Bounded elementary generation of Chevalley groups

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Abstract. We state several results on bounded elementary generation and bounded commutator width for Chevalley groups over Dedekind rings of arithmetic type in positive characteristic. In particular, Chevalley groups of rank ≥ 2 over polynomial rings $\mathbb{F}_q[t]$ and Chevalley groups of rank ≥ 1 over Laurent polynomial rings $\mathbb{F}_q[t, t^{-1}]$, where $\mathbb{F}_q$ is a finite field of q elements, are boundedly elementarily generated. In both cases we establish explicit bounds, and in the latter case they are quite sharp. Using these bounds we can also produce explicit bounds of the commutator width of these groups. We also mention some applications, possible generalisations and several related open problems, whose solution would require explicit computations. The complete text of the present talk is available in [14].

Introduction

In the present talk, we consider Chevalley groups $G = G(\Phi, R)$ and their elementary subgroups $E(\Phi, R)$ over various classes of rings, mostly over Dedekind rings of arithmetic type (we refer to [34] for notation and further references pertaining to Chevalley groups, and to [2] for the number theory background).

Primarily, we are interested in the classical problems of estimating the width of $E(\Phi, R)$ with respect to the two following generating sets.

- The elementary generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$. We say that a group G is **boundedly elementarily generated** if $E(\Phi, R)$ has finite width $w_E(G)$ with respect to elementary generators.

- Commutators $[x, y] = xyx^{-1}y^{-1}$, where $x, y \in G$. We say that G has **finite commutator width** if every element of $E(\Phi, R)$ is a product of $\leq w_C(G)$ commutators $[x, y]$, $x \in G(\Phi, R)$, $y \in E(\Phi, R)$.

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For the case of Chevalley groups of rank ≥ 2, in which we are mostly interested, bounded generation in terms of elementary generators, and bounded generation in terms of commutators are essentially equivalent. Indeed, in this case the Chevalley commutator formula readily implies that every elementary generator can be presented as a product of a bounded number of commutators.

Conversely, a very deep result by Alexei Stepanov and others, see, in particular, [30], and in final form [28], implies that every commutator in $E(\Phi, R)$ is a product of not more than L elementary generators, with the bound $L = L(\Phi)$ depending on Φ alone. But of course the actual estimates of $w_E(G)$ and $w_C(G)$ can be very different.

Both problems have attracted considerable attention over the last 40 years or so. Very roughly, the situation is as follows.

• Bounded elementary generation always holds with obvious *small* bounds for 0-dimensional rings. This follows from the existence of such short factorisations as Bruhat decomposition, Gauß decomposition, unitriangular factorisation of length 4, and the like. On the other hand, bounded generation usually fails for rings of dimension ≥ 2. But for 1-dimensional rings it is problematic.

• Existence of arbitrary long division chains in Euclidean algorithm implies that $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{SL}(2, \mathbb{F}_q[t])$ are not boundedly elementary generated [6]. But this could be attributed to the exceptional behaviours of rank 1 groups.

• What came as a shock, was when Wilberd van der Kallen [13] established that bounded elementary generation — and thus also finite commutator width — fail even for $\mathrm{SL}(3, \mathbb{C}[x])$, a group of Lie rank 2 over a Euclidean ring! Compare also [8], for a slightly simplified proof.

An emblematic example of 1-dimensional rings are Dedekind rings of arithmetic type $R = \mathcal{O}_S$, for which bounded elementary generation of $G(\Phi, R)$ is intrinsically related to the positive solution of the congruence subgroup problem in that group. This connection was first noted by Vladimir Platonov and Andrei Rapinchuk, see [20].

For the **number case** the situation is well understood, even for rank 1 groups. Without attempting to give a detailed survey, let us mention some high points of this development. Apart from the rings $R = \mathcal{O}_S$, $|S| = 1$, with finite multiplicative group, such finiteness results are even available for $\mathrm{SL}(2, R)$.

• For all Chevalley groups of rank ≥ 2, after the initial breakthrough by Douglas Carter and Gordon Keller, [3, 4], later explained and expanded by Oleg Tavgen [31], and many others, we now know bounded elementary generation with excellent bounds depending on the type of Φ and the class number of R alone.

