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MECHANICS

On the Frequency Spectrum of Free Vibrations of Membranes and Plates in Contact with a Fluid

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Abstract—A parallelepiped-shaped container, which is completely filled with a perfect incompressible fluid, is considered. The container is covered with an elastic lid, which is modeled by a membrane or a constant-thickness plate. The other faces of the container are nondeformable. The frequency spectrum of small free vibrations of the lid has been obtained taking into account the apparent mass of the fluid the movement of which is assumed to be potential. The main specific feature of the problem formulation is that the volume of the fluid under the cover remains unchanged in the course of vibrations. As a result, the shape of the deflection of the lid should satisfy the equation of constraint, which follows from the condition of preservation of the volume of the fluid under the lid.

Keywords: membrane, plate, incompressible fluid in container, free constraint vibrations.

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1. INTRODUCTION

The model problem under consideration belongs to the extensive class of dynamic hydroelasticity problems. These problems are encountered in shipbuilding and aviation, in transport of liquids, in describing natural phenomena, and in many other cases. Various approaches to solving these problems and the extensive literature are presented in [1–4]. As the pioneer studies, we mention work of Rayleigh [5] devoted to waves in an infinite plate in contact with a fluid and work of Lamb [6], which deals with vibrations of a circular plate in water. Vibrations of elastic solids in a compressible fluid are accompanied by the emission of sound waves [4], while vibrations of plates on the surface of a fluid induce surface waves [7]. These waves remove the energy of vibrations, which yields a complex spectrum. The frequency spectrum of vibrations of elastic containers filled with a perfect incompressible fluid is real and discrete [1]. As a rule, problems in which fluid has the free surface are considered [8, 9].

Below, a container is considered that is shaped as a rectangular parallelepiped that is completely filled with an incompressible fluid and covered with an elastic rectangular lid. The lid is modeled by an elastic membrane or a plate with the unrestrained sides. The frequency spectrum of free vibrations of this lid together with the fluid is studied provided that the volume of the fluid under the cover remains unchanged. This condition produces the constraint on the shape of the deflection of the lid. If there is a similar constraint in the shape of the deflection of a string and a beam, the frequency spectra of vibrations of these objects are also obtained.

A similar formulation of the problem was used in [10]. In that formulation, no constraint was introduced on the shape of the deflection of the lid (plate); however, an analysis of the graphs of the proper functions presented in [10] indicates that the condition of the preservation of the volume of the fluid under the lid is satisfied. In the problems under consideration, it is assumed that the characteristic period of free vibrations is substantially longer than the travel time of the fluid volume deformation wave. Therefore, the fluid is considered to be incompressible.

2. VIBRATIONS OF THE MEMBRANE TOGETHER WITH THE FLUID

Let us consider the parallelepiped-shaped container $0 \le x \le a$, $0 \le y \le b$, and $0 \le z \le c$, which is completely filled with a perfect incompressible fluid. A membrane is drawn on the face $z = 0$ with the tension force T. The other faces of the container are stationary and smooth. It is required to find the frequency spectrum of vibrations of the membrane taking into account the apparent mass of the fluid. The velocity of the points of the membrane is equal to the normal component of the velocity of the fluid at $z = 0$, i.e., $V_z(x, y, 0, t) = \psi(x, y)\sin(\omega t)$.

Let us expand the function $\psi(x, y)$ into the double Fourier series as follows:

$$\psi(x, y) = \sum_{m, n = 1}^{\infty} \psi_{mn}\cos p_m x \cos q_n y, \quad \psi_{mn} = \frac{4}{ab}\int_0^a\int_0^b \psi(x, y)\cos p_m x \cos q_n y\,dx\,dy, \tag{2.1}$$

where $p_m = m\pi/a$ and $q_n = n\pi/b$. Then the potential of the velocities of the fluid, which satisfies the Laplace equation and the above-presented boundary conditions, can be written as follows after the multiplier sin(ωt) has been separated out:

$$\Phi(x, y, z) = -\sum_{m, n = 1}^{\infty} \frac{\psi_{mn}}{r_{mn}}\cos p_m x \cos q_n y \frac{\cosh(r_{mn}(z-c))}{\sinh(r_{mn}c)}, \quad r_{mn}^2 = p_m^2 + q_n^2. \tag{2.2}$$

The kinetic energy of the fluid can be described by the following equation:

$$T_f = \frac{\rho_f}{2}\int_0^a\int_0^b\int_0^c(\Phi_{,x}^2 + \Phi_{,y}^2 + \Phi_{,z}^2)\,dx\,dy\,dz = \frac{\rho_0 ab}{8}\sum_{m, n = 1}^{\infty} q_{mn}\psi_{mn}^2, \quad q_{mn} = \frac{\coth(r_{mn}c)}{r_{mn}}, \tag{2.3}$$

where ρ_f is the density of the fluid.

