Buckling Analysis of Thin Anisotropic Shells

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Abstract: This paper is concerned with the study of the localised buckling of anisotropic shells. For a shell consisting of the matrix reinforced by threads, the elasticity relations are obtained for the general anisotropic case. By considering the assumptions usually made in the Donnell theory, the equilibrium equations are simplified. For one parametric loading the critical pressure and the buckling modes are obtained by means of the asymptotic method. As an example the buckling of an elliptical shell under internal and external uniform pressure is analysed.

Résumé: L'étude du flambage local des coques minces anisotropes est presentée dans cet article. Les équations de l'élasticité anisotrope sont générées pour le cas d'une coque composite fibrée. La théorie de Donnell permet la simplification des équations de l'équilibre. Pour obtenir les modes de flambage et les pressions critiques dans le cas d'une charge parametrique, une méthode assymptotique est utilisée. A titre d'example, le flambage d'une coque élastique élliptique mise en pression uniforme interne et externe est analysée.

Introduction: The purpose of this paper is the generalisation for the case of anisotropic shells of the results of asymptotic analysis of thin shell buckling provided in [1]. In the expressions obtained in this paper the boundary conditions are not taken into account. For this reason, the results of the study may be applied in the cases when localisation of buckling occurs, for example, for the buckling of a convex shell under hydrostatic pressure or under torsion.

Elasticity Equations for Thin Anisotropic Shells: We consider a thin shell made of composite material, consisting of the matrix reinforced by threads situated in planes parallel to the midsurface. On the shell midsurface we introduce the curvilinear coordinates α_1, α_2 coinciding with the curvature lines. The coordinate z is directed along the normal to the midsurface. We assume that the shell is reinforced with N systems of threads, inclined at angles $\theta^{(k)}$ to the axis α_1, where $k = 1,2,\ldots,N$.

The shell stress σ_{ij} can be expressed as the sum of the matrix stress $\sigma_{ij}^{(0)}$ and the average stress $\sigma_{ij}^{(k)}$, caused by the extensions of the threads

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sum_{k=1}^{N} \sigma_{ij}^{(k)} . \tag{1}$$

Considering Kirchhoff's hypotheses, $\sigma_{33} = 0$ and the stresses σ_{i3} are determined from the equations of equilibrium. The stress-strain relations connecting σ_{ij} and the strains ε_{ij} for $i, j = 1,2$ are given next. For the matrix, these are [2]

$$
\begin{aligned}
&\sigma_{11}^{(0)} = F_0\left(\varepsilon_{11} + \nu_0\varepsilon_{22}\right),\\
&\sigma_{12}^{(0)} = \frac{1-\nu_0}{2}F_0\varepsilon_{12},\\
&\sigma_{22}^{(0)} = F_0\left(\varepsilon_{22} + \nu_0\varepsilon_{11}\right),\\
&F_0 = \frac{E_0\delta_0}{1-\nu_0^2},
\end{aligned}
\tag{2}
$$

where E_0 is the Young modulus and ν_0 is the Poisson ratio of the matrix. The coefficient $\delta_0 < 1$ takes into account the ratio of the volume filled by the matrix.

The stress-strain relations for the threads are

$$
\begin{aligned}
&\sigma_{11}^{(k)} = F_k\left(c_k^4\varepsilon_{11} + c_k^3 s_k\varepsilon_{12} + c_k^2 s_k^2\varepsilon_{22}\right),\\
&\sigma_{12}^{(k)} = F_k\left(c_k^3 s_k\varepsilon_{11} + c_k^2 s_k^2\varepsilon_{12} + c_k s_k^3\varepsilon_{22}\right),\\
&\sigma_{22}^{(k)} = F_k\left(c_k^2 s_k^2\varepsilon_{11} + c_k s_k^3\varepsilon_{12} + s_k^4\varepsilon_{22}\right),
\end{aligned}
\tag{3}
$$

where
$F_k = E_k\delta_k, \quad s_k = \sin\theta_k, \quad c_k = \cos\theta_k,$
$k = 1,2,\ldots,N.$
Here E_k is Young's modulus for the threads of the k-th system, and δ_k is the ratio of the volume filled with threads for the k-th system. Poisson's effect in the threads under stretching is neglected.