This leaves us with the analysis of the group $\mathrm{SL}(2, R)$, for a Dedekind ring $R = \mathcal{O}_S$, with infinite multiplicative group.

• At about the same time, jointly with Paige, Carter and Keller gave a model theoretic proof [unpublished], [5], somewhat refashioned by Dave Morris [18]. But as all model theoretic proofs, this proof gives no bounds whatsoever.

• On the other hand, another important advance was made by Cooke and Weinberger [7], who got excellent bounds, modulo the Generalised Riemann Hypothesis. The explicit unconditional bounds obtained thereafter seemed to be grossly exaggerated [16].

• Some 10 years ago Maxim Vsemirnov and Sury [36] considered the key example of $\mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{p}\right]\right)$, obtaining the bound $w_E(\mathrm{SL}(2,R)) = 5$ *unconditionally*.

• This was a key inroad to the first complete unconditional solution of the general case with a good bound, in the work of Alexander Morgan, Andrei Rapinchuk and Sury [17]. The bound they gave is ≤ 9, but for the case when S contains at least one real or non-Archimedean valuation was almost immediately improved [with the same ideas] to ≤ 8 by Jordan and Zaytman [12].

However, the **function case** turned out to be much more recalcitrant, and is up to now not fully solved, apart from some important but isolated results. On the one hand, an analogue of Riemann's Hypothesis was known in this case for quite some time. Also, the function case analogue of Dirichlet's theorem on primes in arithmetic progressions, the Kornblum—Artin theorem for $\mathbb{F}_q[t]$, is much precise than the Dirichlet theorem itself.

On the other hand, in the positive characteristic additional arithmetic difficulties occur, that have no obvious counterparts in characteristic 0. They reflect in particular in the structure of arithmetic subgroups in the function case. For instance, it is well known that the group $\mathrm{SL}(2, \mathbb{F}_q[t])$ is not even finitely generated, whereas the groups $\mathrm{SL}(2, \mathbb{F}_q[t, t^{-1}])$ and $\mathrm{SL}(3, \mathbb{F}_q[t])$ are finitely generated but not finitely presented.

• Until very recently the only published result was that by Clifford Queen [23]. Queen's main result implies that under some additional assumptions on R — which hold, for instance, for Laurent polynomial rings $\mathbb{F}_q[t, t^{-1}]$ with coefficients in a finite field — the elementary width of the group $\mathrm{SL}(2, R)$ is 5. As we shall see this implies, in particular, bounded elementary generation of all Chevalley groups $G(\Phi, R)$ with plausible bounds.

Queen's proof is mainly based on the same principles proposed by Cooke and Weinberger [7] in the number field case. Namely, it uses subtle analytic ingredients, such as a function field analogue of Artin's primitive root conjecture, in order to obtain short division chains. In contrast to the number field case where the validity of Artin's conjecture is only known conditionally on the Generalized Riemann Hypothesis (GRH), its function field analogue, developed by Bilharz in the 1930's, became an unconditional theorem after Weil's work. See the paper of Lenstra [15] for more details, and for a strengthening of Queen's theorem.

• The case of the groups over the usual polynomial ring $\mathbb{F}_q[t]$ long remained open. Only in 2018 has Bogdan Nica established the bounded elementary generation of $\mathrm{SL}(n, \mathbb{F}_q[t])$, $n \geq 3$. Part of the problem is that in characteristic $p > 0$ bounded elementary generation is not the same as bounded generation in terms of cyclic subgroups. For instance, the groups $\mathrm{SL}(n, \mathbb{F}_q[t])$ do not have bounded generation in this abstract sense, see [1].

• After the preliminary version of the present work has been finished, there appeared a preprint of Alexander Trost [32] where the statement of our Theorem A was established for the ring of integers R of an arbitrary global function field K, with a bound of the form $L(d, q) \cdot |\Phi|$, where the factor L depends on q and of the degree d of K. His method is similar to Morris' approach in [18].