In addition to expansion (2.1), in relation to the membrane fixity conditions $\psi = 0$ at $x = 0$, a and $y = 0$, b we present the function $\psi(x, y)$ in the following form:

$$\psi(x, y) = \sum_{m, n = 1}^{\infty} u_{mn}\sin p_m x \sin q_n y. \tag{2.4}$$

The coefficients ψ_{mn} and u_{mn} are related as follows:

$$\psi_{\hat{m}\hat{n}} = \sum_{m, n = 1}^{\infty} \alpha_{\hat{m}m}\alpha_{\hat{n}n}u_{mn}, \quad \alpha_{\hat{m}m} = \frac{2m(1-(-1)^{\hat{m}+m})}{\pi(m^2 - \hat{m}^2)}. \tag{2.5}$$

The fluid incompressibility requirement imposes the following constraint on the function $\psi(x, y)$:

$$\int_0^a\int_0^b \psi(x, y)\,dx\,dy = \frac{ab}{\pi^2}\sum_{m, n = 1}^{\infty} u_{mn}\gamma_{mn} = 0, \quad \gamma_{mn} = \frac{(1-(-1)^m)(1-(-1)^n)}{4mn}. \tag{2.6}$$

The frequencies and shapes of vibrations are determined when minimizing the functional

$$J_* = \frac{1}{2}\int_0^a\int_0^b(T(\psi_{,x}^2 + \psi_{,y}^2) + \rho_f g\psi^2 - \lambda\rho\psi^2)\,dx\,dy - \lambda T_f - \mu\int_0^a\int_0^b \psi(x, y)\,dx\,dy. \tag{2.7}$$

in ψ, where $\lambda = \omega^2$, ω is the unknown frequency of free vibrations, and μ is the Lagrange multiplier. The term $\rho_f g\psi^2$ takes into account the weight of the fluid and is only introduced when the vibrating membrane is horizontal; here, g is the acceleration of gravity and ρ is the surface density of the membrane. Integration yields the following result:

$$J_* = \frac{ab}{8}\left(\sum_{m, n = 1}^{\infty}(Tr_{mn}^2 + \rho_f g - \lambda\rho)u_{mn}^2 - \lambda\rho_f\sum_{\hat{m}, \hat{n} = 1}^{\infty} q_{\hat{m}\hat{n}}\psi_{\hat{m}\hat{n}}^2\right) - \mu\frac{ab}{\pi^2}\sum_{m, n = 1}^{\infty} u_{mn}\gamma_{mn}. \tag{2.8}$$

After differentiation with respect to u_{mn}, we obtain the following relations:

$$\frac{4}{ab}\frac{\partial J_*}{\partial u_{mn}} = (Tr_{mn}^2 + \rho_f g - \lambda\rho)u_{mn} - \lambda\rho_f\sum_{r, s = 1}^{\infty}\beta_{mn}^{rs}u_{rs} - \hat{\mu}\gamma_{mn} = 0, \quad m, n = 1, 2, \ldots, \tag{2.9}$$

$$\beta_{mn}^{rs} = \sum_{\hat{m}, \hat{n} = 1}^{\infty} q_{\hat{m}\hat{n}}\alpha_{\hat{m}m}\alpha_{\hat{n}n}\alpha_{\hat{m}r}\alpha_{\hat{n}s}, \quad \hat{\mu} = \frac{ab}{4\pi^2}\mu. \tag{2.10}$$

Because of relations (2.5), the coefficients $\alpha_{rs} = 0$ if the sum of the subscripts $r + s$ is even and the coefficients $\gamma_{mn} = 0$ if at least one of these subscripts is even. Therefore, system (2.9) decomposes into four subsystems with odd m and n, even m and n, and even m and odd n, as well as odd m and even n.