The strains ε_{ij} are according to Kirchhoff's hypotheses linear functions of coordinate z

$$
\begin{aligned}
&\varepsilon_{11} = \varepsilon_1 + \kappa_1 z,\\
&\varepsilon_{12} = \omega + 2\tau z,\\
&\varepsilon_{22} = \varepsilon_2 + \kappa_2 z.
\end{aligned}
\tag{4}
$$

Here ε_1, ω, and ε_2 are the stretching-shear strains and κ_1, τ, and κ_2 are the bending-twisting strains of the midsurface that are related to its displacements by the following relations [1]

$$
\begin{aligned}
&\varepsilon_1 = \varepsilon_1^l + \frac{1}{2}\gamma_1^2,\\
&\varepsilon_1^l = \frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2 - \frac{w}{R_1},\\
&\omega = \omega^l + \gamma_1\gamma_2,\\
&\omega^l = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_{21}}\left(\frac{u_1}{A_1}\right),\\
&\varepsilon_2 = \varepsilon_2^l + \frac{1}{2}\gamma_2^2,\\
&\varepsilon_2^l = \frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_1 - \frac{w}{R_2};
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\kappa_1 = -\frac{1}{A_1}\frac{\partial\gamma_1}{\partial\alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\gamma_2,\\
&\tau = -\frac{1}{A_2}\frac{\partial\gamma_1}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_{12}}\gamma_2 +\\
&\qquad \frac{1}{A_1R_2}\frac{\partial u_2}{\partial\alpha_1} - \frac{1}{A_1A_2R_2}\frac{\partial A_1}{\partial\alpha_2}u_1,\\
&\kappa_2 = -\frac{1}{A_2}\frac{\partial\gamma_2}{\partial\alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\gamma_1;
\end{aligned}
\tag{6}
$$

$$
\gamma_1 = -\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u_1}{R_1},\ \ \gamma_1 = -\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{u_2}{R_2}. \tag{7}
$$

Here γ_1 and γ_2 are the angles of rotation of the normal to the midsurface with respect to the coordinate lines α_1 and α_2; u_1, u_2, and w are the components of the midsurface displacement; R_1 and R_2 are the principal curvature radii; A_1 and A_2 are the metric coefficients. In relations (5) the main nonlinear terms are included with the multipliers γ_i, and the rest of the expressions - marked with ^{l} - are linear with respect to the displacements.

Neglecting the small values h/R_1 and h/R_2 in comparison to 1, where h is the shell thickness, we obtain the expressions for stress resultants T_1, T_2, and S and for the moment resultants M_1, M_2, and H [2]

$$T_1 = \int_{-h/2}^{h/2} \sigma_{11} dz, \qquad T_2 = \int_{-h/2}^{h/2} \sigma_{22} dz,$$
$$M_1 = \int_{-h/2}^{h/2} \sigma_{11} z dz, \quad M_2 = \int_{-h/2}^{h/2} \sigma_{22} z dz,$$
$$S = \int_{-h/2}^{h/2} \sigma_{12} dz, \qquad H = \int_{-h/2}^{h/2} \sigma_{12} z dz.$$

Here σ_{ij} are calculated from (1).

Assuming that the shell is symmetric with respect to the midsurface, and taking into account relations (2), (3) and (4) we obtain

$$\begin{aligned} T_1 &= K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{13}\omega, \\ T_2 &= K_{21}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\omega, \\ S &= K_{31}\varepsilon_1 + K_{32}\varepsilon_2 + K_{33}\omega; \end{aligned} \tag{8}$$

$$\begin{aligned} M_1 &= D_{11}\kappa_1 + D_{12}\kappa_2 + 2D_{13}\tau, \\ M_2 &= D_{21}\kappa_1 + D_{22}\kappa_2 + 2D_{23}\tau, \\ H &= D_{31}\kappa_1 + D_{32}\kappa_2 + 2D_{33}\tau. \end{aligned} \tag{9}$$

Here the coefficients K_{ij} and D_{ij} have the form [3]

$$\begin{bmatrix} K_{11} \\ D_{11} \end{bmatrix} = \int_{-h/2}^{h/2} \left(F_0 + \sum_{k=1}^{N} F_k c_k^4 \right) \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz,$$
$$\begin{bmatrix} K_{22} \\ D_{22} \end{bmatrix} = \int_{-h/2}^{h/2} \left(F_0 + \sum_{k=1}^{N} F_k s_k^4 \right) \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz,$$
$$\begin{bmatrix} K_{33} \\ D_{33} \end{bmatrix} = \int_{-h/2}^{h/2} \left(\frac{1-\nu_0}{2} F_0 + \sum_{k=1}^{N} F_k c_k^2 s_k^2 \right) \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz,$$
$$\begin{bmatrix} K_{12} \\ D_{12} \end{bmatrix} = \begin{bmatrix} K_{21} \\ D_{21} \end{bmatrix} = \int_{-h/2}^{h/2} \left(F_0 \nu_0 + \sum_{k=1}^{N} F_k c_k^2 s_k^2 \right) \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz,$$
$$\begin{bmatrix} K_{13} \\ D_{13} \end{bmatrix} = \begin{bmatrix} K_{31} \\ D_{31} \end{bmatrix} = \int_{-h/2}^{h'2} \sum_{k=1}^{N} F_k c_k^3 s_k \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz,$$
$$\begin{bmatrix} K_{23} \\ D_{23} \end{bmatrix} = \begin{bmatrix} K_{32} \\ D_{32} \end{bmatrix} = \int_{-h/2}^{h'2} \sum_{k=1}^{N} F_k c_k s_k^3 \begin{bmatrix} 1 \\ z^2 \end{bmatrix} dz. \tag{10}$$

For convenience, we introduce the notation $\omega = \varepsilon_3$, $\tau = \kappa_3$, $S = T_3$, $H = M_3$, and re-write relations (8) and (9) as

$$T_i = K_{ij}\varepsilon_j, \quad M_i = D_{ij}\kappa_j, \tag{11}$$

where we use Einstein's summation convention.

