On the Free Vibrations Spectrum of Thin Shells

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Abstract: The free vibration spectrum of thin shells is discussed. A shell model based on the Kirchhoff-Love hypotheses is used. The free vibrations of the shell are described by a singularly perturbed boundary value problem. The peculiarities of the spectrum of this problem are analyzed. All results are obtained by means of asymptotic methods, where the relative shell thickness is considered as the main small parameter. Both classical results, such as the classification of the vibration modes, the spectrum density, the turning points and lines, etc., as well as new results, such as the asymptotically multiple frequencies and the localization of the vibration modes near the weakly supported edge are discussed.

Résumé: Le spectre des vibrations libres des coques minces sont discutée. Le modéle présenté s'appuie sur la théorie des coques minces de Kirchhoff-Love. Les vibrations libres des coques sont décrites par perturbation d'une problème aux limites. L'analyse présentée se concentre sur le spectre des vibrations libres. Tous les résultats sont obtetu par l'application des méthodes assymptotiques, dans lesquelles l'épaisseur relative est le petit parmètre principal. On discute les résultats classiques, comme la classification de forme de la vibration, la densite de spectre, les points et les lignes de tournant etc., ainsi que les résultats nouvaux, tels comme les fréquences assymptotiques multiples et la localisation des modes de vibrations au voisinage des extrèmités faiblement fixée.

1. Variational Principle and Boundary Value Problem: We consider a thin elastic shell of small constant thickness h made of a homogeneous isotropic elastic material. We analyze the behavior of the shell by employing a 2D model of the Kirchhoff–Love type. We introduce the orthogonal curvilinear coordinates $x = \{x_1, x_2\} \in \Omega \subset \mathbf{R}^2$ on the shell midsurface, and the unit vectors

$$\boldsymbol{e}_j = \frac{1}{A_j}\frac{\partial \boldsymbol{r}}{\partial x_j}, \quad j = 1, 2, \qquad \boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2, \tag{1.1}$$

where $\boldsymbol{r} \in \mathbf{R}^3$. The displacement vector is $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{n}$ (below we will denote $u_3 = w$). The natural frequencies ω_k and the corresponding eigenfunctions may be obtained from the following variational principle expressed in its dimensionless form as (see [1, 2, 3])

$$\lambda = \frac{\rho\,\omega^2 R^2}{E} = \min_{u_j} \frac{(1-\nu^2)^{-1}\Pi_s + \mu^4 \Pi_b}{T}. \tag{1.2}$$

Here Π_s and Π_b are proportional to the stretching-shear and bending-twisting potential energy of the shell, respectively, and T is proportional to the kinetic energy of the shell

$$\begin{aligned}
\Pi_s &= \int_\Omega \left(\varepsilon_{11}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}^2 + (1-\nu)\varepsilon_{12}^2/2\right)\, d\Omega,\\
\Pi_b &= \int_\Omega \left(\varkappa_{11}^2 + 2\nu\varkappa_{11}\varkappa_{22} + \varkappa_{22}^2 + 2(1-\nu)\varkappa_{12}^2\right)\, d\Omega,\\
T &= \int_\Omega \left(u_1^2 + u_2^2 + u_3^2\right)\, d\Omega.
\end{aligned} \tag{1.3}$$

All linear variables are related to the characteristic length R of the midsurface, $h_* = h/R$ is the small relative shell thickness, λ is the unknown frequency parameter, E is Young's modulus, ν is Poisson's ratio, and ρ is the shell density; $\mu = h_*^{1/2}(1-\nu^2)^{-1/4}$ is a small parameter; ε_{ij} and $\varkappa_{ij}$ are the linearized tensors of membrane and bending

midsurface deformations. Their expressions through u_j are given in [1, 3, 4, 5].
The displacements u_j in relation (1.2) satisfy the constraints at the shell edges $\partial\Omega$. If the edge is clamped then four conditions are to be fulfilled

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad \gamma = 0, \quad (1.4)$$

where γ is the angle of rotation about the tangent to the edge. For other types of boundary conditions some of the constraints (1.4) may be skipped. For the free edge no constraints are imposed on functions u_j. If the number of constraints is less than four then the linear combinations of conditions (1.4) may be given.
The following boundary value problem corresponds to the variational problem (1.2)

$$\big(\boldsymbol{L}(x_1,\, x_2) + \mu^4 \boldsymbol{N}(x_1,\, x_2)\big)\, \boldsymbol{u} + \lambda \boldsymbol{u} = \boldsymbol{0}, \quad (1.5)$$

where

$$\boldsymbol{L} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}, \quad \boldsymbol{N} = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix}$$

are the linear differential operators. Their explicit forms are given in [5, 6].
System (1.5) is of 8th order, therefore, four boundary conditions are to be imposed on each shell edge. If the number of the given constraints on an edge is less than four we should complete them with the natural boundary conditions for the variational problem (1.2). Then, problem (1.5) becomes the self-conjugate one. Hence, the boundary conditions may be separated into two groups: the geometrical boundary conditions of type (1.4) and the natural boundary conditions for problem (1.2).
Relations (1.2) and (1.5) were first derived by using the Kirchhoff–Love hypotheses when the 3D problems were considered. One may get the same problems considering the first approximation of the asymptotic expansions in powers of the small parameter h_* (see for example [3]). One can also directly postulate the 2D problems without any connection with the 3D ones [7].

