hold

$$\int_{\Omega_\pm} |u^\pm(r,\varphi)|^2 \exp(2\delta_\pm r) r dr d\varphi < \infty. \tag{2.5}$$

However, provided $0 > E \geqslant -4\gamma^2$, i.e. E belongs to the negative essential spectrum, the corresponding 'generalised' eigenfunctions do not vanish at infinity and the conditions (2.5) fail for them.

Any eigenfunction, i.e. a nontrivial solution of the problem (2.2)–(2.5), can be split into even and odd w.r.t. φ parts. In this work, we study even w.r.t. φ eigenfunctions. It can be shown,

[1] The upper or lower signs in (2.4) and henceforth are only simultaneously taken.

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however, not dealing with herein, that all eigenfunctions are even for the problem at hand. As a result, the desired solutions satisfy boundary conditions

$$\left.\frac{\partial u^{\pm}}{\partial n}\right|_{l_{\pm}} = 0. \tag{2.6}$$

Therefore, it looks natural to find the eigensolutions for $0 < \varphi < \pi$ for some E then to continue them onto the values $-\pi < \varphi < 0$ due to evenness.

We are looking for such values E that there exist nontrivial classical solutions of equations (2.2) in the corresponding domains $(\Omega_+ \cup \Omega_-) \cap \{y > 0\}$ that satisfy boundary conditions (2.3) on S_+ and (2.4) on $l_\pm$, belong to $H^1_{loc}(\Omega_\pm)$ (see (2.3)) and obey the estimate (2.5).

2.2 Kontorovich–Lebedev (KL) integral representation for solutions and functional difference (FD) equation of the second order

We are looking for classical solutions of the equations (2.2) as $0 < \varphi < \pi$ in the form of KL integral representation

$$u^{\pm}(r,\varphi) = \frac{1}{i\pi}\int_{-i\infty}^{i\infty} \sin \pi\nu K_\nu(\kappa r)\, u^{\pm}_\nu(\varphi) d\nu, \tag{2.7}$$

where the sign $+$ in both sides of the equality is for $0 < \varphi < \Phi$ and the sign $-$ is for $\Phi < \varphi < \pi$, $\kappa = \sqrt{-E}$, K_ν is Macdonald function [11]. This representation separates variables r and φ, ν is the variable of separation.

In order to formulate and prove the main statement of this section, we first introduce a class of functions $\mathcal{M}$ consisting of meromorphic functions H such that

- $H(\nu) = H(-\nu)$ is even,
- H is holomorphic in the strip $\Pi_{1+\delta} = \{\nu \in C: \ |\mathrm{Re}\, \nu| < 1+\delta\}$ for some $\delta > 0$ excluding possibly $\nu = \pm 1$
- $|H(\nu)| < \mathrm{Const}\, |\exp(i\nu[\pi/2 + \delta_0])|, \quad \nu \to i\infty, \quad \nu \in \Pi_{1+\delta}$ as $\delta_0 \in (0, \pi/2)$.

We intend to determine the unknown functions $u^{\pm}_\nu(\varphi)$ in the integrand of (2.7) such that the integrals would absolutely and uniformly w.r.t. r, φ converge. Moreover, we have

Theorem 2.1 *Let the functions $u^{\pm}_\nu$ take on the form*

$$u^{+}_\nu(\varphi) = H^{+}(\nu)\,\frac{\cos(\nu\varphi)}{\sin(\nu\Phi)} \tag{2.8a}$$

as $\varphi \in [0, \Phi]$ and

$$u^{-}_\nu(\varphi) = H^{-}(\nu)\,\frac{\cos(\nu\bar{\varphi})}{\sin(\nu\bar{\Phi})} \tag{2.8b}$$

as $\varphi \in [\Phi, \pi]$,

$$\bar{\Phi} = \pi - \Phi, \quad \bar{\varphi} = \pi - \varphi,$$

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where $H^{\pm}$ belong to the class $\mathcal{M}$ and satisfy the equations

$$H^{+}(\nu+1)-H^{+}(\nu-1)+\frac{2\gamma}{\kappa}\left(H^{+}(\nu)\cot(\nu\Phi)-H^{-}(\nu)\cot(\nu\bar{\Phi})\right)=0 \tag{2.9a}$$

and

$$H^{+}(\nu)+H^{-}(\nu)=0. \tag{2.9b}$$

Then the representations (2.7) fulfill the equations (2.2) and the boundary conditions (2.3) on S^{+} and (2.6) in classical sense.

Proof of Theorem 2.1 We take into account the behaviour of $u_\nu^{\pm}$ as $\nu\to i\infty$ and of

$$K_\nu(z)\sim \text{Const}\,\frac{\nu^{-1/2}\cos(\nu[\pi/2+|\arg(z)|])}{\sin(\pi\nu)},$$

as $\nu\to i\infty$ and $\mathrm{Re}\,(\nu)=0$ for $|\arg z|\leqslant\pi/2$, $|z|$ is arbitrarily fixed. The KL integrals absolutely and uniformly converge so that they can be substituted into the equations and boundary conditions.

The equations are verified by the direct substitution and taking into account that $\left\{\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-\left(1+\frac{\nu^2}{z^2}\right)\right\}K_\nu(z)=0$ and from (2.8a), (2.8b) $\left(\frac{d^2}{d\varphi^2}+\nu^2\right)u_\nu^{\pm}(\varphi)=0$. In order to verify the first boundary condition (2.3) on S^{+}, we make use of the identity $\frac{K_\nu(z)}{z}=\frac{K_{\nu+1}(z)-K_{\nu-1}(z)}{2\nu}$ then new integration variables $\nu+1\to\nu$ and $\nu-1\to\nu$ correspondingly. Thus, we obtain

$$\frac{1}{i\pi}\int_{-i\infty}^{i\infty}d\nu\,\sin\pi\nu\left(\frac{K_\nu(\kappa r)}{\kappa r}H^{+}(\nu)(-\nu)-\frac{\gamma}{\kappa}\left(H^{+}(\nu)\cot(\nu\Phi)-H^{-}(\nu)\cot(\nu\bar{\Phi})\right)K_\nu(\kappa r)\right)$$

$$=\frac{1}{i\pi}\int_{-i\infty}^{i\infty}d\nu\,\sin\pi\nu\left(-\frac{1}{2}\right)\left(K_{\nu+1}(\kappa r)-K_{\nu-1}(\kappa r)\right)H^{+}(\nu)$$

$$-\frac{1}{i\pi}\int_{-i\infty}^{i\infty}d\nu\,\sin\pi\nu\,\frac{\gamma}{\kappa}\left(H^{+}(\nu)\cot(\nu\Phi)-H^{-}(\nu)\cot(\nu\bar{\Phi})\right)K_\nu(\kappa r)$$

$$=\frac{1}{i\pi}\int_{-i\infty+1}^{i\infty+1}d\nu\left(\frac{\sin\pi\nu}{2}\right)K_\nu(\kappa r)H^{+}(\nu-1)-\frac{1}{i\pi}\int_{-i\infty-1}^{i\infty-1}d\nu\left(\frac{\sin\pi\nu}{2}\right)K_\nu(\kappa r)H^{+}(\nu+1)$$

$$-\frac{1}{2i\pi}\int_{-i\infty}^{i\infty}d\nu\,\sin\pi\nu\,\frac{2\gamma}{\kappa}\left(H^{+}(\nu)\cot(\nu\Phi)-H^{-}(\nu)\cot(\nu\bar{\Phi})\right)K_\nu(\kappa r)$$

$$=-\frac{1}{2i\pi}\int_{-i\infty}^{i\infty}d\nu\,\sin\pi\nu\,\Big(H^{+}(\nu+1)-H^{+}(\nu-1)$$

$$+\frac{2\gamma}{\kappa}\left(H^{+}(\nu)\cot(\nu\Phi)-H^{-}(\nu)\cot(\nu\bar{\Phi})\right)\Big)K_\nu(\kappa r)=0,$$