Here we merely state our main results. There are many interesting aspects of the proof, especially in the case of the group $\mathrm{Sp}(4, \mathbb{F}_q[t])$ that requires tons of explicit calculations, related to stability theorems, reciprocity laws, Mennicke symbols, Chebyshev polynomials, etc. Obviously, in the talk we can only present an outline, all details can be found in our paper [14].

1. Bounded generation of $G(\Phi, \mathbb{F}_q[t])$

Here we establish similar results for all Chevalley groups over $\mathbb{F}_q[t]$, with explicit uniform bounds that only depend on type Φ. The first major new result of the present work treats the most difficult example, polynomial rings $\mathbb{F}_q[t]$ with coefficients in finite fields.

Theorem A. *Let $G(\Phi, R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi) \geq 2$ over $R = \mathbb{F}_q[t]$. Then the width of $G(\Phi, R)$ with respect to elementary generators is bounded.*

One of the main points of the present work is that, unlike the proofs based on model theory, here we get *efficient* realistic estimates for the number of factors. In some cases, like for reduction to smaller rank, our bounds are the best possible ones. For small ranks, there might be still some gap between the counter-examples and the estimates we obtain, but our upper bounds are fairly close to the theoretically best possible ones. And the lower bounds in such similar problems are usually quite difficult to obtain, anyway.

Roughly, the leading idea of our proof still follows Tavgen's general scheme, and is based on his rank reduction trick. It is very general and beautiful, and works in many other related situations. Tavgen himself used the fact that for systems of rank ≥ 2 every fundamental root falls into the subsystem of smaller rank obtained by dropping either the first or the last fundamental root. However, as was pointed out by the referee of [26], the argument applies without any modification in a much more general setting. Namely, it suffices to assume that the required decomposition holds for *some* subsystems $\Delta = \Delta_1, \ldots, \Delta_t$, whose union contains all fundamental roots of Φ. These subsystems do not have to be terminal.

Some bound in the bounded generation for all Chevalley groups can be easily derived from the case of rank two systems by a version of the usual Tavgen's trick [31], Theorem 1, described in [35] and [26]. Let us state it in a slightly more general form.

Theorem B. *Let Φ be a reduced irreducible root system of rank $l \geq 2$, and R be a commutative ring. Further, let $\Delta_1, \dots, \Delta_t$ be some subsystems of Φ, whose union contains all fundamental roots of Φ. Suppose that for all $\Delta = \Delta_1, \dots, \Delta_t$, the elementary Chevalley group $E(\Delta, R)$ admits a unitriangular factorisation*

$$E(\Delta, R) = U(\Delta, R)U^-(\Delta, R) \dots U^{\pm}(\Delta, R)$$

of length L. Then the elementary Chevalley group $E(\Phi, R)$ itself admits unitriangular factorisation

$$E(\Phi, R) = U(\Phi, R)U^-(\Phi, R) \dots U^{\pm}(\Phi, R)$$

of the same length L.

Thus, we are left with the analysis of rank 2 cases.

• For A_2 bounded generation of $\mathrm{SL}(3, \mathbb{F}_q[t])$ is precisely the main result of Nica [Ni]. In fact, Nica establishes that

$$w_E(\mathrm{SL}(3, \mathbb{F}_q[t])) \leq 41.$$

This bound 41 is obtained as follows. Over a Dedekind ring one needs 7 elementary operations to reduce a 3×3 matrix to a 2×2 matrix (one would need 8 for a general ring subject to $\mathrm{sr}(R) \leq 2$). The elementary length of any matrix $g \in \mathrm{SL}(2, R)$ inside $\mathrm{SL}(3, R)$ is at most 34. Interestingly, the main arithmetic ingredient of his proof is the Kornblum—Artin functional version of Dirichlet's theorem on primes in arithmetic progressions.

An interesting aspect of Nica's work [19] is that he avoids the usual Mennicke type calculations [2], and carries the proof using the so-called "swindling lemma" instead. This allows him to obtain somewhat better bounds for the number of elementary generators.