The first subsystem has the following form:

$$(Tr_{mn}^2 + \rho_f g - \lambda\rho)u_{mn} - \lambda\rho_f \sum_{r,s=1,3,\ldots}^{\infty} \beta_{mn}^{rs} u_{rs} - \frac{\hat{\mu}}{mn} = 0, \quad m, n = 1, 3, \ldots, \quad \sum_{m,n=1,3,\ldots} \frac{u_{mn}}{mn} = 0; \tag{2.11}$$

in the other subsystems, the last term is absent.

Let us reduce system (2.9) to the dimensionless form. We assume that $a \le b$ and take a as the unit length. Let us set

$$\hat{p}_m^2 = m^2, \quad \hat{q}_n^2 = n^2\delta, \quad \delta = (a/b)^2 \le 1, \quad \hat{r}_{mn}^2 = m^2 + \delta n^2, \quad \hat{q}_{mn} = \frac{\coth(\hat{r}_{mn}\hat{c})}{\hat{r}_{mn}}, \quad \hat{c} = \frac{\pi c}{a},$$
$$\lambda = \frac{T\pi^2}{a^2}\hat{\lambda}, \quad \hat{\rho}_f = \frac{\rho_f a}{\pi\rho} = \frac{\rho_f a}{\pi\rho_p h}, \quad \hat{g} = \frac{\rho_f g a^2}{T\pi^2}, \quad \hat{\beta}_{mn}^{rs} = \sum_{\hat{m},\hat{n}=1}^{\infty} q_{\hat{m}\hat{n}}\alpha_{\hat{m}m}\alpha_{\hat{n}n}\alpha_{\hat{m}r}\alpha_{\hat{n}s}. \tag{2.12}$$

Then, system (2.9) can be rewritten as follows:

$$(\hat{r}_{mn}^2 + \hat{g} - \hat{\lambda})u_{mn} - \hat{\lambda}\hat{\rho}_f \sum_{r,s}^{\infty} \hat{\beta}_{mn}^{rs} u_{rs} - \hat{\mu}\gamma_{mn} = 0. \tag{2.13}$$

with the subscripts at the unknown u_{mn} taking on even or odd values, depending on the subsystem under consideration.

When calculating $\hat{\lambda}_{m_0 n_0}$, we consider the final system (2.13) at $m, n \le K$ and replace the infinite sums with finite sums; the number $K > \max(m_0, n_0)$ is selected using the condition of attaining the required accuracy of the variable $\hat{\lambda}_{m_0 n_0}$.

System (2.10) comprises a number of particular cases. Using the parameter $\hat{\rho}_f$ in which ρ_f/ρ_p is the ratio of the volume densities of the fluid and the membrane and h is the thickness of the membrane, the apparent mass of the fluid can be taken into account. Because $a/h \gg 1$, the value of the parameter $\hat{\rho}_f$ can be high. If the apparent mass of the fluid is neglected, it should be taken that $\hat{\rho}_f = 0$.

The case $\hat{c} \gg 1$ corresponds to the infinite depth c; in this case, $q_{mn} = 1/r_{mn}$.

If $a \ll b$, it can be assumed that $\delta = 0$, which corresponds to the passage to the plane problem formulation, i.e., to vibrations of a string.

With slight modifications, system (2.13) can also be used when the membrane is replaced by a plate unrestrained over all its edges. In this case, the variable $\hat{r}_{mn}^2$ in the first term of Eqs. (2.13) should be replaced with the variable $\hat{r}_{mn}^4$ and the following relations should be used instead of (2.9):

$$\lambda = \frac{Da^4}{\rho\pi^4}\hat{\lambda}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \quad \hat{g} = \frac{\rho_f q a^2}{T\pi^2}, \tag{2.14}$$

where D is the cylindrical rigidity, h is the thickness, and E and ν are Young's modulus and Poisson's ratio of the plate.

For the other variants of the fixing of the plate, the form of system (2.13) remains unchanged, but the coefficients, especially those related to formula (2.5), vary. An additional obstacle is that, under boundary conditions different from the hinged-edge condition, the shapes of vibrations of the plate have no analytical representation.

At $\delta = 0$, the passage from vibrations of the plate to vibrations of a strip-like beam in the fluid occurs.

The problem at hand has two specific features, i.e., the consideration of incompressibility, which leads to the restriction on the deflection (the constraint), and the consideration of the effect of the apparent mass of the fluid. Below (in section 3), the effect of this constraint on the frequencies and shapes of the free vibrations is analyzed using an example of the simplest problem on the free vibrations of a string (beam) without taking into account the apparent mass. This problem has an explicit solution. In addition to its independent significance, this model problem supports the discussion of the more complex problems considered in sections 2, 4, and 5.