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where we used (2.9a) on the last step. The second boundary condition (2.3) on S^+ is considered in a similar way by using (2.9b). It is obvious that, in view of the conditions of the theorem, our formal calculations are easily justified. □

It is worth commenting on validity of the Meixner's conditions (2.4) which are to be satisfied by the KL integrals in (2.7). The verification can be done by means of reduction of the KL integrals to the so called Watson–Bessel integral representation, see analogous derivations in Section 5.2.2 of [19]. The same result follows from Sommerfeld integral representations for solutions exploited below; however, for breivity, we omit-related technical discussion.

Excluding the function $H^-(\nu) = -H^+(\nu)$ (see (2.9b)) from equation (2.9a), we arrive at the FD equation for $H^+ \in \mathcal{M}$

$$H^+(\nu+1) - H^+(\nu-1) - 2i\Lambda\, W(\nu)\, H^+(\nu) = 0 \tag{2.10}$$

with meromorphic potential

$$W(\nu) = \frac{\cot(\nu\Phi) + \cot(\nu\bar{\Phi})}{-2i},$$

where Λ is spectral parameter, $\Lambda = \frac{2\gamma}{\kappa}$. The equation (2.10) is a main object of our further studies. Making use of this FD equation, we determine Λ for which the equation admits appropriate nontrivial solutions such that KL representations (2.7) exponentially vanish at infinity and specify the desired eigenfunctions. To this end, the asymptotic behaviour of $u^{\pm}(r,\varphi)$ as $r \to \infty$ is carefully investigated below.

3 Reduction to an integral equation and spectral properties of the FD equation

3.1 Reduction to an integral equation

We, first, notice that meromorphic coefficient W in (2.10) is singular at $\nu = 0$. We make use of the following auxiliary Lemma.

Lemma 1 *Let $q(\nu)$ be holomorphic as $\nu \in \Pi_\delta$ except zero, where it has a simple pole, and $|q(\nu)| \leqslant c_H e^{-\varkappa|\nu|}$, $|\nu| \to \infty$, $\varkappa > 0$ in this strip, $q(\nu) = -q(-\nu)$. Then an even solution $s(\nu)$ of the equations*

$$s(\nu+1) - s(\nu-1) = 2iq(\nu),$$

which is regular (holomorphic) in the strip $\nu \in \Pi_{1+\delta} \setminus \{\pm 1\}$ and exponentially vanishes as $|\nu| \to \infty$ there, is given by

$$s(\nu) = \frac{1}{4}\int_{-i\infty}^{i\infty} d\tau\, q(\tau)\, \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu}, \qquad \nu \in \Pi_{1+\delta}.$$

Proof Actually, proof of this Lemma is a direct consequence of application of the known approach (see Section 7.3.1 in [2]), which is equivalent to making use of the Fourier transform along the imaginary axis

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$$s(\nu) = -\frac{1}{2\pi}\, v.p. \int_{-i\infty}^{i\infty} \widetilde{s}(t) e^{-i\nu t}\, dt\,, \qquad \widetilde{s}(t) = \int_{-i\infty}^{i\infty} s(\nu) e^{i\nu t}\, d\nu.$$

($v.p.$ stands for Cauchy principal value.) Expoiting functional equation for s, we find

$$\widetilde{s}(t) = \frac{\widetilde{q}(t)}{-2i \sin t} = -\frac{1}{\sin t}\, v.p. \int_{-i\infty}^{i\infty} q(\tau) e^{i\tau t}\, d\tau.$$

We make use of the inverse Fourier transform

$$s(\nu) = -\frac{1}{2\pi} \int_{-i\infty}^{i\infty} dt \frac{e^{-i\nu t}}{-\sin t} \left(v.p. \int_{-i\infty}^{i\infty} q(\tau) e^{i\tau t}\, d\tau \right)$$

$$= \int_{-i\infty}^{i\infty} d\tau\, q(\tau) \left(\int_{-i\infty}^{i\infty} \frac{i \sin(t[\tau - \nu]) + i \sin(t[\tau + \nu])}{2\pi \sin t} dt \right) = \frac{1}{4} \int_{-i\infty}^{i\infty} d\tau\, q(\tau) \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu},$$

where we took into account that q is odd, changed orders of integration and exploited formula 3.981(1) in [11]. □

We use this Lemma and obtain

$$H^+(\nu) = -\frac{\Lambda}{2} \int_{-i\infty}^{i\infty} d\tau \frac{W(\tau) \sin \pi\tau}{\cos \pi\tau + \cos \pi\nu} H^+(\tau)\,, \qquad \nu \in \Pi_1. \tag{3.1}$$

This representation enables one to specify $H^+(\cdot)$ in the strip Π_1 as a holomorphic function provided $H^+(\tau)$ is known and is summable on the imaginary axis in the integrand of the right-hand side in (3.1). It can be shown that $H^+(\cdot)$ is continued from the domain Π_1 onto the whole complex plane $\mathbf{C}$ as a meromorphic function from the desired class. Remark that $W(\tau) = -W(-\tau)$, $W(\tau) > 0$ as $\tau \in iR_+$ and $W(\tau) = 1 + O(e^{2i\bar{\Phi}\tau/\pi})$, $\quad \tau \to i\infty$.

We let $\nu \in iR_+$ in the representation (3.1), make use of evenness of H^+ then introduce the new unknown D,

$$H^+(\nu) = \frac{D(\nu)}{\sqrt{W(\nu)}}$$

and obtain the integral equation ($\nu \in iR_+$)

$$D(\nu) = -\Lambda \int_{0}^{i\infty} d\tau \frac{\sin \pi\tau \sqrt{W(\nu)\, W(\tau)}}{\cos \pi\tau + \cos \pi\nu} D(\tau). \tag{3.2}$$

It is obvious that D satisfies the estimate

$$|D(\nu)| \;<\; \text{Const}\, |\exp(i\nu[\pi/2 + \delta_0])|,$$

$\nu \to i\infty, \quad \nu \in \Pi_{1+\delta}$ as $\delta_0 \in (0, \pi/2)$.

Having solution D of the equation (3.2) on the semiaxis and, therefore, on the whole imaginary axis due to parity, we specify summable solution H^+ on the imaginary axis and then by means of

(3.1) in the strip $\Pi_{1+\delta}$. It can be verified that such a constructed function $H^+(\nu)$ is continued as the desired meromorphic solution of the FD equation (2.10). Meromorphic continuation of H^+ from $\Pi_{1+\delta}$ onto the whole complex plane can be performed by means of the FD equation (2.10). In equation (3.2), the parameter Λ plays the role of characteristic parameter.