• Luckily, we do not have to imitate Tavgen's proof [31], section 5, for the case of the Chevalley group of type G_2. Instead of a difficult direct calculation, we show that this case can be derived from the case of A_2 by the usual stability arguments. Stability of the embeddings $\mathrm{A}_1 \subseteq \mathrm{A}_2 \subseteq \mathrm{G}_2$ under $\mathrm{asr}(R) \leq 2$ was established by Michael Stein, see [27]. We had just to go over the proof to trace all elementary operations.

Over a Dedekind ring one needs 20 elementary operations to reduce any element of $E(\mathrm{G}_2, R)$ to an element of $\mathrm{SL}(2, R)$ in a long root embedding — one would need 24 for a general ring subject to $\mathrm{asr}(R) \leq 2$, which gives us

$$w_E(G(\mathrm{G}_2, \mathbb{F}_q[t])) \leq 54.$$

• A large part of the actual proof of theorem A is the analysis of the most difficult case of $\mathrm{Sp}(4, \mathbb{F}_q[t])$, which is the Chevalley group of type C_2. The difficulty is that now we have to take two types of embeddings of $\mathrm{A}_1 \leq \mathrm{C}_2$, the long root embedding and the short root embedding.

Here, we again take the proof in Tavgen's paper [31], section 4, as a prototype. But there is a substantial difference, since now we have to verify many arithmetic lemmas that are well known in the number case, but for which we could not find any obvious reference in the function case. Apart from a strong version of Kornblum—Artin theorem, we had to carry through rather meticulous calculations depending on the explicit formula of the reciprocity law for power-residue symbols.

Now, we have to first reduce the long root embedding to such an embedding whose entry is a square, then (following Bass—Milnor—Serre and Tavgen) reduce it to a short root embedding, and, finally, perform (more difficult!) calculations to express a matrix from $\mathrm{SL}(2,R)$ in the short root embedding as a product of elementary unipotents in $\mathrm{Sp}(4,R)$. As a result, the bound we get is worse than for other rank 2 cases.

This eventually leaves us with the [exaggerated] bound

$$w_E(G(\mathrm{C}_2,\mathbb{F}_q[t]))\leq 79.$$

and we challenge the reader to improve it, along the lines of [19].

Quite amazingly, C_2 is the only difficult case! For groups of types B_l and C_l, $l\geq 3$, we have found *much* easier proofs, based on the fact that *either* a long root, *or* a short one can be embedded in a root system of type A_2, so that we can proceed directly from [19].

In particular, for groups of rank 3 one gets *better* bounds than for C_2, viz.

$$w_E(G(\mathrm{C}_3,\mathbb{F}_q[t]))\leq 72,\qquad w_E(G(\mathrm{B}_3,\mathbb{F}_q[t]))\leq 65.$$

Some bounds for the elementary bounded generation for all Chevalley groups can be easily derived from the above form of Tavgen rank reduction theorem. For instance, it can be derived from the existence of two types of embeddings of $\mathrm{A}_2\leq\mathrm{F}_4$, the long root embedding and the short root embedding, that

$$w_E(G(\mathrm{F}_4,\mathbb{F}_q[t]))\leq 216,$$

but this bound seems not to be the best possible.

For $\mathrm{SL}(n,R)$ there is a realistic bound of the width in elementary generators, in terms of stability conditions, taking into account the elementary fact that for Dedekind rings $\mathrm{sr}(R)=1.5$. The above proof of Theorem A gives us occasion to return to the stability arguments for all Chevalley groups, and obtain bounds which are substantially better than the ones that could be obtained via Tavgen's trick.

Alternatively, Theorem A can be restated in the following equivalent form. The difference is that in this case the computations of many authors, subsumed and expanded by Andrei Smolensky [25], allow to produce short explicit bounds.

Theorem C. *Let $G(\Phi,R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi)\geq 2$ over $R=\mathbb{F}_q[t]$. Then $G(\Phi,R)$ is of finite commutator width L, where*

- $L\leq 5$ *for $\Phi=A_l$, for $l\geq 2$, or $\Phi=F_4$,;*
- $L\leq 6$ *for $\Phi=B_l,C_l,D_l$, for $l\geq 3$ or $\Phi=E_7,E_8$, or, finally, $\Phi=C_2,G_2$ under the additional assumption that* 1 *is the sum of two units in R* (*which is automatically the case, provided $q\neq 2$*);
- $L\leq 7$ *for $\Phi=E_6$.*

We believe that the bound for E_6 could be also improved to $L \leq 6$, but we were strongly discouraged by the extent of explicit calculations needed to do that.