2. General Properties of Free Vibration Spectrum of Shell: The operator $\boldsymbol{L} + \mu^4 \boldsymbol{N}$ is elliptic and positive so problem (1.5) has the discrete spectrum with the single point of accumulation at infinity.
That is a singularly perturbed problem. If we assume $\mu = 0$ (which corresponds to membrane-like shell vibrations), system (1.5) is simplified to the 4th order system

$$\boldsymbol{L}\boldsymbol{u} + \lambda \boldsymbol{u} = 0. \quad (2.1)$$

Imposing two (tangential) boundary conditions on the shell edges we again obtain the self-conjugate boundary value problem. System (2.1) is not of elliptic type. Its type may depend on the point x and on the parameter λ. As a consequence its spectrum is much more complex than the spectrum of problem (1.5). It may have finite points of accumulation of discrete spectrum and intervals of continuous spectrum [1, 6].

Sometimes the difference $\omega_{k+1} - \omega_k$ between the neighboring natural frequencies is very small, so, the density $p(\omega)$ of the shell natural frequencies distribution may be introduced by relation

$$n(\omega,\, \omega + \Delta\omega) \simeq p(\omega)\Delta\omega \quad (2.2)$$

where $n(\omega,\, \omega + \Delta\omega)$ is the number of frequencies contained in interval $[\omega,\, \omega + \Delta\omega)$. The density $p(\omega)$ was first found in [8] for shallow shells for which the curvature radii R_1 and R_2 are (approximately) constant.

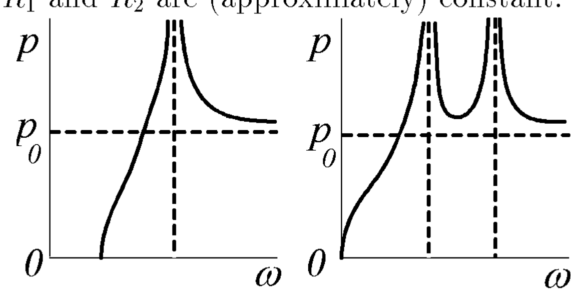

Fig. 1. Density $p(\omega)$.

In Fig. 1 the density $p(\omega)$ for the positive (left) and for the negative Gaussian curvature is presented. The density is not constant, it has one or two points of concentration. The density $p(\omega) \to p_0$ if $\omega \to \infty$. The value p_0 is equal to the constant density of Kirchhoff's transverse vibrations of a plate

$$p_0 = \frac{1}{4\pi}\sqrt{\frac{\Omega m}{D}}, \quad D = \frac{Eh^3}{12(1-\nu^2)}. \quad (2.3)$$

Here Ω is the midsurface area and m is the shell mass. This formula was obtained by Courant.

Let us introduce the function $n(\lambda)$ which is equal to the number of the eigenvalues λ_k of problem (1.5) for which $\lambda_k \leq \lambda$. The following expression for $n(\lambda)$ as $h_* \to 0$ was reported in [6] (see also [9]) for the curvilinear coordinates coinciding with the lines of the curvatures

$$n(\lambda) = \frac{\sqrt{3(1-\nu^2)}}{4\pi h_*}\left[\int_\Omega \int_0^{2\pi} \Re\sqrt{\lambda - F}\, d\theta\, d\Omega + O(h_*^\gamma)\right], \quad (2.4)$$

$$F = F(x,\theta) = \left(R_1^{-1}(x)\sin^2\theta + R_2^{-1}(x)\sin^2\theta\right)^2. \quad (2.5)$$

Here $R_j(x)$ are the main radii of the midsurface curvature. The residual term $O(h_*^\gamma)$, $\gamma > 0$, depends on the boundary conditions at the edges $\partial\Omega$. For the general case relation (2.4) for $\gamma = 1/5$ has been proved.

If we assume that the radii $R_j(x)$ are constant then the derivative $n(\lambda)$ with respect to ω provides the spectral density obtained in [8] and shown in Fig. 1.