We reduce equation (3.2) to a more convenient form by means of introduction of new variables

$$x = \frac{1}{\cos \pi \nu}, \quad y = \frac{1}{\cos \pi t}, \quad \frac{dy}{\pi} = \frac{\sin \pi t}{\cos^2 \pi t} dt,$$

and new unknown

$$\rho(x) = \cos \pi \nu \, D(\nu)|_{x=\frac{1}{\cos \pi \nu}},$$

$x, y \in [0, 1]$,

$$\rho(x) - \frac{\Lambda}{\pi} \int_0^1 dy \, \frac{\sqrt{v(x)v(y)}}{x+y} \, \rho(y) = 0, \tag{3.3}$$

where

$$v(y) = W(t)|_{y=\frac{1}{\cos \pi t}}$$

and

$$v(y) = 1 + O(y^b) \text{ as } y \to 0, \quad b = \frac{2\bar{\Phi}}{\pi} < 1, \quad v(y) = O(1/\sqrt{1-y}), \ \ y \to 1 - 0.$$

Spectral properties of the integral operator in (3.3) play a crucial role for further analysis.

3.2 Spectral properties of the integral operator

We study a positive integral operator $K: \ L_2(0, 1) \to L_2(0, 1)$ which is defined by

$$(K\rho)(x) = \frac{1}{\pi} \int_0^1 dy \, \frac{\sqrt{v(x)v(y)}}{x+y} \, \rho(y)$$

and the integral equation in (3.3)[2] is written in the form

$$(K\rho)(x) = \lambda \rho(x), \tag{3.4}$$

$\lambda = \Lambda^{-1}$. The operator K is selfadjoint and bounded (compare the following estimate with analogous one in Section 2.10, [5]),

$$|(Kh, g)_{L_2(0,1)}| \leqslant V^* \|h\| \|g\|,$$

for some $V^* \geqslant 1$. We also remark that v is positive, $(v > 1)$, $v(y) = 1 + O(y^b)$ as $y \to 0$, $0 < b < 1$, $v(y) = O(1/\sqrt{1-y})$, $\quad y \to 1 - 0$.

[2] This operator belongs to a class of weighted Hankel operators that are considered in the related literature.

We turn to the study of spectrum $\sigma(K) \subset [0, V^*]$ of the operator K in (3.4). To this end, it is useful to represent it in the form

$$K = D + [K - D],$$

where

$$(D\rho)(x) = \frac{1}{\pi} \int_0^1 \frac{dy}{x+y}\, \rho(y)$$

is the so called Dixon's operator (see Section 11.18 in [24]) and

$$([K - D]\rho)(x) = \frac{1}{\pi} \int_0^1 dy\, \frac{\sqrt{v(x)v(y)} - 1}{x+y}\, \rho(y)$$

is of the Hilbert–Schmidt class in view of the estimates $|v(x)v(y) - 1| = O\left(x^b + y^b\right), \quad x, y \to 0,$ $\sqrt{v(x)v(y)} = O\left((1-x)^{-\frac{1}{4}}(1-y)^{-\frac{1}{4}}\right), \quad x \to 1-0,\ y \to 1-0.$

The selfadjoint Dixon's operator is bounded and admits explicit characterisation. In particular, its spectrum is essential coinciding with the segment [0, 1], which follows, e.g. from Section 11.17 in [24].[3] Indeed, it is possible to reduce the equation $(D\rho)(x) = \lambda\rho(x)$ to that specified by a convolution-type operator on the semi-axis. The latter equation has nontrivial solutions for each $\lambda \in [0, 1]$. The operator $K = D + [K - D]$ is a compact pertubation of the Dixon's operator so that its essential spectrum $\sigma_{ess}(K)$ also coincides with [0, 1]. Asserting the latter, we actually made use of the aponimous Weyl theorem saying that essential spectrum is preserved under compact self-adjoint perturbation of a self-adjoint operator.

However, the operator K may also have the discrete component $\sigma_d(K)$ of the spectrum. Indeed, the operator K is positive so that its discrete spectrum, if exists, is located on the interval $(1, V^*]$, i.e. $\sigma_d(K) \subset (1, V^*]$. We can also expect that $\sigma_d(K)$ is not empty in view of the properties of the potential v which follow from those of the potentials W. In order to prove this, it is sufficient to find a nontrivial $\rho \in L_2(0, 1)$ and a positive d such that

$$\frac{(K\rho, \rho)}{(\rho, \rho)} \geqslant 1 + d,$$

which follows from the variational (minimax) principle [5], see, e.g. Theorem 6 in Section 9.2. Indeed, this theorem means, in particular, that, provided a self-adjoint operator $B = B^*$ is compact and $\mathcal{L}$ is a subspace in a Hilbert space such that $(B\rho, \rho) > \lambda(\rho, \rho)$, as $\rho \in \mathcal{L}$, then the number of eigenvalues of B on the right-hand side from λ is not less than $\dim \mathcal{L}$. As a result, existence of at least one eigenvalue of the operator K on the right-hand side from $\lambda = 1$ follows from this theorem if there exists at least one nontrivial $\rho \in L_2(0, 1)$ such that the above-mentioned estimate for K is valid.

In order to get the desired estimate, we take the following test function

$$h(x) = [v(x)]^{-1/2}.$$

[3] Making use of the spectral measure for the Dixon's operator, one could compute Weyl singular sequences corresponding to any point from the segment [0, 1], i.e. from the essential spectrum.

Table 1. *The values of* $F(\Phi)$. *Positive values of* $F(\Phi)$ *correspond to that the sufficient condition for existence of the discrete spectrum is satisfied.*

Φ	$19\pi/20$	$9\pi/10$	$17\pi/20$	$4\pi/5$	$3\pi/4$	$7\pi/10$	$13\pi/20$	$\pi/2$
$F(\Phi)$	0.3018	0.1918	0.1045	0.0354	−0.1880	−0.0606	−0.0915	−0.1295

Then we consider the normalised function $\rho(x)=\frac{h(x)}{\|h(x)\|}$, where $\|h(x)\|^2=\int_0^1\frac{dx}{v(x)}$. We have

$$(K\rho,\rho)=\frac{1}{\pi}\int_0^1 dx\int_0^1 dy\,\frac{\sqrt{v(x)v(y)}}{x+y}\,\rho(y)\,\overline{\rho(x)}$$
$$=\frac{1}{\pi\|h(x)\|^2}\int_0^1 dx\int_0^1 dy\,\frac{1}{x+y}=\frac{2\log 2}{\pi\int_0^1\frac{dx}{v(x)}}.$$

Now, impose an additional restriction on the potential v assuming that

$$\frac{2\log 2}{\pi}>\int_0^1\frac{dx}{v(x)} \tag{3.5}$$

and conclude

$$(K\rho,\rho)>1.$$

It is important to be sure that the sufficient condition (3.5) is satisfied for given $v(x)$. It can be written in the form

$$\pi\int_0^\infty\frac{dt}{\cosh^2(\pi t)}\frac{\sinh(\pi t)}{W(it)}<\frac{2\log 2}{\pi}.$$

For given $v(y)=W(t)|_{y=\frac{1}{\cos\pi t}}$, the condition (3.5) is satisfied at least as $\bar{\Phi}$ is small enough. It is obvious that (3.5) is only a sufficient condition of existence of the discrete spectrum. Table 1 shows dependence $F(\Phi):=2\log 2/\pi-\int_0^1\frac{dx}{v(x)}$ as function of Φ. For negative values of F, the sufficient condition is not valid. However, for the original problem, it is known that the discrete spectrum exists for all $\Phi\in(\pi/2,\pi)$.