The values of δ_0 and δ_k in (10), as well as the elastic constants of the materials may depend on z. It is more reliable to find the coefficients K_{ij} and D_{ij} directly from experiments, such that the previous considerations just define the structure of (11).

We can express the elastic energy Π of the shell as a sum of the stretching energy Π_ε and the bending energy Π_κ [2] $\Pi = \Pi_\varepsilon + \Pi_\kappa$, where Π_ε and Π_κ are given by

$$\Pi_\varepsilon = \frac{1}{2} \iint (T_1\varepsilon_1 + T_2\varepsilon_2 + S\omega) d\Sigma = \frac{1}{2} \iint K_{ij}\varepsilon_i\varepsilon_j d\Sigma,$$
$$\Pi_\kappa = \frac{1}{2} \iint (M_1\kappa_1 + M_2\kappa_2 + 2H\tau) d\Sigma.$$

Here $d\Sigma = A_1 A_2 d\alpha_1 d\alpha_2$ is the area element and the integration in Π_ε and Π_κ is performed on the entire midsurface.

In the case the reinforcing threads are symmetric with respect to the directions α_1 and α_2, i.e. for each thread system with an angle θ_k there corresponds a system with an angle $\theta_l = -\theta_k$, then the sums in the last two relations in (10) vanish and $K_{i3} = D_{i3} = K_{3i} = D_{3i} = 0, \quad i = 1,2$. As a result we obtain the orthotropic shell, for which relations (8) and (9) have the form

$$\begin{aligned} T_1 &= K_{11}\varepsilon_1 + K_{12}\varepsilon_2, & M_1 &= D_{11}\kappa_1 + D_{12}\kappa_2, \\ T_2 &= K_{21}\varepsilon_1 + K_{22}\varepsilon_2, & M_2 &= D_{21}\kappa_1 + D_{22}\kappa_2, \\ S &= K_{33}\omega; & H &= 2D_{33}\tau. \end{aligned}$$

Equations of Equilibrium: The equilibrium equations for a shell element have the form [4]

$$
\begin{aligned}
&\frac{\partial(A_2T_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}T_2+\frac{\partial(A_1S)}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}S+ \\
&\qquad A_1A_2\left(q_1-\frac{Q_1}{R_1}\right)=0, \\
&\frac{\partial(A_1T_2)}{\partial\alpha_2}-\frac{\partial A_1}{\partial\alpha_2}T_1+\frac{\partial(A_2S)}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}S+ \\
&\qquad A_1A_2\left(q_2-\frac{Q_2}{R_2}\right)=0, \\
&\frac{\partial(A_2Q_1)}{\partial\alpha_1}+\frac{\partial(A_1Q_2)}{\partial\alpha_2}+ \\
&A_1A_2\left(q_3+\left(\frac{1}{R_1}+\kappa_1\right)T_1+2\tau S+\left(\frac{1}{R_2}+\kappa_2\right)T_2\right)=0, \\
&\frac{\partial(A_2M_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}M_2+\frac{\partial(A_1H)}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}H+ \\
&\qquad A_1A_2Q_1=0, \\
&\frac{\partial(A_1M_2)}{\partial\alpha_2}-\frac{\partial A_1}{\partial\alpha_2}M_1+\frac{\partial(A_2H)}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}H+ \\
&\qquad A_1A_2Q_2=0,
\end{aligned}
\tag{12}
$$

where q_i are the components of the external pressure (external loading per unit area of the midsurface).

The third equation in (12) contains the main nonlinear terms $\kappa_1 T_1$, τS, and $\kappa_2 T_2$.

The last two equations in (12) express the fact that the total moment of the internal stresses (the sum of the moment resultants, and the moments of stress and transverse shear resultant) is equal to zero for a shell element. The transverse shear resultants Q_1 and Q_2 may be found from these equations and then substituted into the first three equations in (12). As a result we obtain the system of three equations in the displacements u_1, u_2, and w.

We simplify equations (12) and the equations of the first Section by considering the assumptions usually made for the Donnell equations for shallow shells [5]. The metric of the midsurface is described as the metric of a plane, and we assume that the values of A_1, A_2, R_1, and R_2 are constant.