3. Classification of Vibration Modes:

Unlike the spectra of elastic bodies of simple geometrical form such as strings, beams, membranes, and plates, the shell spectrum is essentially more sophisticated. To study the shell spectrum one should use a classification based on the asymptotic properties of the vibration modes. To classify the vibration modes it is convenient to use the index of variation p, $0 \leq p < 1$

$$\max\left\{\left|\frac{\partial w}{\partial x}\right|, \left|\frac{\partial w}{\partial y}\right|\right\} \sim h_*^{-p}|w|, \quad l \sim h_*^p, \quad (3.1)$$

introduced by Goldenweiser [5]. Due to (3.1) the index p is related to the characteristic length l of the picture of deformation.

There are four main types of vibration modes, which are characterized by the following asymptotic relations:

(i) the quasi-tangential vibrations
$|w| \ll |u|$, $0 < p < 1$, $r = p$;
(ii) the Rayleigh type vibrations
$|w| \gg |u|$, $0 < p < 1/2$, $r = -1 + 2p$;
(iii) the quasi-transversal vibrations with small index of variation
$|w| \gg |u|$, $0 < p < 1/2$, $r = 0$;
(iv) the quasi-transversal vibrations with large index of variation
$|w| \gg |u|$, $1/2 < p < 1$, $r = -1 + 2p$.

Here $|u| = \max\{|u_1|, |u_2|\}$ and parameter r denotes the asymptotic order of the natural frequency $\omega \sim h_*^{-r}$.

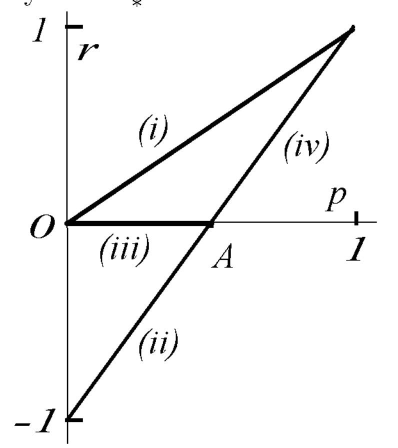

Fig. 2. Index of variation p vs. frequency order r

In Fig. 2 the dependence of the index of variation p on the frequency order r for four types of vibration is shown.

For each type of vibrations problem (1.5) may be simplified. For type (i) we may assume $\mu = 0$ and neglect the transversal inertia, i. e. the term λw. For the other types we may neglect the tangential inertia, i. e. λu_1 and λu_2. For type (iii) we may assume $\mu = 0$. If $p = 0$ all components of the displacements u_j, and w have equal orders and all inertia terms are essential.

At point A in Fig. 1, where $p = 1/2$ and $r = 0$, the three types of vibration modes coincide. In the neighborhood of this point system (1.5) may be essentially simplified and rewritten in the Donnell form

$$\begin{aligned} &\mu^4 \Delta\Delta w + \mu^2 \Delta_R \Phi - \lambda w = 0, \\ &\mu^4 \Delta\Delta \Phi - \mu^2 \Delta_R w = 0, \end{aligned} \tag{3.2}$$

where the differential operators are

$$\Delta_R w = \frac{1}{A_1 A_2} \left[\frac{\partial}{\partial x_1} \left(\frac{A_2}{R_2 A_1} \frac{\partial w}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{A_1}{R_1 A_2} \frac{\partial w}{\partial x_2} \right) \right] \tag{3.3}$$

and Δ is the Laplace operator given by relation (3.3) for $R_1 = R_2 = 1$. Here Φ is the unknown stress function.

It is interesting to note that only vibration modes of type (iv) contribute to the spectral density $p(\omega)$. The number of natural frequencies of the other types is asymptotically smaller compared to the number for the (iv) type.

For $p \geq 1$ the 2D model is inapplicable.

4. Lower Part of Spectrum and Midsurface Bending: Sometimes it is necessary to know the lowest natural frequency of a structure. Only for the Rayleigh type vibrations $r < 0$ and the frequencies $\omega \to 0$ as $h_* \to 0$. Such vibrations are studied in [1, 10].

As it follows from relation (1.2), the natural frequencies of the lowest order

$$\lambda \sim \mu^4 \quad \text{or} \quad \omega \sim h_* \tag{4.1}$$

exist if there exist displacements $u_j(x)$ for which $\Pi_s = 0$ or

$$\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{22} = 0 \tag{4.2}$$

and which satisfy all imposed constraints of type (1.4). Such displacements are called pure bending and for them λ may be obtained from Rayleigh's formula

$$\lambda = \mu^4 \frac{\Pi_b}{T}. \tag{4.3}$$

The existence of the non-zero solutions of equations (4.2) depends on the boundary conditions. In [1] many cases were revealed for which the pure bending does not exist but there exist displacements for which the value Π_s is small.

For example, if there is a constraint $\gamma = 0$ among the others, and the pure bending exists without and does not exist with this constraint, then $\lambda \sim \mu^3$. An other example is the case, when there are constraints $w = 0$ and/or $\gamma = 0$ among the others, and the pure bending exists without and does not exist with these constraints, then $\lambda \sim \mu$.