3. VIBRATIONS OF THE CONSTRAINED STRING

The equation of free vibrations of a string with the length a can be written as follows:

$$T\frac{\partial^2 w}{\partial x^2} + \rho\omega^2 = 0, \quad w(0) = w(a) = 0, \tag{3.1}$$

where T is the tension force, ρ is the mass of the unit length of the string, and ω is the frequency of the vibrations. Let us determine the frequency spectrum of the proper vibrations of the constrained string as follows:

$$\int_0^a w(x)dx = 0. \tag{3.2}$$

Without the constraint, the frequencies and shapes of free vibrations have the following form:

$$\omega_m = \omega_* m, \quad \omega_* = \frac{\pi}{a}\sqrt{\frac{T}{\rho}}, \quad \varphi_m(x) = \sin\frac{m\pi x}{a}, \quad m = 1, 2, \dots . \tag{3.3}$$

Let us expand the unknown shape of the vibration into a series of functions (3.3) as follows:

$$w = \psi(x) = \sum_{m=1}^{\infty} u_m\varphi_m(x). \tag{3.4}$$

Then, the problem of the minimization of functional (2.8) can be written in the following form:

$$\min_{u_m}\sum_{m=1}^{\infty}(\omega_m^2 u_m^2 - \lambda u_m^2 - \mu c_m u_m), \quad c_m = \int_0^a \varphi_m(x)dx = \begin{cases} 2a/(m\pi), & m = 1, 3, \dots, \\ 0, & m = 2, 4, \dots, \end{cases} \tag{3.5}$$

where $\lambda = \omega^2$ are the unknown squared frequencies of the vibrations and μ is the Lagrange multiplier. As a result, we obtain the following relation:

$$(\omega_m^2 - \lambda)u_m = c_m\mu/2. \tag{3.6}$$

Because of (3.5), at even values of m, the frequencies and shapes of the vibrations of the constrained string remain unchanged, while at odd values of m, the substitution of u_m into the equation of constraint yields the following equation for λ:

$$\sum_{m=1,3,\dots}^{K}\frac{1}{m^2(\omega_m^2 - \lambda)} = 0, \tag{3.7}$$

which can be written in dimensionless form as follows:

$$\sum_{m=1,3,\dots}^{K}\frac{1}{m^2(m^2 - \hat{\lambda})} = 0, \quad \lambda = \omega_*^2\hat{\lambda}. \tag{3.8}$$

The following shapes of the vibrations correspond to the roots $\hat{\lambda}_k$ of Eq. (3.8):

$$\psi_k(x) = \sum_{m=1,3,\dots}^{K}\frac{\varphi_m(x)}{m(m^2 - \hat{\lambda}_k)}. \tag{3.9}$$

The first roots are $\hat{\omega}_k = \sqrt{\hat{\lambda}_k} = 2.8606, 4.9181, 6.9418$, and 8.9548. The asymptotic formula $\hat{\omega}_k = 2k + 1 - 0.405/(2k+1) + O(k^{-2})$, $k \longrightarrow \infty$ is true.

Thus, in the case of the constraint, the first frequency described by formula (3.3) vanishes; at even values of m, the frequencies remain unchanged, while at odd values of m, they decrease, and the above-presented values of $\hat{\omega}_m$ are taken instead the values 3, 5, ...

Figure 1 shows the graphs of the first two proper functions $\psi_1(x)$ and $\psi_2(x)$ normalized using the formula $\int_0^1 \psi_k^2(x)dx = 1/2$, as well as (for comparison) the similar shapes of vibrations $\sin(3\pi x)$ and $\sin(5\pi x)$ for the string without taking into account constraint (3.2).

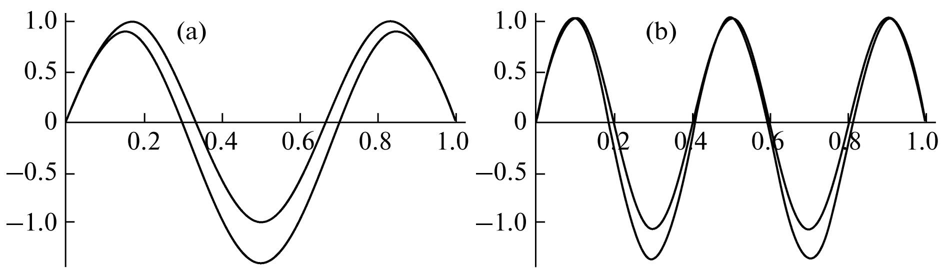

Fig. 1. Graphs of the first two proper functions.