Let $dx_1 = A_1 d\alpha_1$ and $dx_2 = A_2 d\alpha_2$. In (6) and (7) we neglect the displacements u_1 and u_2 compared to w. We also neglect the terms containing the transverse shear resultants Q_1 and Q_2 in the first and second equations in (12). As a result, relations (5)–(7) and equations (12) become:

$$
\begin{aligned}
&\varepsilon_1^l=\frac{\partial u_1}{\partial x_1}-\frac{w}{R_1},\quad \varepsilon_2^l=\frac{\partial u_2}{\partial x_2}-\frac{w}{R_2}, \\
&\omega^l=\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2},\quad \tau=\frac{\partial^2 w}{\partial x_1\partial x_2}, \\
&\gamma_1=-\frac{\partial w}{\partial x_1},\quad \gamma_2=-\frac{\partial w}{\partial x_2}, \\
&\kappa_1=\frac{\partial^2 w}{\partial x_1^2},\quad \kappa_2=\frac{\partial^2 w}{\partial x_2^2}.
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\frac{\partial T_1}{\partial x_1}+\frac{\partial S}{\partial x_2}+q_1=0,\quad \frac{\partial T_2}{\partial x_2}+\frac{\partial S}{\partial x_1}+q_2=0, \\
&\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 H}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}+ \\
&\left(q_3+\left(\frac{1}{R_1}+\kappa_1\right)T_1+2\tau S+\left(\frac{1}{R_2}+\kappa_2\right)T_2\right)=0,
\end{aligned}
\tag{14}
$$

If the loads q_1, q_2 and q_3 have the same order or $\{q_1, q_2\} \ll q_3$, then with the same error it may be assumed that $q_1 = q_2 = 0$ in (14).

The simplified system (13)-(14) may be used not only for the analysis of shallow shells but also for the vibration and buckling analysis of arbitrary thin shells. In that case the stress-strain state of a shell may consists of many waves of deformations, but in the limit of one deformation wave the shell should be considered as shallow.

Local Buckling Modes: In this section, the results known for local buckling of isotropic shells are generalized for the case of anisotropic shells. Let consider that as a result of loading, there exists in a shell a momentless stress-strain state determined by the initial stress resultants T_1^0, T_2^0, S^0. The stress-strain state is referred to as momentless or membrane-like if the moment resultants $M_1 = M_2 = H = 0$. Next we analyse the stability of such a state.

The bifurcation equations for the equilibrium equations (14) become

$$\frac{\partial T_1}{\partial x_1} + \frac{\partial S}{\partial x_2} = 0, \quad \frac{\partial T_2}{\partial x_2} + \frac{\partial S}{\partial x_1} = 0,$$

$$T_1^0 \frac{\partial^2 w}{\partial x_1^2} + 2S^0 \frac{\partial^2 w}{\partial x_1 \partial x_2} + T_2^0 \frac{\partial^2 w}{\partial x_2^2} - \left(\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 H}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} \right) + \frac{T_1}{R_1} + \frac{T_2}{R_2} = 0. \quad (15)$$

These equations together with the strain-displacement relations (13) and the elasticity relations (8) and (9) form a closed system, provided linear approximation for tangential deformations is assumed in (8), i.e. $\varepsilon_j = \varepsilon_j^l$.

Next we study the one parametric loading by introducing the loading parameter λ as $\{T_1^0, T_2^0, S^0\} = -\lambda\{t_1, t_2, t_3\}$. The minus sign is necessary in order to seek $\lambda > 0$, since buckling is possible only if there exist directions in which compressive stresses are developed. Such directions exist if and only if at least one of the following inequalities is satisfied [1]

$$T_1^0 < 0 \quad \text{or} \quad T_2^0 < 0 \quad \text{or} \quad \left(S^0\right)^2 > T_1^0 T_2^0. \quad (16)$$

We seek the displacements under bifurcation in the form

$$u_1 = u_1^0 \sin z, \; u_2 = u_2^0 \sin z, \; w = w^0 \cos z, \quad z = k_1 x_1 + k_2 x_2, \quad (17)$$

where the amplitudes u_1^0, u_2^0, w^0 and the wave numbers k_1, and k_2 must be determined.

From the first two equations in (15) we find u_1^0 and u_2^0 as functions of w^0

$$u_1^0 = \frac{\Delta_1}{\Delta} w^0, \quad u_2^0 = \frac{\Delta_2}{\Delta} w^0,$$

where

$$\Delta = A_{22}k_1^4 - 2A_{23}k_1^3 k_2 + (2A_{12} + A_{33})k_1^2 k_2^2 - 2A_{13}k_1 k_2^3 + A_{11}k_2^4,$$

$$\Delta_1 = \left(\frac{A_{22}}{R_1} - \frac{A_{12}}{R_2} \right) k_1^3 + \left(\frac{A_{13}}{R_2} - \frac{2A_{23}}{R_1} \right) k_1^2 k_2 + \left(\frac{A_{12} + A_{33}}{R_1} - \frac{A_{11}}{R_2} \right) k_1 k_2^2 - \frac{A_{13}}{R_1} k_2^3,$$

$$\Delta_2 = \left(\frac{A_{11}}{R_2} - \frac{A_{12}}{R_1} \right) k_2^3 + \left(\frac{A_{23}}{R_1} - \frac{2A_{13}}{R_2} \right) k_1 k_2^2 + \left(\frac{A_{12} + A_{33}}{R_2} - \frac{A_{22}}{R_1} \right) k_1^2 k_2 - \frac{A_{23}}{R_2} k_1^3.$$