In these two cases the constraints that are ignored may be satisfied with functions that exponentially decrease away from the edge.

If the shell has a positive Gaussian curvature and part of its edge is free or weakly supported then again $\lambda \to 0$ when $h_* \to 0$. The console shells of revolution with weakly supported parallel edges are studied in [11, 12]. In these cases the vibration modes are localized near the weakly supported edge.

Let us return to Fig. 1. One can see in the left side of Fig. 1 that for the shell of positive Gaussian curvature the non-zero spectral density begins with the frequency $\omega_0 > 0$. Therefore, the frequency interval $0 \leq \omega \leq \omega_0$ contains a comparatively small number of natural frequencies. For the shell of negative Gaussian curvature the non-zero spectral density begins with $\omega = 0$ (in the right side of Fig. 1). Hence there exist frequencies such that $\omega \to 0$ as $h_* \to 0$.

Let us analyze the shell with negative or zero Gaussian curvature. One can easily construct the deflection in the form (see [1])

$$w(x_1, x_2) = w_0(x_1, x_2) \sin\left(h_*^{-p} f(x_1, x_2)\right) \tag{4.4}$$

and the corresponding functions $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ such that the tangential deformations $\varepsilon_(ij)$ and therefore the potential energy Π_s converge to 0 as $h_0 \to 0$. The function w_0 is introduced to satisfy the arbitrary constraints at the edge $\partial\Omega$. Such displacement is called pseudo-bending.

For the negative Gaussian curvature

$$p=1/3, \quad \lambda_{min}=O\left(\mu^{4/3}\right), \; \omega_{min}=O\left(h_*^{1/3}\right) \tag{4.5}$$

and for the zero Gaussian curvature (for the cylindrical and the conic shells)

$$p=1/4, \quad \lambda_{min}=O\left(\mu^2\right), \; \omega_{min}=O\left(h_*^{1/2}\right). \tag{4.6}$$

These upper estimates do not depend on the boundary conditions. If the shell has weakly supported edges then the value of ω_{min} may be smaller.

5. On Asymptotic Solutions: In the general case the construction of the asymptotic solution for the singularly perturbed problem (1.5) is very complex (see [4, 13]). We consider, firstly, the shells of revolution. For these shells the variables are separated as

$$w(x_1, x_2) = w(s)\cos m\varphi, \quad m = 0, 1, 2, \dots \tag{5.1}$$

and problem (1.5) is reduced to a one-dimensional one, containing two main parameters: the small thickness parameter μ and the waves number in the circumferential direction m. As curvilinear coordinates we use the generatrix length s and the angle φ in the circumferential direction. The form of the asymptotic solution essentially depends on the value m. We discuss the axisymmetric vibrations ($m = 0$) and the cases with large value m.

6. Axisymmetric Vibrations: In this case system (1.5) may be reduced to an equation of the 6th order

$$-\mu^4 \sum_{k=1}^{6} a_k(s)\frac{d^k w}{ds^k} + \sum_{k=1}^{2} b_k(s)\frac{d^k w}{ds^k} = 0. \tag{6.1}$$

All coefficients a_k and b_k are assumed to be regular for $s_1 \le s \le s_2$ and $a_6(s) = 1$,

$$b_2(s) = \lambda - R_2^{-2}(s). \tag{6.2}$$

Let us denote

$$\Lambda^- = \min_{s_1\le s\le s_2}\{R_2^{-2}(s)\}, \quad \Lambda^+ = \max_{s_1\le s\le s_2}\{R_2^{-2}(s)\}. \tag{6.3}$$

In the intervals of s, where $b_2(s) \neq 0$, equation (6.1) has four solutions with the large index of variation $p = 1/2$. Formal asymptotic expansions of these solutions are

$$w_n(s,\mu) = \sum_{k=0}^{\infty}\mu^k A_{kn}(s)\exp\left(\frac{1}{\mu}\int q_n(s)\,ds\right) \tag{6.4}$$

where for $n = 1, 2, 3, 4$ functions $A_{kn}(x)$ are regular and

$$A_{0n} = B^{-1/2} b_2^{-3/8}, \quad q_n = b_2^{1/4} e^{n\pi i/2}. \tag{6.5}$$

Here $B(s)$ is the distance between the axis of rotation and the point s on the midsurface.

The other two solutions have asymptotic expansions in powers of μ^4

$$w_n(s) = \sum_{k=0}^{\infty}\mu^{4k} w_{kn}(s), \quad n = 5, 6 \tag{6.6}$$

where w_{05} and w_{06} are the solutions of the membrane equation

$$b_2\frac{d^2 w}{ds^2} + b_1\frac{dw}{ds} + b_0 w = 0. \tag{6.7}$$

asymptotic expansions (6.4) and (6.6) are valid near the turning points x_* where $b_2(s_*) = 0$, since $A_{0n}(s) \to \infty$ as $s \to s_*$ and equation (6.6) has the singular point $s = s_*$. Turning points appear in the frequency interval $\lambda \in [\Lambda^-, \Lambda^+]$.