Let us denote the corresponding eigenvalues λ_m, $m\in M=\{1,2,3\ldots N_\Phi\}$ counting[4] them in the order of their decrease and taking into account their multiplicity then

$$\sigma_d(K)=\cup_{m\in M}\lambda_m.$$

The corresponding eigenfunctions are denoted ρ_m.

[4]We notice that the set M of natural numbers may be also infinite for some other classes of potentials. We use that finiteness of the discrete spectrum is known from cited literature.

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Theorem 3.1 *The spectrum $\sigma(K)$ of the operator K consists of the essential spectrum $\sigma_{ess}(K) = [0, 1]$ and, provided that (3.5) is valid, of a non-empty finite discrete part $\sigma_d(K) = \cup_{m\in M}\lambda_m$.*

It is worth noticing that similar results can be also obtained for a special class of weights v including $v(x) = W(t)|_{x=\frac{1}{\cos\pi t}}$; however, this is not addressed herein. In this work, we are interested in behaviour of some 'individual' eigenfunction $u_m^{\pm}$, i.e. assume that m is fixed. On the other hand, complete description of the negative spectrum of the operator A_γ is closely related with the Theorem 3.1. It is known that the discrete part of the operator A_γ is finite, which follows from [15] and [3].

3.3 Solutions of the FD equation (2.10)

We have that for $\Lambda_m = \lambda_m^{-1}$, where $\lambda_m \in \sigma_d(K)$, equation (2.10) has a solution

$$H_m^+(\nu) = \frac{D_m(\nu)}{\sqrt{W(\nu)}} = \frac{1}{\sqrt{W(\nu)}}\frac{(\rho_m(x))|_{x=\frac{1}{\cos\pi\nu}}}{\cos\pi\nu}$$

recalling that $H_m(\nu)$ solves the equation (2.10) with $\Lambda = \Lambda_m := \sin\tau_m$. From $\rho_m \in L_2(0, 1)$, we obtain

$$\int_0^{i\infty} \left|H_m^+(\nu)\right|^2 \, |\sin(\pi\nu)W(\nu)| \, |d\nu| < \infty. \tag{3.6}$$

We can assert that H_m is meromorphic and even. It is also holomorphic in Π_ϵ for some $\epsilon > 0$. The latter assertions require some explanations. The corresponding logical implications are as follows. Having an eigensolution $\rho_m \in L_2(0, 1)$, we obtain H_m^+ defined on the positive part of the imaginary axis and then, by oddness, on the whole axis. We take into account the representation (3.1) and observe that, provided $H_m^+(t)$ is specified in the integrand, the left-hand side is a holomorphic function in the strip Π_1. It is also useful to notice that the representation (3.1) can be substituted into the functional equation (2.10) so that the left-hand side satisfies the latter. Such a verification requires some work based on continuation of the integral in the right-hand side of (3.1) onto a wider strip and, actually, is analogous to that in Section 7.3.2 of [2]. In this sense, reduction from H_m^+ to ρ_m (and, in particular, the Lemma 1 in Section 3.1) can be reversed. From the strip Π_1 the solution $H_m^+(\cdot)$ is continued onto the whole complex plane as a meromorphic function by use of the functional equation (2.10), where the potential $W(\nu)$ is meromorphic. This procedure enables one to reconstruct $H_m^+(\cdot)$, satisfying the functional equation, by means of ρ_m.

We have already discussed existence of the desired solutions H_m and now we turn to their estimate as $\nu \to i\infty, \nu \in \Pi_\epsilon$. The idea of such study is rather natural. It is well established in the Fourier analysis that the behaviour of a function at infinity is closely related to the regularity domain of its Fourier transform on the complex plane. As a result, it is sometimes more efficient to identify positions of singularities of the Fourier transform on the complex plane in order to terminate its regularity domain. Then the asymptotics of a function at infinity is specified by position of singularities of its Fourier transform. Indeed, consider the Fourier transform of H_m along the imaginary axis

$$F_m(\alpha) = \frac{1}{i}\int_{iR} H_m(\nu)e^{i\nu\alpha}\, d\nu,$$

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where, in view of the estimate (3.6), $F_m(\alpha)$ is holomorphic in the strip $\Pi_{\pi/2}$ and is even. We apply the transform to the functional equation (2.10) with $\Lambda_m = \sin\tau_m$ and after some calculations obtain

$$F_m(\alpha) = \frac{-\sin\tau_m}{\sin\alpha - \sin\tau_m}\,\frac{v.p.}{i}\int_{iR}[W(\nu)+1]H_m(\nu)e^{i\nu\alpha}\,d\nu.$$

Simple analysis of the latter representation enables us to assert that $F_m(\alpha)$ is holomorphic as $\alpha \in \Pi_{\pi-\tau_m}$ and the nearest to the imaginary axis singularities are on the real axis at $\alpha = \pi - \tau_m$ and also at $\alpha = -\pi + \tau_m$ by parity. (The integral factor is holomorphic in the strip $\Pi_{\pi+\bar{\Phi}-\tau_m}$ provided F_m is regular in $\Pi_{\pi-\tau_m}$.) Taking into account the inverse Fourier transform and position of the leading singularities, we arrive at the estimate

$$H_m(\nu) = O\left(\frac{1}{\cos(\nu[\pi-\tau_m])}\right),$$

$\nu \to i\infty$, $\tau_m \in (0, 1)$. We omitted some simple calculations.

As a result, the solutions $H_m^+ = H_m$, $m \in M$ of the equation (2.10) corresponding to $\Lambda = \sin\tau_m$ belong to the class $\mathcal{M}$ of meromorphic functions.

4 Asymptotics of the eigenfunctions $u_m^\pm$ as $r \to \infty$

We found solutions in the form of the KL integrals

$$u_m^+(r,\varphi) = \frac{1}{i\pi}\int_{-i\infty}^{i\infty}\sin(\pi\nu)K_\nu(\kappa_m r)\,H_m^+(\nu)\,\frac{\cos(\nu\varphi)}{\sin(\nu\Phi)}\,d\nu\,, \quad \varphi \in [0, \Phi], \tag{4.1}$$

$$u_m^-(r,\varphi) = \frac{1}{i\pi}\int_{-i\infty}^{i\infty}\sin(\pi\nu)K_\nu(\kappa_m r)\,H_m^-(\nu)\,\frac{\cos(\nu\bar{\varphi})}{\sin(\nu\bar{\Phi})}\,d\nu\,, \quad \bar{\varphi} \in [0, \bar{\Phi}], \tag{4.2}$$

where m is arbitrarily fixed,

$$\kappa_m = \sqrt{-E_m} = \frac{2\gamma}{\sin\tau_m}.$$

Representations (4.1),(4.2) solve the equations (2.2) and satisfy the boundary conditions (2.3). It can be shown that they also satisfy the conditions (2.4).

In this section, we determine the asymptotics of these solutions, which enables one also to verify the estimates (2.5). However, it is not possible to make use of the asymptotics $K_\nu(\kappa r) = \sqrt{\frac{\pi}{2}}\,\frac{\exp(-\kappa r)}{\sqrt{\kappa r}}\,(1 + O(1/[\kappa r]))$ as $\kappa r \to \infty$ in (4.1), (4.2) as the resulting integrals would diverge. It is natural to transform the KL integral representations to the Sommerfeld integrals that are well adapted to asymptotic evaluation as $r \to \infty$.