Here we denote as A_{ij} the minors for the elements of matrix $\{K_{ij}\}$, given by (10)

$$A_{11} = K_{22}K_{33} - K_{23}^2,$$
$$A_{12} = K_{13}K_{23} - K_{12}K_{33},$$
$$A_{22} = K_{11}K_{33} - K_{13}^2,$$
$$A_{23} = K_{12}K_{13} - K_{11}K_{23},$$
$$A_{33} = K_{11}K_{22} - K_{12}^2,$$
$$A_{13} = K_{12}K_{23} - K_{22}K_{13}.$$

Now we can cancel w^0 in the third equation in (15) since all functions are only of w^0 and find λ as

$$\lambda = f(k_1, k_2) = \frac{B_\varepsilon + B_\kappa}{B_t}, \quad (18)$$

where

$$B_\varepsilon = \frac{\Delta_k}{\Delta}\left(\frac{k_1^2}{R_2}+\frac{k_2^2}{R_1}\right)^2,$$
$$\Delta_k = K_{11}A_{11}+K_{12}A_{12}+K_{13}A_{13},$$
$$B_\kappa = D_{11}k_1^4+4D_{13}k_1^3k_2+2(D_{12}+2D_{33})k_1^2k_2^2+ 4D_{23}k_1k_2^3+D_{22}k_2^4,$$
$$B_t = t_1k_1^2+2t_3k_1k_2+t_2k_2^2.$$

Here we denote by Δ_k the determinant of the matrix $\{K_{ij}\}$. The variables B_κ, B_ε and B_t are proportional respectively to the bending-twisting shell energy Π_κ, the stretching-shear shell energy for additional displacements Π_ε, and the work of the initial momentless stress resultants on the additional rotations of the normal.

Since Π_κ and Π_ε are positive definite, the matrix $\{K_{ij}\}$ is also positive definite and therefore

$$\Delta_k > 0,\quad K_{ii} > 0,\quad A_{ii} > 0,\quad B_\kappa > 0,$$
$$\Delta > 0 \quad \text{for} \quad k_1^2+k_2^2 \neq 0.$$

Analysis of Expression for Critical Loading: Expression (18) is rather general. It may be used for estimation of the value of a critical loading and expected buckling mode in many problems. We obtain the critical value λ_0 for the parameter λ by minimizing the function $f(k_1,k_2)$ in all real k_1 and k_2, such that $B_t > 0$. Due to condition (16) such values of k_1 and k_2 exist.

Let $k_1 = r\cos\varphi,\ \ k_2 = r\sin\varphi.$. Taking into account that the functions in (18) are homogeneous in k_1 and k_2 we introduce

$$B_\varepsilon = B_\varepsilon^*(\varphi),\quad B_\kappa = r^4B_\kappa^*(\varphi),$$
$$B_t = r^2B_t^*(\varphi),\quad \Delta = r^4\Delta^*(\varphi).$$

Minimizing the function (18) we obtain

$$\lambda_0 = \min_\varphi\{f^*(\varphi)\} = f^*(\varphi_0),$$

$$f^*(\varphi) \equiv 2\frac{\sqrt{B_\varepsilon^*(\varphi)B_\kappa^*(\varphi)}}{B_t^*(\varphi)},\quad r_0^4 = \frac{B_\varepsilon^*(\varphi_0)}{B_\kappa^*(\varphi_0)}.$$

Due to (17), the pits are significantly inclined at angle $-\varphi_0$ to the axis x_2.

In fact, the algorithm described above may be applied only for shells of positive Gaussian curvature ($R_1R_2 > 0$). For shells of negative Gaussian curvature ($R_1R_2 < 0$), due to (18) for B_ε we get

$$\lambda_0 = \min_\varphi\{f^*(\varphi)\} = 0,$$
$$r_0 = 0 \quad \text{for} \quad \tan\varphi_0 = \sqrt{-R_1/R_2}. \qquad (19)$$

Similarly, for shells of zero Gaussian curvature ($R_1^{-1} = 0$), i.e. cylindrical and conical, we obtain from (18)

$$\lambda_0 = \min_\varphi\{f^*(\varphi)\} = 0,\ \ r_0 = 0 \ \ \text{for}\ \varphi_0 = 0. \quad (20)$$

Relations (19) and (20) mean that for shells of zero or negative Gaussian curvature the order of the critical loading ($\lambda_0 = 0$) decreases and the buckling mode is not localized ($r_0 = 0$). To obtain the critical loading and buckling modes for such shells one should apply the method of the asymptotic integration that is described below for a circular cylindrical shell as an example. The case of the axially compressed cylindrical shell $t_2 = t_3 = 0$, $t_1 > 0$, is the only one, when the application of relations (18) provides a nontrivial result.