For the conic shell of revolution the asymptotic representation of the integrals near the turning point was first obtained in [14]. In [15] the general case for the simple turning point ($b_2'(s_*) \neq 0$) is studied. Near the turning point the solution $w_6(x,\mu)$ has the expansion (6.6) and the other 5 solutions have the following asymptotic expansions ($n = \overline{1,5}$)

$$w^{(n)}(s,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon A_k(s) v_k^{(n)}(\eta) + \varepsilon\delta_{n5} w^*(s,\mu), \tag{6.8}$$

where δ_{n5} is the Kroneker symbol,

$$\varepsilon = \mu^{4/5}, \quad \eta(s) = \frac{1}{\varepsilon}\left(\frac{5}{4}\int_{s_*}^{s} b_2^{1/4}(s)\,ds\right)^{4/5}. \tag{6.9}$$

The standard functions v_0^n satisfy equation

$$\frac{d^5 v}{d\eta^5} - \eta\frac{dv}{d\eta} - v = 0 \tag{6.10}$$

and for the functions v_k^n

$$v_{k+1}^{(n)}(\eta) = \int v_k^{(n)}(\eta)\,d\eta, \quad k = 0, 1, \ldots \tag{6.11}$$

All functions A_k, η, w^*, and w_6 are regular at $s = s_*$. Series (6.8) are not asymptotic any more for $|s - s_*| \sim 1$. Hence one must find the relations between solutions (6.8) in the neighborhood of the turning point and solutions (6.4) and (6.6). These relations are given in [1, 14, 15]. Unfortunately, the uniform asymptotic expansion that is valid in the entire interval $[s_1, s_2]$ is not found. It seems that such expansion does not exist.

Now we are ready to study the spectrum of axisymmetrical vibrations. If $\lambda < \Lambda^-$ then membrane solutions (6.6) are the main. Solutions (6.4) decrease exponentially away from the ends of the interval $[s_1, s_2]$ and they help satisfy the non-tangential boundary conditions. Such singular degeneracy is termed in [16] as "regular".

If $\lambda > \Lambda^+$ the frequency equation has the form (for the clamped edges)

$$\Delta_0(\lambda)\cos z + O(\mu) = 0, \quad z = \frac{1}{\mu}\int_{s_1}^{s_2} b_2^{1/4}\,ds. \tag{6.12}$$

Here two types of eigenfunctions appear. The first of them are the membrane modes ($\Delta_0(\lambda) = 0$). The others are the bending modes ($\cos z = 0$). They are described by solutions (6.4) that oscillate fast in s. We note that the density of the bending axisymmetric frequencies is μ^{-1} times larger than the density of the membrane frequencies.

In the frequency interval $\Lambda^- < \lambda < \Lambda^+$ the eigenfunction $w(s)$ consists of two parts (see Fig. 3). In the part of the interval where $b_2(s) < 0$ the function $w(s)$ changes slowly, whereas in the part where $b_2(s) > 0$ it oscillates fast. The shell has the maximal deflection near the turning point s_*.

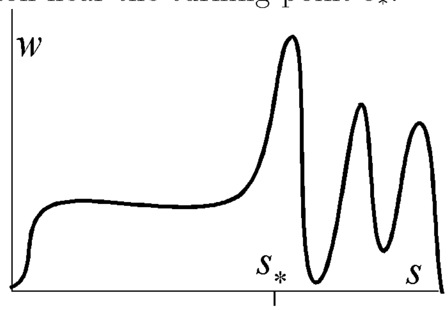

Fig. 3. Mode with turning point.

Qualitatively, the non-symmetric vibrations with fixed $m \sim 1$ do not differ essentially from the axisymmetrical case. The bending solutions (6.4) are the same and there are four membrane solutions. Near the turning point $s = s_*$ relations (6.8) hold and there are three regular membrane solutions of type (6.6).

7. Non-symmetric Vibrations with Large $\boldsymbol{m}$: Let us assume that the number of waves in circumferential direction m is large and $m \sim \mu^{-1}$. We assume that $\rho = \mu m$, then system (1.5) may be reduced to the standard form

$$\mu\frac{d\boldsymbol{y}}{ds} = \boldsymbol{A}(s,\mu)\boldsymbol{y}, \quad \boldsymbol{A}(s,\mu) = \sum_{k=0}^{\infty}\boldsymbol{A}_k(s)\mu^k \tag{7.1}$$

where $\boldsymbol{A}$ is an 8×8 matrix.