Vibrations of an unrestrained beam taking into account constraint (3.2) are studied using the same procedure. The difference is that, for the beam, the following formula is used instead of formula (3.3):

$$\omega_m = \omega_* m^2, \quad \omega_* = \frac{\pi^2}{a^2}\sqrt{\frac{EI}{\rho}}, \tag{3.10}$$

where EI is the flexural rigidity. For odd values of m, the dimensionless frequency parameter $\hat{\lambda}$ can be found using the following equation:

$$\sum_{m = 1, 3, \ldots} \frac{1}{m^2(m^4 - \hat{\lambda})} = 0, \quad \lambda = \omega_*^2\hat{\lambda}. \tag{3.11}$$

The first roots of this equation $\hat{\omega}_k = \sqrt{\hat{\lambda}_k}$, which are equal to 8.541, 24.566, 48.375, and 80.55, are close to the squared odd numbers $(2k + 1)^2$. The shapes of the free vibrations of the beam taking into account the constraint are similar to those of the string. Graphs of these shapes of the vibrations are not shown here since they cannot visually be distinguished from the graphs presented in Fig. 1.

4. VIBRATIONS OF THE CONSTRAINED STRING TAKING INTO ACCOUNT THE APPARENT MASS OF THE FLUID

In fact, the plane movement of the membrane and the fluid in the planes parallel to the Oxz plane is considered. Taking into account $\delta \longrightarrow 0$, Eq. (2.9) can be written in the following form:

$$(Tp_m^2 - \lambda\rho)u_m - \lambda\rho_f\sum_{r=1}^{\infty}\beta_m^r u_r - \mu c_m = 0, \quad m = 1, 2, \ldots, \tag{4.1}$$

where

$$\beta_m^r = \sum_{\hat{m}=1}^{\infty} q_{\hat{m}}\alpha_{\hat{m}m}\alpha_{\hat{m}r}, \quad q_{\hat{m}} = \frac{a}{\hat{m}\pi}\coth(\hat{m}\pi c/a).$$

As in the general case, system of equations (4.1) decomposes into two subsystems for even and odd values of m since $\beta_m^r = 0$ if the indices m and r have different evenness.

In dimensionless form, system of equations (4.1) has the following form:

$$(m^2 - \hat{\lambda})u_m - \hat{\lambda}\hat{\rho}_f\sum_{r=1}^{\infty}\hat{\beta}_m^r u_r - \mu c_m = 0, \tag{4.2}$$

where $\hat{\beta}_m^r = (\beta_m^r\pi/a)$ and the other designations are the same as those used in formula (2.12).

Let us initially consider system (4.2) at even values of m. In this case, $c_m = 0$ and system (4.2) takes on the following form:

$$(m^2 - \hat{\lambda})u_m - \hat{\lambda}\hat{\rho}_f\sum_{r = 2, 4, \ldots}\hat{\beta}_m^r u_r = 0, \quad m = 2, 4, \ldots, \tag{4.3}$$

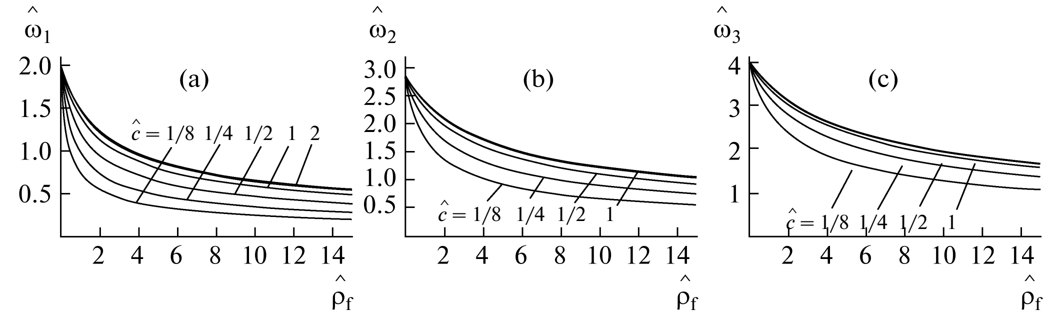

Fig. 2. Dependences of the dimensionless frequencies on the apparent mass parameter for various values of the container depth parameter.