Anisotropic Ellipsoid under External Uniform Pressure: As an example we consider an elliptical shell of revolution with the semi-axes *(a,a,b)*. The angle between the axis of symmetry and the normal to the surface is denoted as θ. We select the parameter $R=a$ as a characteristic length. Then for the principal curvatures

$$\rho_2 = R/R_2 = \sqrt{\sin^2\theta + d\cos^2\theta}, \qquad (21)$$
$$\rho_1 = R/R_1 = \rho_2^3/d^2, \quad d = b/a.$$

Here d is the coefficient of the ellipsoid compression.

The elliptical shell consists of the matrix made of the uniform material of thickness h, Young's modulus E and Poisson's ration ν The shell is reinforced with two similar systems of threads. The angles between the threads and the meridional direction are equal to $\pm\alpha$. The threads occupy the volume $(1-\delta_0)V$, where V is the entire volume of the structure; Young's modulus of the thread material is e times larger than E.

The elliptical shell is under uniform normal pressure λ. The relations for the initial stresses are well-known [1]:

$$t_1 = \frac{1}{2}\rho_2 \operatorname{sign}\lambda, \quad t_2 = \frac{2\rho_2 - \rho_1}{2\rho_2^2}\operatorname{sign}\lambda, \quad t_3 = 0.$$

For the external pressure sign $\lambda>0$, and for the internal pressure sign $\lambda<0$. Note that for the external pressure the buckling may occur due to (14) for elliptical shells of arbitrary form, whereas for the internal pressure only for such shells that satisfy the condition $2\rho_2 < \rho_1$, i.e.

$$\rho_2^2 > 2d^2. \qquad (22)$$

It follows from (21) that for ρ_2 the following relations hold

$1<\rho_2<d$, for $d>1$, and $d<\rho_2<1$, for $d<1$. (23)

The simultaneous inequalities (22) and (23) are satisfied only for $2d^2 < 1$.

For the system of threads described above the shell is orthotropic and the relation for λ has the following form

$$\lambda_0 = \min_{\varphi,\theta} f^*(\gamma,\theta),$$
$$f^*(\gamma,\theta) = \frac{2(B_\varepsilon^*(\varphi,\theta)B_\kappa^*(\varphi,\theta))^2}{B_t^*(\varphi,\theta)}, \qquad (24)$$
$$r_0^4 = \frac{B_\varepsilon^*(\varphi_0,\theta_0)}{B_\kappa^*(\varphi_0,\theta_0)},$$

where

$$B_\varepsilon^* = \frac{\Delta_k}{R^4\Delta}(\rho_2\cos^2\varphi + \rho_1\sin^2\varphi)^2$$

$\Delta_k = K_{11}A_{11}+K_{12}A_{12}$,
$B_k^* = D_{11}\cos^4\varphi + 2(D_{12}+2D_{33})\cos^2\varphi\sin^2\varphi + D_{22}\sin^4\varphi$,
$B_t^* = t_1\cos^2\varphi + t_2\sin^2\varphi$,
$\Delta = A_{22}\cos^4\varphi + (2A_{12}+A_{33})\cos^2\varphi\sin^2\varphi + A_{11}\sin^4\varphi$,
$A_{11} = K_{22}K_{33} - K_{23}^2$, $A_{12} = K_{13}K_{23} - K_{12}K_{33}$,
$A_{22} = K_{11}K_{33} - K_{13}^2$, $A_{33} = K_{11}K_{22} - K_{23}^2$.

We start with the analysis of the isotropic elliptical shell under external pressure ($\delta_0 = 0$). In this case relation (18) may be written as

$$\lambda_0 = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\min_{\varphi,\theta}\frac{\rho_2\cos^2\varphi + \rho_2\sin^2\varphi}{t_1\cos^2\varphi + t_2\sin^2\varphi}.$$

Minimising by φ the above expression we obtain

$$\lambda = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{\rho_1}{t_2}, \text{ for } \varphi_0=\pi/2 \text{ if } t_2\rho_2>t_1\rho_1.$$

$$\lambda = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{\rho_2}{t_1}, \text{ for } \varphi_0=0 \text{ if } t_2\rho_2<t_1\rho_1.$$

$$\lambda = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{\rho_1}{t_2} = \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{\rho_2}{t_1}$$

if $t_2\rho_2=t_2\rho_1$.