In the general case the solutions of system (7.1) may be represented in the form of asymptotic series

$$\boldsymbol{w}^{(n)}(x,\mu) = \sum_{k=0}^{\infty}\mu^k \boldsymbol{w}_k(x)\exp\left(\frac{1}{\mu}\int i q_n(s)\,ds\right) \tag{7.2}$$

where $q_n(s)$ satisfies the algebraic equation of the 4th order with respect to q^2

$$\det(\boldsymbol{A}(s) - iq\boldsymbol{E}) = 0. \tag{7.3}$$

We consider the case when all 8 roots of equation (7.3) are simple and there exist only one pair of real roots $\pm q_1(s)$. Then eigenvalues λ may be found from the frequency equation

$$\tan z(\lambda, \mu) = d, \quad z = \frac{1}{\mu}\int_{s_1}^{s_2} q_1(s)\, ds \quad (7.4)$$

where $d = d(\lambda, \mu)$ is a slowly varying function of λ, which depends on the boundary conditions at $s = s_1$ and $s = s_2$.
The lowest part of spectrum for a fixed m corresponds to the case when there exists a turning point $s = s_*$ at which two roots coincide $q_1(s_*) = q_2(s_*)$. Let for $s < s_*$ equation (7.3) not have the real roots and for $s > s_*$ let it have two real roots $\pm q_1(s)$. Then, the frequency equation has the form (7.3) but the integral is calculated in the interval $[s_*, s_2]$ and function d depends only on the boundary conditions at $s = s_2$. The eigenfunction is shown in Fig. 4.

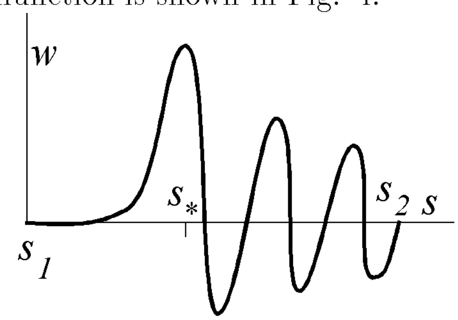

Fig. 4. The mode with turning point.

The parallel $s = s_*$ divides the shell surface into two parts. One of them is intensively vibrating, while in the other part there are practically no vibrations.
We study vibrations of the prolate ellipsoid of revolution. For the frequency set containing the minimal frequency there exist two turning points $s_*^{(1)}$ and $s_*^{(2)}$, and the vibration mode concentrates between these points near the shell equator. In this case the frequency equation is (see [1, 4, 13])

$$\int_{s_*^{(1)}}^{s_*^{(2)}} q_1(s)\, ds = \mu\pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots \quad (7.5)$$

where q_1 is the real root of equation (7.3).

8. Asymptotic Separation of Variables: For cylindrical and conic shells of revolution limited by two parallels the separation of variables (5.10) may be performed. For the shell of zero Gaussian curvature, if it is non-circular and/or its edges are slanted the exact separation of variables is impossible. But for low-frequency vibrations of such shells the variables may be asymptotically separated [17]. For simplicity, we consider here only the cylindrical shell and seek the low frequencies (see Fig. 5).

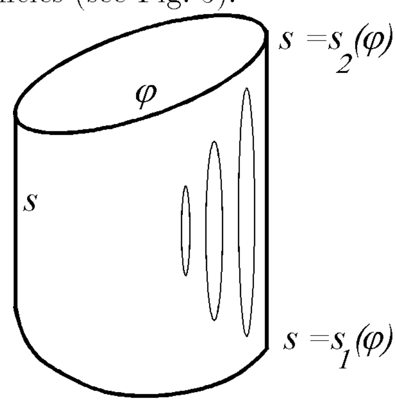

Fig. 5. Localized mode.

Due to estimations (4.6) we rewrite system (3.2) for the cylindrical shell, for which $1/R_1 = 0$, and $R_2 = R_2(\varphi)$, in the form

$$\begin{aligned} &\mu_* e^4 \Delta\Delta w + \frac{1}{R_2}\frac{\partial^2 \Phi}{\partial s^2} - \lambda_* w = 0, \\ &\mu_*^4 \Delta\Delta\Phi - \frac{1}{R_2}\frac{\partial^2 w}{\partial s^2} = 0, \end{aligned} \quad (8.1)$$

where

$$\Delta w = \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \varphi^2}, \quad \lambda = \mu^2\lambda_*, \quad \mu_*^2 = \mu. \quad (8.2)$$