4.1 Sommerfeld integral representation and its properties

We exploit Sommerfeld integral representation for the Macdonald function that takes the form

$$K_\nu(z) = \frac{1}{4i}\int_{\gamma_0} e^{z\cos\alpha}\,\frac{\sin(\nu\alpha)}{\sin(\pi\nu)}\,d\alpha$$

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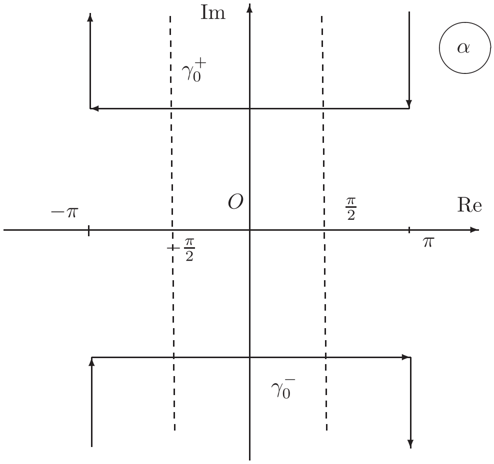

FIGURE 2. The Sommerfeld double-loop contour $\gamma_0 = \gamma_0^+ \cup \gamma_0^-$.

as $\operatorname{Re} z > 0$, whereas the integration contour γ_0 is shown in Figure 2. The latter integral rapidly converges due to vanishing of the exponent in the integrand at the ends of γ. We substitute the latter representation of the Macdonald function into (4.1),(4.2), change the orders of integration, which is justified (see also Section 5.6 in [19]), and arrive at

$$u_m^{\pm}(r,\varphi) = \frac{1}{2\pi i}\int_{\gamma_0} d\alpha\, e^{\kappa_m r\cos\alpha}\, F_m^{\pm}(\alpha,\varphi), \tag{4.3}$$

where

$$F_m^{+}(\alpha,\varphi) = \frac{1}{2i}\int_{-i\infty}^{i\infty} \sin(\nu\alpha)\, H_m^{+}(\nu)\,\frac{\cos(\nu\varphi)}{\sin(\nu\Phi)}\, d\nu \;= f_m^{+}(\alpha+\varphi) + f_m^{+}(\alpha-\varphi),$$

$$f_m^{+}(\alpha) = \frac{1}{4i}\int_{-i\infty}^{i\infty} H_m^{+}(\nu)\,\frac{\sin(\nu\alpha)}{\sin(\nu\Phi)}\, d\nu \;= \frac{v.p.}{4i}\int_{-i\infty}^{i\infty} H_m^{+}(\nu)\,\frac{e^{i\nu\alpha}}{i\sin(\nu\Phi)}\, d\nu \tag{4.4}$$

and

$$F_m^{-}(\alpha,\varphi) = \frac{1}{2i}\int_{-i\infty}^{i\infty} \sin(\nu\alpha)\, H_m^{-}(\nu)\,\frac{\cos(\nu\bar\varphi)}{\sin(\nu\bar\Phi)}\, d\nu \;= f_m^{-}(\alpha+\bar\varphi) + f_m^{-}(\alpha-\bar\varphi),$$

$$f_m^{-}(\alpha) = \frac{1}{4i}\int_{-i\infty}^{i\infty} H_m^{-}(\nu)\,\frac{\sin(\nu\alpha)}{\sin(\nu\bar\Phi)}\, d\nu. \tag{4.5}$$

From the expressions for $f_m^{\pm}$ in (4.4), (4.5), one can assert that these functions are odd, are holomorphic in the strips $\alpha \in \Pi_{\pi+\Phi-\tau_m}$ for f_m^{+} and $\alpha \in \Pi_{\pi+\bar\Phi-\tau_m}$ for f_m^{-}, which follows from the estimates for $H_m^{\pm}$ as $\nu \to i\infty$ given at the end of Section 3. It can be also verified that $f_m^{\pm}(\alpha) = c_{\pm} + O(\exp(-d_{\pm}|\alpha|))$ as $\alpha \to i\infty$, $(d_{\pm} > 0)$ in these strips correspondingly. (Remark that these asymptotics enables one to verify Meixner's conditions from Sommerfeld representations, see

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Section 5.2.2 in [19].) It can be shown that $f_m^{\pm}$ are also holomorphic in the upper and lower halfplanes of the complex plane and their singularities (poles) are on the real axis outside the basic strips $\Pi_{\pi+\Phi-\tau_m}$ and $\Pi_{\pi+\bar{\Phi}-\tau_m}$. (They are actually on the boundaries of these strips.) From (4.3) we find

$$u_m^+(r,\varphi) = \frac{1}{2\pi i}\int_{\gamma_0} d\alpha\, e^{\kappa_m r\cos\alpha}\, [f_m^+(\alpha+\varphi) + f_m^+(\alpha-\varphi)], \tag{4.6}$$

$$u_m^-(r,\varphi) = \frac{1}{2\pi i}\int_{\gamma_0} d\alpha\, e^{\kappa_m r\cos\alpha}\, [f_m^-(\alpha+\bar{\varphi}) + f_m^-(\alpha-\bar{\varphi})], \quad \bar{\varphi} = \pi - \varphi, \tag{4.7}$$

where $f_m^{\pm}$ are meromorphic Sommerfeld transformants having properties described above.

The Sommerfeld integral representations (4.6), (4.7) for the eigenfunctions $u_m^{\pm}$ are further used in order to derive asymptotics as $r \to \infty$. To this end, we notice that the integrals obey equations (2.2) and are even w.r.t. φ, which means that the conditions (2.6) are satisfied. Moreover, the following Lemma is valid.

Lemma 2 *Let odd functions $f_m^{\pm}$, defined by the equalities (4.4), (4.5) in the basic strips $\Pi_{\pi+\Phi-\tau_m}$ and $\Pi_{\pi+\bar{\Phi}-\tau_m}$ correspondingly, be analitically continued on the complex plane as meromorphic functions with properties described above. Then, provided $f_m^{\pm}$ obey functional equations*

$$\begin{aligned}(\sin\alpha - \sin\tau_m) f_m^+(\alpha+\Phi) - (\sin\alpha + \sin\tau_m) f_m^+(\alpha-\Phi)\\ = (\sin\alpha - \sin\tau_m) f_m^-(\alpha+\bar{\Phi}) - (\sin\alpha + \sin\tau_m) f_m^-(\alpha-\bar{\Phi}),\end{aligned} \tag{4.8a}$$

$$f_m^+(\alpha+\Phi) - f_m^+(\alpha-\Phi) = -f_m^-(\alpha+\bar{\Phi}) + f_m^-(\alpha-\bar{\Phi}), \tag{4.8b}$$

the Sommerfeld representations (4.6), (4.7) fulfill the boundary (2.3) in classical sense.