In the last case the angle φ is undefined. It means that there exist multiple buckling modes. At the same time the value of the buckling loading is unique.
For the case under consideration the condition $t_2\rho_2=t_1\rho_1$ may be written as

$\rho_1/(2\rho_2)=(2\rho_2-\rho_1)/(2\rho_2)$, or $\rho_1=\rho_2$, which corresponds to d=1, i.e. spherical shell. For d>1, $t_2\rho_2>t_1\rho_1$, and for d<1, $t_2\rho_2<t_1\rho_1$.

Therefore the relation for the critical loading is given as

$$\lambda_0=\min_\theta \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\,2\rho_2^2 \qquad \text{for } d\le 1,$$

$$\lambda_0=\min_\theta \frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{2\rho_1\rho_2^2}{2\rho_2-\rho_1} \quad \text{for } d\ge 1.$$

Now minimising by θ we obtain

$$\lambda_0=\frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\,2d^2 \qquad \text{for } d\le 1,$$

$$\lambda_0=\frac{Eh^2}{R^2\sqrt{3(1-\nu^2)}}\frac{2}{2d^2-1} \qquad \text{for } d\ge 1.$$

For d>1 the weakest parallel is on the equator $\theta_0=\pi/2$, and the pits are stretched in the direction of the meridian $\varphi_0=\pi/2$.

For d<1 the weakest parallel is the pole θ=0. Note, that in the last case the value of λ_0 does not depend on the angle φ and, therefore, angle φ_0 is undetermined.

Now we consider the orthotropic shell. Unlike the isotropic case for the orthotropic shell relation (18) cannot be simplified and one should seek the minimum of function (24) numerically. For that we fix the parameters α and δ_0 and find the minimum of the function

$$\lambda_0=\min_{\varphi,\theta} f^*(\varphi,\theta),$$

$$f^*(\varphi,\theta)= 4\sqrt{\frac{B_\kappa^*\Delta_k}{\Delta}}\,\frac{\rho_2^2 d^2\cos^2\varphi+\rho_2^4\sin^2\varphi}{d^2\cos^2\varphi+(2d^2-\rho_2^2)\sin^2\varphi}.$$

The domain of the parameters (φ, ρ_2) where we seek the minimum is:

$$\begin{aligned}&\varphi\in[0,\pi/2] \text{ and } \rho_2\in[1,d] \text{ for } d\ge 1,\\ &\varphi\in[0,\pi/2] \text{ and } \rho_2\in[d,1] \text{ for } \tfrac{\sqrt2}{2}\le d\le 1, \quad (25)\\ &\varphi\in[0,\varphi^*] \text{ and } \rho_2\in[d,1] \text{ for } d\le\tfrac{\sqrt2}{2},\end{aligned}$$

where

$$\varphi^*=\begin{cases}\dfrac{\pi}{2}, & \text{if } \rho_2^2<2d^2,\\ \mathrm{Atan}\dfrac{d}{\sqrt{\rho_2^2-2d^2}}, & \text{if } \rho_2^2>2d^2.\end{cases}$$

The numerical calculations revealed that the function attains its minimum at θ_0=0 for d <1, and at $\theta_0=\pi/2$ for d >1. This result does not depend on the values of the other parameters.

The effect of the parameter of the shell compression d on the relative critical loading is shown in Figure 1. The value of the critical loading for the isotropic sphere is assumed to be equal to 1. Poisson's coefficient is ν=0.3. Line 1 corresponds to the isotropic case δ_0=1, line 2 to the case δ_0=0.9, α=0, line 3 to the case δ_0=0.9, $\alpha=\pi/16$, line 4 to the case δ_0=0.9, $\alpha=\pi/8$, line 5 to the case δ_0=0.9, $\alpha=\pi/4$.

As it might be expected, increasing the threads' stiffness and their relative volumes leads to an increase of the critical loading.

The angle φ_0 depends on the values of the parameters d, α and δ_0. For large values of d the pits are stretched in the direction of the meridian, that angle φ_0 converges to $\pi/2$ as d increases and for highly prolate orthotropic elliptical shells the buckling modes are similar to those for the isotropic shells. The increase of the thread stiffness leads to a smaller angle φ_0

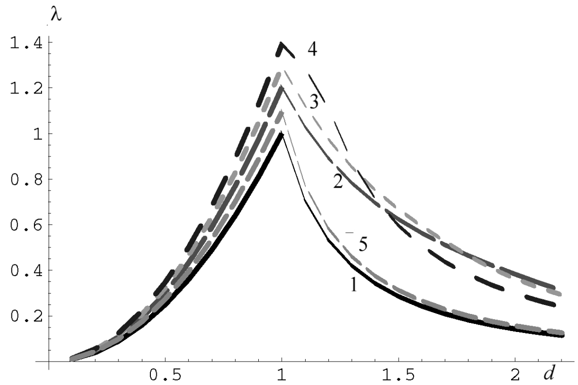

Figure 1 Critical loading under external pressure vs. shell compression d.