Here λ_* is the new frequency parameter and μ_* is the new small parameter. Boundary conditions are imposed at the edges $s = s_k(\varphi)$, $k = 1, 2$.
The vibration modes are stretched in the longitudinal direction (Fig. 5). In the general case there exist modes localized near the generatrix $\varphi = \varphi_*$, which are called the weakest. Such modes may be represented in

the form

$$w(s,\varphi,\mu_*)=\sum_{n=0}^{\infty}\mu_*^{\frac{n}{2}}w_n(s,\xi)e^{iz},\ \ \xi=\frac{\varphi-f_*}{\sqrt{\mu_*}},$$
$$z=\mu_*^{-1/2}q\xi+(1/2)a\xi^2,\quad q>0,\ \ \Im a>0,$$
$$\lambda_*=\lambda_0+\mu_*\lambda_1+\mu_*^2\lambda_2+\dots, \tag{8.3}$$

where $w_n(s,\xi)$ are the polynomials in ξ with the coefficients depending on s and satisfying the one-dimensional boundary value problems.

In particular, $w_0(s,\xi)=H_m(\xi)W_0(s)$, where $H_m(\xi)$ is a Hermit polynomial, and $W_0(x)$ satisfies the equation

$$\frac{1}{R_2^2(\varphi)}\frac{d^4W_0}{ds^4}+(q^8-\Lambda q^4)W_0=0 \tag{8.4}$$

and two boundary conditions at $s=s_k(\varphi)$. The problem of choosing two boundary conditions from the four given conditions is discussed in [18].

The first two values λ_0 and λ_1 in expansion (8.3) for λ_* may be expressed through the eigenvalue $\Lambda(\varphi,q)$ of problem (8.4)

$$\lambda_0=\min_{y,q}\Lambda(y,q),\quad \lambda_1=\left(m+\frac{1}{2}\right)\sqrt{\Lambda_{\varphi\varphi}\Lambda_{ss}-\Lambda_{\varphi s}^2} \tag{8.5}$$

where $\Lambda_{\varphi\varphi}$, Λ_{ss}, and $\Lambda_{\varphi s}$ are the partial derivatives.

The eigenvalues λ_* are asymptotically double. It means, that there exist two real eigenfunctions $C_1\,\mathrm{Re}\,w+C_2\,\mathrm{Im}\,w$ with the constants C_1 and C_2 and two corresponding eigenvalues $\lambda_*^{(1)}$ and $\lambda_*^{(2)}$ which have the same expansion (8.3) and

$$|\,\lambda_*^{(1)}-\lambda_*^{(2)}\,|=O\left(e^{-c/\mu_*}\right),\quad c>0. \tag{8.6}$$

As an example, the low-frequency vibrations of a thin elliptical cylinder are studied [19]. The vibration modes are localized near the generatrix with the largest radius of curvature $R_2(\varphi)$. Since two such generatrices exist, each natural frequency is fourfold. The vibration modes are even or odd with respect to the ellipse diameters (see Fig. 5 where these four modes are schematically plotted).

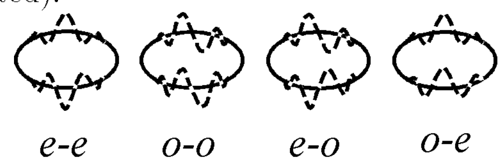

Fig. 6. Eigenfunctions scheme.

Using the symmetrical property of the problem one can calculate numerically these frequencies and examine the peculiarities of the fourfold frequencies. It was revealed that for some set of parameters the frequencies differ in the 5th decimal digits. Moreover, frequencies "e-e" and "o-o" (and also "e-o" and "o-e") differ in 10th decimal digits.

9. Rectangular Panel Vibrations: As a second example of asymptotic separation of variables we consider the low-frequency vibrations of a rectangular cylindrical panel [20] (see Fig. 7, right side).

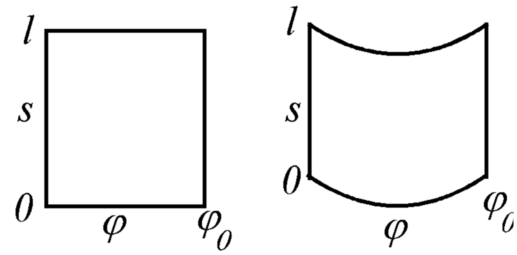

Fig. 7. Plate and cylindrical panel.

We study the arbitrary boundary conditions at the curvilinear edges and assume that one of the rectilinear edges ($\varphi=0$) is free or weakly supported, so there exists a vibration mode localized near this edge. Under these assumptions for the lowest eigenvalue λ_* the following asymptotic relation holds

$$\lambda_*=\alpha^2\left(\lambda_0+\mu_*\lambda_2+O(\mu_*^3)\right). \tag{9.1}$$

Here α is the eigenvalue of the beam vibrations equation

$$X^{IV}-\alpha^4X=0 \tag{9.2}$$

with two so-called main boundary conditions (see [18]) at the curvilinear edges.