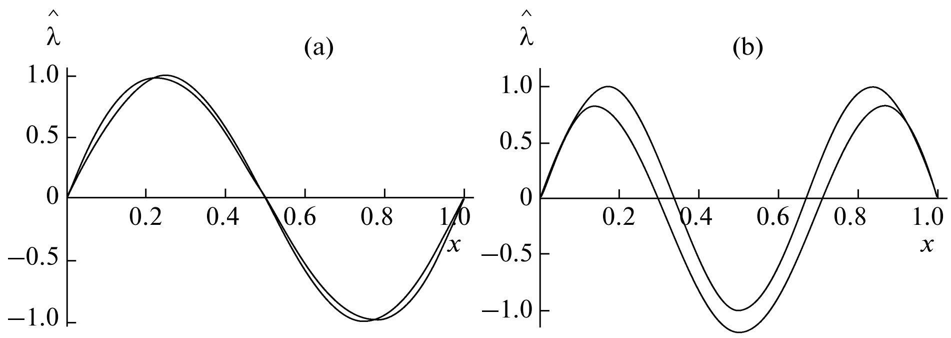

Fig. 3. Comparison of the shapes of the deflection obtained with and without consideration of the apparent mass of the fluid at $\hat c = 0.5$ and $\hat\rho_f = 10$.

where

$$\hat\beta_m^r = \frac{16mr}{\pi^2}\sum_{\hat m = 1, 3, \dots}^{\infty}\frac{\coth(\hat m c).}{\hat m(m^2-\hat m^2)(r^2-\hat m^2)}, \quad \hat c = \frac{\pi c}{a}. \tag{4.4}$$

The proper values $\hat\lambda_k$ of the parameter $\hat\lambda$ are the roots of the determinant of system of equations (4.3). The dimensionless proper frequencies $\hat\omega_k = \sqrt{\hat\lambda_k}$ depend on the dimensionless parameters $\hat c$ and $\hat\rho_f$. With decreasing parameter $\hat c$, the frequencies diminish and with an increase in the parameter $\hat c$ they tend to the limiting values, which correspond to a deep container. As the parameter $\hat\rho_f$ increases, the effect of the mass of the membrane on the frequencies vanishes; at $\hat\rho_f \gg 1$, the frequencies decrease proportionally to $1/\sqrt{\hat\rho_f}$. Figures 2a and 2c show the graphs of the functions $\hat\omega_k(\hat\rho_f)$ for the first two roots of Eq. (4.3) at $\hat c$ = 1/8, 1/4, 1/2, 1, 2, and 4. It can be seen that the increase in parameter $\hat c$ ceases to affect $\hat\omega$ at $\hat c = 2$ for the first root and at $\hat c = 1$ for the second root.

At an odd value of m, $c_m \neq 0$ and the equation of constraint, which follows from the condition of the incompressibility of the fluid, should be additionally satisfied. In this case, we obtain the following system of equations instead of the system described by Eqs. (4.3) and (4.4):

$$(m^2-\hat\lambda)u_m - \hat\lambda\hat\rho_f\sum_{r=1,3,\dots}\hat\beta_m^r u_r - \frac{\mu}{m} = 0, \quad m = 1, 3, \dots, \quad \sum_{m=1,3,\dots}\frac{u_m}{m} = 0, \tag{4.5}$$

where

$$\hat\beta_m^r = \frac{16mr}{\pi^2}\sum_{\hat m = 2, 4, \dots}^{\infty}\frac{\coth(\hat m\hat c).}{\hat m(m^2-\hat m^2)(r^2-\hat m^2)}.$$

The proper values $\hat{\lambda}_k$ are the roots of the determinant of linear system of equations (4.5) in the unknown $u_1, u_3, \ldots, u_{2K-1}, \mu$. As in the case without considering the apparent mass, the roots of this equation lie between the roots of the determinant of system of equations (4.3). The dependence $\hat{\omega}(\hat{\rho}_f, \hat{c})$ for the first root is shown in Fig. 2b.