Proof We verify the first condition in (2.3), whereas the second condition is treated in a similar way. We take into account that $\frac{\partial}{\partial\varphi} f(\alpha+\varphi) = \frac{\partial}{\partial\alpha} f(\alpha+\varphi)$ and integrate by parts, which leads to

$$\frac{1}{2\kappa_m r}\left(\frac{\partial}{\partial\varphi}u_m^+ + \frac{\partial}{\partial\varphi}u_m^-\right)\Bigg|_{\varphi=\Phi} - \frac{\gamma}{\kappa_m}\left(u_m^+ - u_m^-\right)|_{\varphi=\Phi}$$
$$= \frac{1}{4\pi i}\int_{\gamma_0} d\alpha\, e^{\kappa_m r\cos\alpha}\, 2\left\{\sin\alpha\left[f_m^+(\alpha+\Phi) - f_m^-(\alpha-\bar{\Phi})\right] - \frac{2\gamma}{\kappa_m}\left(f_m^+(\alpha+\Phi) - f_m^-(\alpha-\bar{\Phi})\right)\right\} = 0.$$

In the latter equation, we used that $f_m^{\pm}$ are odd functions. In a prescribed class of functions, the Sommerfeld integral $\frac{1}{4\pi i}\int_{\gamma_0} d\alpha\, S(\alpha)$ is zero if and only if the even part of its integrand $\{S(\alpha)\}$ is zero, $S(\alpha) - S(-\alpha) = 0$, [2]. As a result, we have the equation (4.8a), where we took into account that $f_m^{\pm}(\alpha) = -f_m^{\pm}(-\alpha)$. The equation (4.8b) is similarly derived. In some natural conditions, the reverse statement to that in the Lemma is also valid. □

It is useful to comment on interconnection between the equations (4.8a), (4.8b) and functional equations in (2.9a), (2.9b). It can be directly shown that these equations are equivalent. In order to prove this, we shall proceed in a formal way; however, our formal calculation can be easily justified. Indeed, direct use of the Fourier-type transform in (4.4), (4.5) enables one to verify,

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by simple substitution, that the relation (2.9b) is equivalent to (4.8b). On the other side, from representations (4.4), (4.5) we can obtain

$$(-\sin\tau_m)[f^+(\alpha+\Phi)+f^+(\alpha-\Phi)] = \frac{(-\sin\tau_m)}{4i}\int_{-i\infty}^{i\infty} H_m^+(\nu)\,2\sin(\nu\alpha)\cot(\nu\Phi)\,d\nu,$$

$$\sin\alpha[f^+(\alpha+\Phi)-f^+(\alpha-\Phi)] = \frac{1}{4i}\int_{-i\infty}^{i\infty} H_m^+(\nu)\,2\sin\alpha\cos(\nu\alpha)\,d\nu.$$

Thus, we have

$$\begin{aligned}
&(\sin\alpha-\sin\tau_m)f_m^+(\alpha+\Phi) - (\sin\alpha+\sin\tau_m)f_m^+(\alpha-\Phi)]\\
&= \frac{1}{4i}\int_{-i\infty}^{i\infty}\left(H_m^+(\nu)\ \sin([\nu+1]\alpha) - H_m^+(\nu)\ \sin([\nu-1]\alpha) - 2\sin\tau_m H_m^+(\nu)\sin(\nu\alpha)\cot(\nu\Phi)\right)d\nu\\
&= \frac{1}{4i}\int_{-i\infty+1}^{i\infty+1} H_m^+(\nu-1)\ \sin(\nu\alpha)d\nu - \frac{1}{4i}\int_{-i\infty-1}^{i\infty-1} H_m^+(\nu+1)\ \sin(\nu\alpha)d\nu\\
&\quad - \frac{1}{4i}\int_{-i\infty}^{i\infty} 2\sin\tau_m H_m^+(\nu)\sin(\nu\alpha)\cot(\nu\Phi)d\nu\\
&= -\frac{1}{2i}\int_{-i\infty}^{i\infty}\left(\frac{1}{2}\left[H_m^+(\nu+1)\ - H_m^+(\nu-1)\right] + \sin\tau_m H_m^+(\nu)\ \cot\nu\Phi\right)\ \sin(\nu\alpha)d\nu,
\end{aligned}$$

where on the last step we deformed the integration contours into the imaginary axis. In the same manner, we verify that

$$\begin{aligned}
&(\sin\alpha-\sin\tau_m)f_m^-(\alpha+\bar\Phi) - (\sin\alpha+\sin\tau_m)f_m^-(\alpha-\bar\Phi)]\\
&= -\frac{1}{2i}\int_{-i\infty}^{i\infty}\left(\frac{1}{2}\left[H_m^-(\nu+1)\ - H_m^-(\nu-1)\right] + \sin\tau_m H_m^-(\nu)\ \cot\nu\bar\Phi\right)\ \sin(\nu\alpha)d\nu.
\end{aligned}$$

Taking into account (2.9b), writing equation (2.9a) in the form

$$\begin{aligned}
&\frac{1}{2}\left[H_m^+(\nu+1)\ - H_m^+(\nu-1)\right] + \sin\tau_m\, H_m^+(\nu)\cot(\nu\Phi)\\
&\quad= \frac{1}{2}\left[H_m^-(\nu+1)\ - H_m^-(\nu-1)\right] + \sin\tau_m\, H_m^-(\nu)\cot(\nu\bar\Phi),
\end{aligned}$$

from the formulae above we arrive at the desired equivalence.

Functional equations (4.8a), (4.8b) are called Malyuzhinets' equations that are traditionally encountered with in diffraction by wedges [21]. These equations can be written in the matrix form

$$\begin{pmatrix} f_m^+(\alpha+\Phi) \\ f_m^-(\alpha+\bar{\Phi}) \end{pmatrix} = \begin{pmatrix} \dfrac{\sin\alpha}{\sin\alpha-\sin\tau_m} & \dfrac{-\sin\tau_m}{\sin\alpha-\sin\tau_m} \\ \dfrac{-\sin\tau_m}{\sin\alpha-\sin\tau_m} & \dfrac{\sin\alpha}{\sin\alpha-\sin\tau_m} \end{pmatrix} \begin{pmatrix} f_m^+(\alpha-\Phi) \\ f_m^-(\alpha-\bar{\Phi}) \end{pmatrix} \tag{4.9}$$

and enable one to continue f_m^+ from the basic strip $\alpha\in\Pi_{\pi+\Phi-\tau_m}$ and f_m^- from the basic strip $\alpha\in\Pi_{\pi+\bar{\Phi}-\tau_m}$ onto the whole complex plane as meromorphic functions. We recall that inside these strips, the Sommerfeld transformants are holomorphic and defined by expressions (4.4), (4.5) correspondingly. In particular, we can specify singularities (poles) of f_m^+ and f_m^-, which are located on the real axis. The closest to the imaginary axis poles of these meromorphic functions are on the boundaries of the strips $\Pi_{\pi+\Phi-\tau_m}$ and $\Pi_{\pi+\bar{\Phi}-\tau_m}$. Consider the first equation in (4.9) written as

$$f_m^+(\alpha+2\Phi) = \frac{\sin(\alpha+\Phi)}{\sin(\alpha+\Phi)-\sin\tau_m} f_m^+(\alpha) - \frac{\sin\tau_m}{\sin(\alpha+\Phi)-\sin\tau_m} f_m^-(\alpha+\Phi-\bar{\Phi}).$$

Indeed, let the argument α of f_m^+ in the right-hand side of this equation belong to the strip $\Pi_{\pi+\Phi-\tau_m}$, where f_m^+ is holomorphic. Then the argument $z=\alpha+\Phi-\bar{\Phi}$ of f_m^- belongs to the strip $\operatorname{Re} z\in(-[\pi+\bar{\Phi}-\tau_m], [\pi+\bar{\Phi}-\tau_m]+2(\Phi-\bar{\Phi}))$. Then simple analysis shows that in the left-hand side $f_m^+(\zeta)$ $(\zeta=2\Phi+\alpha)$ have a pole at $\zeta=a_m^+$ with $a_m^+=\pi+\Phi-\tau_m$ (due to zeros of the denominators in the right-hand side) as well as at $-a_m^+=-(\pi+\Phi-\tau_m)$ due to oddness. The other poles of f_m^+ are also due to the poles of f_m^- in the right-hand side of the latter equation. However, these singularities are located outside the closed strip $\bar{\Pi}_{\pi+\Phi-\tau_m}$.