2. Bounded generation of $G(\Phi, \mathbb{F}_q[t, t^{-1}])$

In fact, ulterior applications to Kac—Moody groups that we have in mind do not need the full power of Theorems A and C. We only need a similar result for the equally classical but *much easier* example of *Laurent* polynomial rings $\mathbb{F}_q[t, t^{-1}]$ with coefficients in finite fields.

For Chevalley groups over such rings bounded generation can be derived from Theorem A. Yet, the bounds thus obtained will not be the best possible ones. However, the multiplicative group of the ring $R = \mathbb{F}_q[t, t^{-1}]$ is *infinite*. This means that bounded generation — with much better bounds! — follows already from the result by Clifford Queen [23]. Let us state the most surpising finiteness result in terms of unitriangular factors obtained along this route.

Theorem D. *Let $R = \mathcal{O}_S$ be the ring of S-integers of K, a function field of one variable over $\mathbb{F}_q$ with S containing at least two places. Assume that at least one of the following holds:*

- *either at least one of these places has degree one,*
- *or the class number of R, as a Dedekind domain, is prime to $q - 1$.*

Then any simply connected Chevalley group $G = G(\Phi, R)$ admits the following decompositions

$$G = UU^-UUU^- = U^-UU^-UU^-.$$

The key case here are the groups $\mathrm{SL}(2, R)$, for which the result follows from Theorem 2 of [23]. It is stated there correctly, but the proof at the very last page contains a minor inaccuracy and would imply that G admits a unitriangular decomposition of length 4, which contradicts the main result of [35]. The reason is that at a certain stage of the calculation one obtains an invertible element $\epsilon \in R^*$, whereas [23] takes this element to be 1. Slightly rearranging the proof, one gets the correct (and best possible!) bound, that any element of $\mathrm{SL}(2, R)$ is a product of ≤ 5 elementary transvections. Theorem C now follows by Tavgen's rank reduction trick.

In particular, this theorem allows to dramatically reduce bounds for groups over $\mathbb{F}_q[t, t^{-1}]$, to

$$w_E(G(\mathrm{A}_2, \mathbb{F}_q[t, t^{-1}])) \leq 15, \qquad w_E(G(\mathrm{C}_2, \mathbb{F}_q[t, t^{-1}])) \leq 20,$$
$$w_E(G(\mathrm{G}_2, \mathbb{F}_q[t, t^{-1}])) \leq 30,$$

via unitriangular factorisations. Stability results that we mentioned before afford even better bounds, such as, for instance,

$$w_E(G(\mathrm{A}_2, \mathbb{F}_q[t, t^{-1}])) \leq 12, \qquad w_E(G(\mathrm{C}_2, \mathbb{F}_q[t, t^{-1}])) \leq 15,$$
$$w_E(G(\mathrm{G}_2, \mathbb{F}_q[t, t^{-1}])) \leq 25.$$

As above, using the technology of [25], we can derive from Theorem D estimates for the commutator width.

Theorem E. *Let R be as in Theorem D. Then the commutator width of the simply connected Chevalley group $G = G(\Phi, R)$ is $\leq L$, where*

- $L = 3$ *for* $\Phi = A_l$, *for* $l \geq 2$, *or* $\Phi = F_4$,;
- $L = 4$ *for* $\Phi = B_l, C_l, D_l$, *for* $l \geq 3$ *or* $\Phi = E_7, E_8$, *or, finally,* $\Phi = C_2, G_2$ *under the additional assumption that* 1 *is the sum of two units in* R (*which is automatically the case, provided* $q \neq 2$);
- $L = 5$ *for* $\Phi = E_6$;

This kind of sharp bounds were quite unexpected for us. In particular, Chevalley groups over such *arithmetic* rings have the same commutator width as Chevalley groups over rings of stable rank 1, see [25].