Figure 3a shows the first shape of the deflection found by solving system of equations (4.2) and (for comparison) the graph of the function $\sin(2\pi x)$. Figure 3b shows the shape of the deflection found by solving system of equations (4.5) and the graph of the function $\sin(3\pi x)$. It follows from the data presented in Fig. 3a and from comparison of the data presented in Figs. 3b and 1a that the consideration of the apparent mass of the fluid only slightly influences the shape of the deflection, while the effect of the incompressibility is more pronounced.

5. TRANSFORMATION OF THE SYSTEM (2.13) AND NUMERICAL RESULTS

As was noted in paragraph 2, the problem on the vibrations of the membrane together with the fluid decomposes into four separate subproblems depending on the values of the wavenumbers m and n. If at least one of these numbers is even, the equation of constraint, which follows from the condition of the incompressibility of the fluid, is automatically satisfied. The calculations show that the shapes of the free vibrations are close to the vibrations of the function $\varphi_{mn} = \sin p_m x \sin q_n y$ like the curves presented in Fig. 3a are close to each other. The dimensionless frequencies $\hat{\omega}_{mn}$ depend on the parameters $\hat{\rho}_f$, $\hat{c}$, δ, and $\hat{g}$. Without taking into consideration the effect of the fluid, these frequencies can be found using the formula $\hat{\omega}_{mn} = \hat{r}_{mn} = \sqrt{m^2 + \delta n^2}$, while with the consideration of the apparent mass the dependence of the frequencies $\hat{\omega}_{mn}$ on the parameters $\hat{\rho}_f$ and $\hat{c}$ is similar to the dependence for the string shown in Fig. 2.

Let us consider the case when both subscripts m and n have odd values in more detail. We renumber the parameters

$$\hat{r}_{mn} = \sqrt{m^2 + \delta n^2}, \quad m, n = 1, 3, \ldots, \tag{5.1}$$

in ascending order and obtain the following result:

$$\hat{r}_1 = \hat{r}_{m_1 n_1} \le \hat{r}_2 = \hat{r}_{m_2 n_2} \le \ldots \le \hat{r}_k = \hat{r}_{m_k n_k} \le \ldots . \tag{5.2}$$

Then, the subscripts m_k and n_k depend on the position k occupied by the values of $\hat{r}_{mn}$ in sequence (5.2). Let us write system of equations (2.13) with the only summation index as follows:

$$(\hat{r}_k^2 - \hat{\lambda})u_k - \hat{\lambda}\hat{\rho}_f \sum_{l=1}^{\infty} \hat{\beta}_k^l u_l - \mu\gamma_k = 0, \quad k = 1, 2, \ldots, \quad \sum_{k=1}^{\infty} \gamma_k u_k = 0, \tag{5.3}$$

where

$$\hat{\beta}_k^l = \left(\frac{4}{\pi}\right)^4 \sum_{m, n = 2, 4, \ldots}^{\infty} \frac{m_k n_k m_l n_l \coth(\hat{r}_{mn} c)}{\hat{r}_{mn}(m_k^2 - m^2)(n_k^2 - n^2)(m_l^2 - m^2)(n_l^2 - n^2)}, \quad \gamma_k = \frac{1}{m_k n_k}. \tag{5.4}$$

As an example, let us consider the square membrane ($\delta = 1$) with the parameters $\hat{c} = 1.5$ and $\hat{\rho}_f = 10$. For odd values of m and n, the first four dimensionless frequencies $\hat{\omega}$ found by solving system of equations (5.3) at $\hat{g} = 0$ are equal to 1.791, 2.738, 2.742, and 3.558; the corresponding shapes of the deflection are shown in Fig. 4 (the upper panels). For comparison, the shapes of the deflection for a square plate with the frequencies $\hat{\omega} = 5.840, 9.016, 11.224, 19.12$ are presented in the lower panels (Fig. 4). These frequencies and shapes have been obtained by solving system of equations (5.3) after performing the replacement of $\hat{r}_k^2$ with $\hat{r}_k^4$. It can be seen that the shapes of the vibrations of the membrane and the plate in the fluid substantially differ, while in the case of vibrations in air, these shapes are identical.

At a constant value of δ, all dimensionless frequencies grow with increasing parameter $\hat{g}$ and diminish with decreasing parameters $\hat{\rho}_f$ and $\hat{c}$, which follows from the Courant theorem [11] on the minimum–maximum property of proper values. In this case, at $\hat{\rho}_f \gg 1$ the frequencies grow proportionally to $\sqrt{\hat{\rho}_f}$, while at $\hat{c} \ge 3$, they are close to the values typical of the infinitely deep fluid ($\hat{c} = \infty$).