In the same manner, from the second equation in (4.9), we find that $a_m^-=\pi+\bar{\Phi}-\tau_m$ and $-a_m^-=-(\pi+\bar{\Phi}-\tau_m)$ are the closest to the imaginary axis poles of f_m^-. It is worth noticing that all singularities (poles) of the Sommerfled transformants are on the real axis.

In neighbourhoods of these poles $a_m^\pm$ the Sommerfeld transformants have representations

$$f_m^+(\alpha) = \frac{A_m^+}{\alpha-a_m^+} + \ldots, \quad f_m^-(\alpha) = \frac{A_m^-}{\alpha-a_m^-} + \ldots, \tag{4.10}$$

where $A_m^\pm=\operatorname{res}_{\alpha=a_m^\pm} f_m^\pm(\alpha)$ and dots mean regular parts, $\alpha=a_m^+=-\alpha=a_m^-$, $A_m^+=-A_m^-$.

We intend to calculate the Sommerfeld integrals asymptotically. To this end, one should deform the integrations contours into the steepest descent (SD) paths $\gamma_0^{\pm\pi}=\{\alpha: \ \operatorname{Re}\alpha=\pm\pi\}$. In this process, some poles of the transformants can be crossed including those in (4.10) and symmetric to them. It turns out that, namely, the singularities discussed above will contribute to the leading terms of the asymptotics in some conditions. The other poles following the 'leading' poles can be also captured; however, they contribute to the asymptotic correction.

4.2 Asymptotic evaluation of the Sommerfeld integrals

We consider the leading terms of the asymptotics. The integrals (4.3) obviously have saddle points $\pm\pi$ ($\frac{d}{d\alpha}\cos\alpha=0$ at $\alpha=\pm\pi$) and are asymptotically computed by means of the saddle point technique like in the Malyuzhinets' problem, Section 6.4 in [2]. They can be written as

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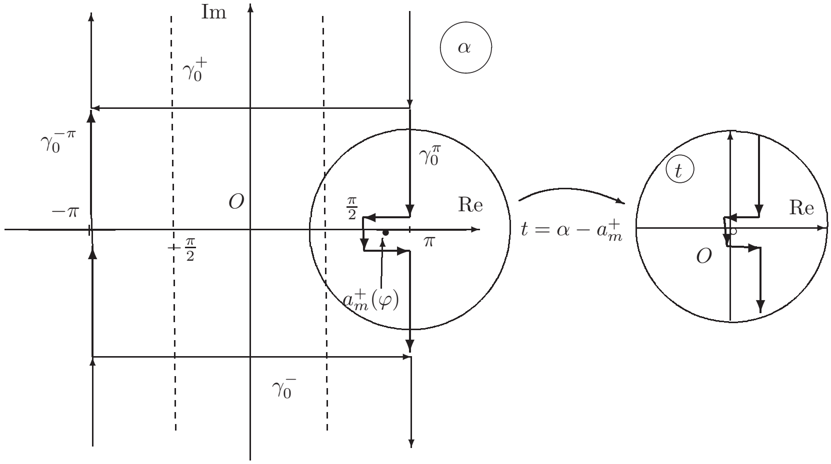

FIGURE 3. Asymptotic evaluation of the Sommerfeld integral, the captured pole $\alpha_m^+(\varphi) = a_m^+ - \varphi$.

$$u_m^+(r, \varphi) = \frac{1}{\pi i} \int_{\gamma_0} d\alpha\, e^{\kappa_m r \cos\alpha} f_m^+(\alpha + \varphi), \tag{4.11}$$

$$u_m^-(r, \varphi) = \frac{1}{\pi i} \int_{\gamma_0} d\alpha\, e^{\kappa_m r \cos\alpha} f_m^-(\alpha + \bar{\varphi}), \quad \bar{\varphi} = \pi - \varphi, \tag{4.12}$$

We mainly discuss asymptotics of (4.11), whereas for (4.12) only some results will be given. In the process of deformation of the integration contours γ_0 into the SD paths $\operatorname{Re}\alpha = \pm\pi$ (Figure 3), some poles of the integrands can be captured and contribute to the asymptotics. However, for some directions φ the pole at $\alpha = a_m^+ - \varphi$ of $f_m^+(\alpha + \varphi)$ in (4.11) can be located in a close neighbourhood of the saddle point π. (Similarly, the poles $\alpha = a_m^- - \bar{\varphi}$ of $f_m^-(\alpha + \bar{\varphi})$ in (4.12) can be located in a close neighbourhood of the saddle points π.)

In the asymptotic evaluation of the integral (4.11) as $r \to \infty$, we distinguish two cases:[5] the first case (I) correponds to either

$$\pi - (a_m^+ - \varphi) \geqslant \frac{\text{const}}{(\kappa_m r)^{1/2-\epsilon}} > 0$$

or

$$-\pi + (a_m^+ - \varphi) \geqslant \frac{\text{const}}{(\kappa_m r)^{1/2-\epsilon}} > 0$$

and the second case (II) is

$$|\pi - (a_m^+ - \varphi)| \leqslant \frac{\text{const}}{(\kappa_m r)^{1/2-\epsilon}}$$

for some small positive ϵ, $a_m^+ = \pi + \Phi - \tau_m$, $0 \leqslant \varphi \leqslant \Phi$.

[5] Recall that parameters m and Φ are assumed to be fixed.

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In the case I, the pole at $\alpha = a_m^+ - \varphi$ is separated from the saddle point π $(\pi > a_m^+ - \varphi)$ and is captured in the process of deformation of the contour γ_0 (Figure 3) into the SD paths $\gamma_0^\pi \cup \gamma_0^{-\pi}$. The corresponding residue gives the main term in the far-field asymptotics

$$u_m^+(r, \varphi) = 2\, A_m^+\, e^{-\kappa_m r \cos[\Phi - \varphi - \tau_m]} + O\left(\frac{e^{-\kappa_m r}}{\sqrt{\kappa_m r}}\right), \tag{4.13}$$

whereas the saddle points are responsible for the remainder terms of $O\left(\frac{e^{-\kappa_m r}}{\sqrt{\kappa_m r}}\right)$. However, if the other poles are also captured, they specify the remainder to the leading terms of the asymptotics (4.13).

On the other hand, if the poles of $f_m^+(\alpha + \varphi)$ are outside the strip $\alpha \in \Pi_\pi$ $(\pi < a_m^+ - \varphi)$, the principal contribution is specified by the saddle points $\pm\pi$

$$u_m^+(r, \varphi) = 2\,[f_m^+(-\pi + \varphi) - f_m^+(\pi + \varphi)]\, \frac{e^{-\kappa_m r}}{\sqrt{2\pi \kappa_m r}} \left(1 + O\left(\frac{1}{\kappa_m r}\right)\right). \tag{4.14}$$

The asymptotics (4.13), (4.14) enable us to assert that the eigenfunction exponentially vanishes in the directions corresponding to the case I, however, with different rates.