3. Applications and possible generalisations

Primarily, we have in mind the following two types of applications, that are described in [14].

- Estimates of the width of Kac—Moody groups defined over a finite field with respect to commutators and other natural generating sets.
- Model-theoretic applications. Bounded generation implies a lot of important logical properties of groups. In our case the groups $G(\Phi, \mathbb{F}_q[t])$ and $G(\Phi, \mathbb{F}_q[t, t^{-1}])$, $\mathrm{rk}(\Phi) > 1$ turn out to be first order rigid, quasi-finitely axiomatisable and logically homogeneous.

Here are some generalisations of the above theorems A—E that we plan to address in the next papers.

- For all ranks $\mathrm{rk}(\Phi) \geq 1$ remove the remaining restrictions on the ring R in Theorems D and E.
- For ranks $\mathrm{rk}(\Phi) \geq 2$ prove analogues of Theorems A and C for all Dedekind rings of arithmetic type. This should be possible, but might be difficult, since many of the requisite arithmetic facts are not as easily available, as in the number case.
- For classical groups, reduction to smaller ranks is well-known. We are in possession of similar reduction results, based on effectivisation of [27, 21, 22, 9]. These results give pretty sharp bounds also for exceptional cases. But calculations with columns of height 26, 27, 56 and 248 are quite a bit more involved, and spread over several dozen pages.

In the next paper we plan to produce all details for the stability reduction for the exceptional cases F_4, E_6, E_7, E_8 in the same spirit as we have done in [14] for G_2, B_l and C_l. The goal is obtain new explicit bounds for the elementary width in these cases, which are better than the known ones even in the number case.

- Another very challenging problem would be to perform scrupulous analysis of the proofs to reduce the number of elementary moves. We are pretty sure that our bounds are far from being optimal. Even without attempting to get sharp bounds, we believe that we could improve the bounds in the present paper, and other related results.

However, to get the *best possible* bounds one might need to perform extensive computer search/computer calculations. However, to get such optimal bounds would be *extremely* difficult, no such bounds are in sight even in the number case, even for such groups as $\mathrm{SL}(3, \mathbb{Z})$.

• Partial positive results, such as bounded expressions of elementary conjugates and commutators in terms of elementary generators — decomposition of unipotents, Stepanov's univeral localisation, and the like, [29, 30, 28]. It seems one should be able to obtain similar results also for other word maps.

• Let us mention yet another extremely pregnant generalisation, **bounded reduction**. In fact, even below the usual stability conditions and even in the absense of the bounded generation for $G(\Phi, R)$, it makes sense to speak of the number of elementary generators necessary to reduce an element g of $G(\Phi, R)$ to an element of $G(\Delta, R)$, for a subsystem $\Delta \subseteq \Phi$.

One such prominent example are polynomial rings $R[t_1, \dots, t_m]$, where bounded reduction holds starting with a rank depending on R alone, not on the number of indeterminates. For the case of $\mathrm{SL}(n, R[t_1, \dots, t_m])$ this is essentially an effectivisation of Suslin's solution of the K_1-analogue of Serre's problem, explicit bounds were obtained in the remarkable paper by Leonid Vaserstein [33], which unfortunately remained unpublished. For other split classical groups such bounds were recently obtained by Pavel Gvozdevsky [10].

• Most of the results so far pertain to the *absolute* case alone. However, it makes sense to ask similar questions for the *relative* case, in other words for the congruence subgroups $G(\Phi, R, I)$, and the elementary subgroups $E(\Phi, R, I)$ of level $I \trianglelefteq R$. The expectation is to get similar *uniform* bounds in terms of the elementary conjugates $x_{-\alpha}(\eta)x_\alpha(\xi)x_{-\alpha}(-\eta)$, $\alpha \in \Phi$, $\xi \in I$, $\eta \in R$. Some results in this direction are contained in the paper by Sinchuk and Smolensky [24]. As a more remote goal one could think of generalisations to birelative subgroups, see [11].

We intend to return to [some of] these subjects in the full version of the present paper, and in its [expected] sequel.

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