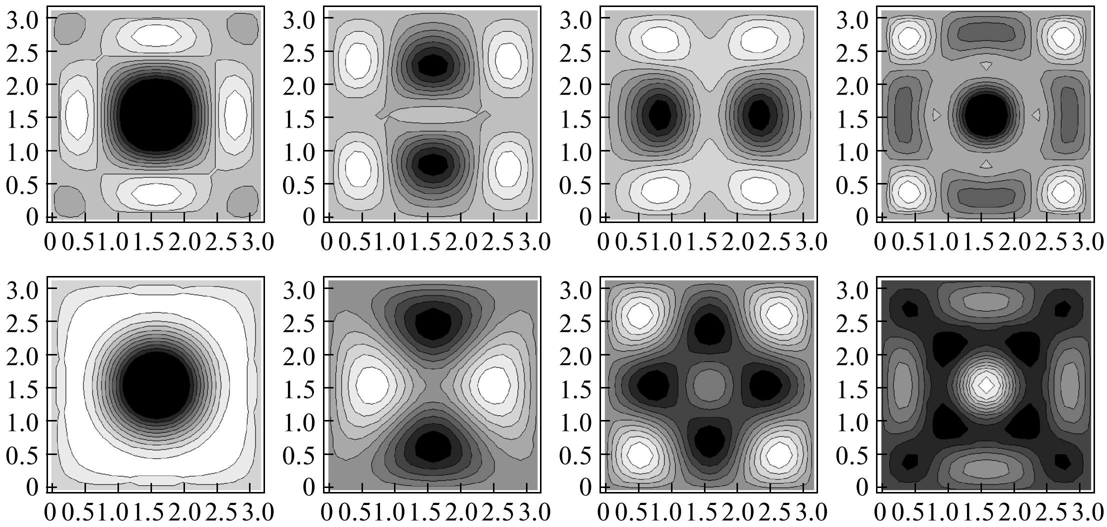

Fig. 4. Shapes of the deflection of the membrane (upper panels) and the plate (lower panels).

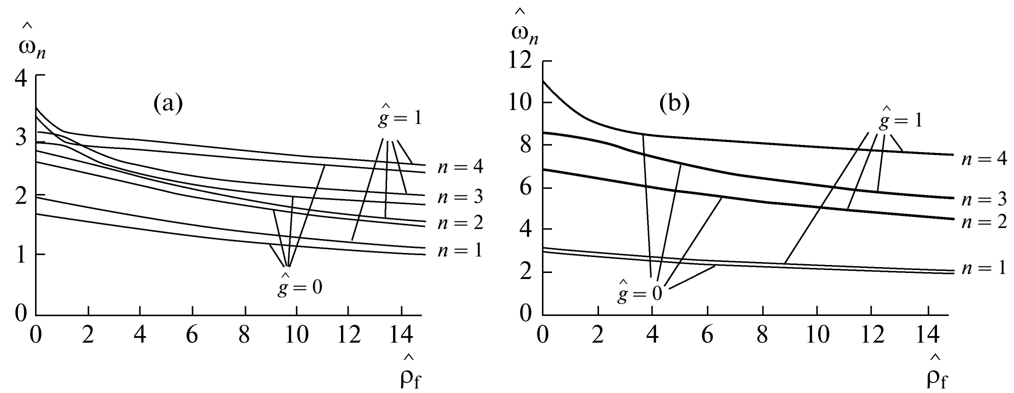

Fig. 5. Dependences of $\hat{\omega}_n(\hat{\rho}_f)$ for the first four frequencies.

Figure 5 shows the dependences $\hat{\omega}_n(\hat{\rho}_f)$ of the first four frequencies of the doubly odd group obtained for the membrane (left panel) and the plate (right panel) at $\delta = 0.25$ and $\hat{c} = 1.5$ with ($\hat{g} = 1$) and without ($\hat{g} = 0$) taking into consideration the effect of the force of gravity in the case of the overhead lid. It can be seen that the consideration of the force of gravity leads to a substantial increase in the frequencies of the proper vibrations of the membrane, but only slightly influences the frequencies of the vibrations of the plate.

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