We turn to the case II. The direction that corresponds to $\pi = a_m^+ - \varphi$ is called singular direction [1]. The angular vicinity of this direction, i.e. those φ satisfying $|\pi - (a_m^+ - \varphi)| \leqslant \frac{\text{const}}{(\kappa_m r)^{1/2-\epsilon}}$, is usually called transition zone. As we remarked, for the transition zone the pole $a_m^+ - \varphi$ is close to the saddle point π. A uniform version of the saddle point technique is to be applied. Consider $O(1/[\kappa_m r]^{1/2-\varepsilon})$ vicinity of the saddle point π displayed $B_\pi([\kappa r]^{-1/2+\epsilon}$ in Figure 3. Taking into account (4.10), we have for the Sommerfeld integral

$$u_m^+(r, \varphi) = \frac{A_m^+}{\pi i} \int\limits_{\gamma_0^\pi \cap B_\pi([\kappa r]^{-1/2+\epsilon})} d\alpha\, \frac{e^{\kappa r \cos\alpha}}{(\alpha + \varphi) - a_m^+} + \delta u^\pm(r, \varphi), \tag{4.15}$$

where the remainder term is

$$\delta u_m^+(r, \varphi) = \frac{1}{\pi i} \int\limits_{\gamma_0^\pi \cap B_\pi([\kappa r]^{-1/2+\epsilon})} d\alpha\, e^{\kappa r \cos\alpha} \delta f_m^+(\alpha + \varphi)$$
$$+ \frac{1}{\pi i} \int\limits_{\gamma_0^\pi \setminus B_\pi([\kappa r]^{-1/2+\epsilon})} d\alpha\, e^{\kappa r \cos\alpha} f_m^+(\alpha + \varphi) + \frac{1}{\pi i} \int\limits_{\gamma_0^{-\pi}} d\alpha\, e^{\kappa r \cos\alpha} f_m^+(\alpha + \varphi)$$

with $\delta f_m^+(\alpha + \varphi) = f_m^+(\alpha + \varphi) - \dfrac{A_m^+}{(\alpha + \varphi) - a_m^+}$. We asymptotically evaluate the integral in (4.15). In the disk $B_\pi([\kappa r]^{-1/2+\epsilon})$, i.e. in this vicinity of the saddle point $\alpha = \pi$, we can approximate $\cos\alpha$ in the exponent of the integrand as $\cos\alpha = -1 + \frac{1}{2}(\alpha - \pi)^2 + \ldots$ then make use of the new variable of integration $t = e^{i\pi/2}(\alpha - \pi)$. Thus, obtain

$$\frac{A_m^+}{\pi i} \int\limits_{\gamma_0^\pi \cap B_\pi([\kappa r]^{-1/2+\epsilon})} d\alpha \, \frac{e^{\kappa r \cos\alpha}}{(\alpha+\varphi)-a_m^+}$$
$$= \frac{A_m^+ e^{-\kappa_m r}}{\pi i} \int\limits_{-\infty}^{\infty} dt \, \frac{e^{-\frac{1}{2}\kappa_m r t^2}}{(t - i[(a_m^+ - \varphi) - \pi])} \left(1 + O([\kappa_m r]^{-1/2+\epsilon})\right),$$

where the limits of integration were changed by $\pm\infty$, which contributes exponentially small relative error as $\kappa_m r \to \infty$. Remark that, if the pole in the denominator of the integrand is in the upper halfplane of complex variable t, integration is conducted along the real axis. However, provided $i[(a_m^+ - \varphi) - \pi]$ is in the lower halfplane, the contour comprises the pole from below. The latter integral is represented in terms of the Fresnel-type integral (the error function) in accordance with Section 6.3.1 of [9]

$$\Psi(z;s) := \int\limits_{-\infty}^{\infty} dt \, \frac{e^{-zt^2}}{(z-s)} = \pi i e^{-s^2 z}[1 - \mathcal{F}(-is\sqrt{z})],$$

where $\mathcal{F}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-t^2} dt$.

As a result, from (4.15), we arrive at

$$u_m^+(r,\varphi) = \frac{A_m^+ e^{-\kappa_m r}}{\pi i} \, \Psi\left(\frac{1}{2}\kappa_m r \, ; \, i[(\Phi-\varphi)-\tau_m]\right) + \delta u^\pm(r,\varphi), \tag{4.16}$$

the remainder term $\delta u^\pm(r,\varphi)$ is of $O\left(\frac{e^{-\kappa_m r}\Psi}{[\kappa_m r]^{1/2-\epsilon}}\right)$ in the transition zone. The asymptotic expression (4.16) describes switching from the asymptotic regime of decay (4.14) to that in (4.13).

Study of the asymptotics in the domain Ω_- is similar. In the process of deformation of the contour into the SD paths, several poles are usually captured. The leading contribution is described by the closest to the imaginary axis poles which has the form

$$u_m^-(r,\varphi) = 2\, A_m^- \, e^{-\kappa_m r \cos[\bar\Phi - \bar\varphi - \tau_m]} \, (1 \,+\, o(1))\,. \tag{4.17}$$

The remainder term in (4.17) is specified by next closest to the imaginary axis poles.

If the poles are not located in close vicinites of the saddle point and are not captured in the process of deformation of γ_0, the asymptotics takes the form

$$u_m^-(r,\varphi) = 2\,[f_m^-(-\pi+\bar\varphi) - f_m^-(\pi+\bar\varphi)]\, \frac{e^{-\kappa_m r}}{\sqrt{2\pi\kappa_m r}} \left(1 \,+\, O\left(\frac{1}{\kappa_m r}\right)\right) \tag{4.18}$$

The saddle points specify the leading terms of the asymptotics in (4.18). In a transition zone, the corresponding asymptotics is derived similarly to (4.16) in terms of the Fresnel integral.

5 Conclusion

In this work, we have constructed integral representations for the eigenfunctions in two adjacent wedges with the boundary conditions that describe a semitransparent thin layer in acoustics . On this way, we also established existence of the discrete spectrum for a class of meromorphic potentials of the FD equation at hand. We made use of the Sommerfeld integral representation for the eigenfunctions in order to study far-field asymptotic behaviour.

Our analysis shows that an eigenfunction exponentially vanishes at infinity as was expected. However, the rate of decay depends on the observation direction φ. Indeed, in the domain Ω_+, there is a singular direction with an asymptotically small angular vicinity, where the leading term of the asymptotics is described by a Fresnel-type integral. The latter plays the role of transition function that is responsible for switching regimes of the exponential decay. These regimes are correspondingly specified by asymptotic contributions of poles or saddle points in the process of asymptotic evaluation of the Sommerfeld integrals. In the supplementary domain Ω_-, the situation is analogous; however, as $0 < \bar{\varphi} < \bar{\Phi}$ several poles of the integrand can usually be captured provided $\bar{\Phi}$ is small enough. In this case, the remainder term to the leading asymptotics is specified by the next poles following those specifying the leading contribution. Provided one computes correction to the leading terms, the next poles are taken into account and contribute to the asymptotics. In this case, these poles may also be close to the saddle points and, therefore, the corresponding transition zones of higher orders can emerge. In particular, if $\bar{\Phi}$ is sufficiently small, the number of captured poles can be large. Then the picture of the asymptotic behaviour of an eigenfunction becomes more complicated because, as φ varies, many poles move inside the strip Π_π going through the saddle points. This generates numerous transition zones. Remark that, as $\bar{\Phi} \to 0$, an additional asymptotic parameter arises and study of the asymptotic behaviour of eigenfunctions requires an alternative approach.

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Conflict of interest

None.

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