The value λ_0 is the eigenvalue of equation

$$\frac{d^8Y}{d\eta^8} - 2\lambda_0 \frac{d^4Y}{d\eta^4} + Y = 0, \quad \eta = \frac{\varphi}{\mu_*} \quad (9.3)$$

with four given boundary conditions at the rectilinear edge $\eta = 0$ and the condition of decreasing $Y \to 0$ as $\eta \to \infty$.

constraints	λ_0
free edge	0.113 and 0.973
$u_1 = 0$ or $\gamma_1 = 0$	0.223
$u_1 = 0$ and $\gamma_1 = 0$	0.419
$u_2 = 0$ or $w = 0$	0.809

(9.4)

The non-zero solution of problem (9.3) exists if the shell edge is free ($\varphi = 0$), or the boundary conditions have the form of any of the 5 types given by (9.4) with the corresponding eigenvalues λ_0. For the free edge two eigenvalues exist.

All boundary conditions on the curvilinear and the rectilinear edges affect parameter λ_2 in expansion (9.1).

10. Vibrations of Rotating Shells of Revolution: In Section 3 we mentioned that for $m \geq 1$ the shell of revolution has double frequencies. These frequencies split due to the rotation of the shell.

We consider the vibrations of the shell of revolution that rotates with the constant angular velocity about the axis of symmetry. Instead of (5.1) we seek in this case the partial solution in the form ($m = 0, 1, 2 \ldots$)

$$w(s, \varphi, t) = w \cos(m\varphi - \omega t + \alpha). \quad (10.1)$$

Then, we obtain the system of equations

$$\sum_{j=1}^{3} \left(\frac{h^2}{12} h_{ij} + l_{ij}\right) u_j = \sum_{j=1}^{3} \left(\omega^2 \delta_{ij} + 2\omega\Omega c_{ij} + \Omega^2 e_{ij}\right) u_j \quad (10.2)$$

where the operators n_{ij} and l_{ij} may be obtained from operators in (1.5), after substitution (10.1). Here δ_{ij} is the Kroneker symbol, the term with c_{ij} takes into account the Coriolis forces, and $c_{12} = c_{21} = \cos\theta$, $c_{23} = c_{32} = -\sin\theta$, the other $c_{ij} = 0$, θ is the angle between the axis of symmetry and the normal to the shell. The last term in (10.2) takes into account the centrifugal forces and the initial asisymmetric stresses in the shell. The specific form of this term may be found in [21].

Let $\omega_k^{(m)}$ be the mth spectrum of the rotating shell. Here $k = \pm 1, \pm 2, \ldots$ since $\omega_k^{(m)}$ may be either positive or negative. For the non-rotating shells $\omega_{-k} = \omega_k$ and one may consider only the positive ω_k. We consider the forced vibrations under an external harmonic force, that is fixed with respect to the shell and has the frequency $\nu > 0$. The resonance conditions is

$$\nu = \pm\omega_k^{(m)} \quad (10.3)$$

if the force is non-orthogonal to the corresponding vibration mode.

The parameters Ω and ν are given in the non-dimensional form similar to (1.2) For the shells made of metals $\Omega \ll 1$. For $\Omega \sim 1$ the initial deformation ε_2^0 caused by the centrifugal forces has the order of Ω^2 and the shell is destroyed. Since Ω is small we can use the perturbation method considering the non-rotating shell as the unperturbed case. For $m \geq 1$ we get

$$\omega_\pm^{(m)} = \pm\alpha_k^{(m)}(\Omega) + \Omega\beta_k^{(m)}(\Omega), \quad k > 0$$

$$\alpha_k^{(m)} = \omega_{k,0}^{(m)} + \Omega^2\alpha_{k,2}^{(m)} + \cdots,$$
$$\beta_k^{(m)} = \beta_{k,0}^{(m)} + \Omega^2\beta_{k,2}^{(m)} + \cdots,$$

where $\alpha_k^{(m)}$ and $\beta_k^{(m)}$ are the even functions of Ω; $\omega_{k,0}^{(m)}$ are the frequencies of the non-rotating shell. Let u_j be the vibration mode for the non-rotating shell corresponding to $\omega_{k,0}^{(m)}$. Then

$$\beta_{k,0}^{(m)} = \frac{2\int_{s_1}^{s_2} u_2(u_1\cos\theta - w\sin\theta)B\,ds}{\int_{s_1}^{s_2} (u_1^2 + u_2^2 + w^2)\,B\,ds}$$

Due to (10.3) the resonance frequencies for small Ω form the couples of the close frequencies, the difference between which is

$2\Omega\beta_k^{(m)}$ [21].
For $m = 0$ the rotation of the shell leads only to the shift of the frequencies, but does not split them.

11. Conclusion: The knowledge of the qualitative properties of shell vibration modes and the approximate asymptotic formulas help us avoid the large errors that may occur when applying numerical analysis.

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