The dependence of the critical loading and buckling mode on the angle between the systems of threads is more complicated. Note that for the angles α larger than $\pi/4$ the critical loading and buckling modes are equal to the critical loading and modes for the isotropic elliptical shell. For the slightly prolate ellipsoid the critical loading attains its maximum for angles close to $\pi/$

For the oblate orthotropic elliptical shell ($d<1$) the value of φ_0 may be determined in an unique way from the conditions:

$$\varphi_0: \quad \lambda(\varphi_0, 0) = \min_{\varphi} \frac{B_\kappa^*}{\Delta},$$

from which it follows that

$$\varphi_0 = \pm \mathrm{Atan} \sqrt[4]{\frac{E_2}{E_1}}.$$

For small and large values of d the following approximate formulas may be used to obtain the critical loading:

$$\lambda_0 = \min_{\theta} \frac{4}{R^2} \sqrt{\frac{\Delta_k D_{11}}{A_{22}}} d^2 = h^2 \frac{2\sqrt{E_1 E_2}}{R^2 \sqrt{3(1-\nu_1\nu_2)}} d^2 \qquad \text{for } d \le 1,$$

$$\lambda_0 = \min_{\theta} \frac{4}{R^2} \sqrt{\frac{\Delta_k D_{11}}{A_{22}}} (2d^2 - 1) = h^2 \frac{2\sqrt{E_1 E_2}}{R^2 \sqrt{3(1-\nu_1\nu_2)}} (2d^2 - 1) \qquad \text{for } d \le 1.$$

Anisotropic Elliptical Shell under Internal Uniform Pressure: We start the consideration with the case of the isotropic shell (δ_0 =0). Since for the shell under internal normal pressure $t_1<0$, and $t_2>0$, then the inequality $t_2\rho_2>t_1\rho_1$ holds for any values of the parameter d. Therefore, we should seek the minimum of the function $\frac{2\rho_1\rho_2^2}{\rho_1 - 2\rho_2} = \frac{2\rho_2^4}{\rho_2^2 - 2d^2}$ under condition $2d^2 < \rho_2^2 < 1$. For $d<1/2$ the function attains its minimum at $\rho_2=2d$, i. e. on the parallel θ=arcsin($3d^2/(1-d^2)$) and this minimum is equal to $16d^2$. For $1/2<d<\sqrt{2}/2$ the minimum is reached at $\rho_2=1$, i.e. on the equator and it is equal to $2/(1-2d^2)$.

Hence, the pits are stretched in the direction of the meridian and they move away from the equator to the pole as d decreases.

Now we consider the orthotropic elliptical shell described in the previous section. Then, the relation for the critical loading may be written as

$$\lambda_0 = \min_{\varphi,\theta} f^*(\varphi, \theta),$$

$$f^*(\varphi,\theta) = 4\sqrt{\frac{B_\kappa^* \Delta_k}{\Delta}} \cdot \frac{\rho_2^2 d^2 \cos^2\varphi + \rho_2^4 \sin^2\varphi}{-d^2\cos^2\varphi - (2d^2 - \rho_2^2)\sin^2\varphi}$$

in the domain

$$\varphi \in \left[\mathrm{Atan}\frac{d}{\sqrt{\rho_2 - 2d^2}}, \frac{\pi}{2}\right], \quad \rho_2 \in [\sqrt{2}\,d, 1].$$

As before, we seek the minimum for all positive λ_0. We recall that the buckling of the elliptical shell under the internal pressure may occur only if $d<\sqrt{2}/2$.

In Figure 2 the dependence of the relative critical loading on the parameter d is plotted. The parameters of the shell are the same as in the previous section. It is assumed that the critical loading for the isotropic shell with d=1/2 is equal to 1.

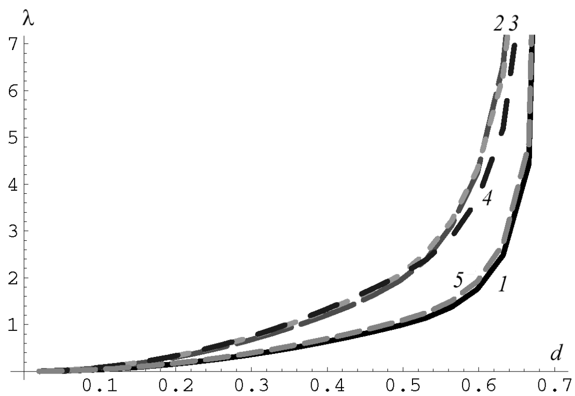

Figure 2 Critical loading under internal pressure vs. shell compression parameter *d*.

For the shell reinforced with threads the critical loading is higher than for the isotropic shell, and the weakest parallel is closer to the equator for $1/2<d<\sqrt{2}/2$. At the same time, the orientation of the pit axis φ_0 changes significantly.

The critical loading decreases as the angle α increases. For $\alpha>\pi/4$ the buckling modes and critical loadings of isotropic and orthotropic shells practically coincide.

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