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Tests of exponentiality based on Arnold–Villasenor characterization and their efficiencies

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ABSTRACT

Two families of scale-free exponentiality tests based on the recent characterization of exponentiality by Arnold and Villasenor are proposed. The test statistics are constructed using suitable functionals of U-empirical distribution functions. The family of integral statistics can be reduced to V- or U-statistics with relatively simple non-degenerate kernels. They are asymptotically normal and have reasonably high local Bahadur efficiency under common alternatives. This efficiency is compared with simulated powers of new tests. On the other hand, the Kolmogorov type tests demonstrate very low local Bahadur efficiency and rather moderate power for common alternatives, and can hardly be recommended to practitioners. The conditions of local asymptotic optimality of new tests are also explored and for both families special "most favourable" alternatives for which the tests are fully efficient are described.

1. Introduction

Exponential distribution plays an essential role in probability and statistics since various models with exponentially distributed observations often appear in applications such as survival analysis, reliability theory, engineering, demography, etc. Therefore, testing exponentiality is one of the most important problems in goodness-of-fit theory.

There exists a multitude of tests for this problem which are based on various ideas (see books and reviews Ahsanullah and Hamedani (2010), Asher (1990), Balakrishnan and Basu (1995), Cox and Oakes (1984), Doksum and Yandell (1985), Henze (1992), Henze and Meintanis (2002a, 2002b), Nabendu et al. (2002)). Among them many tests are based on characterizations of exponential law. In particular, some tests based on lack of memory property can be found in Ahmad and Alwasel (1999), Angus (1982), Koul (1977, 1978) and Nikitin (1996) and some tests based on some other characterizations in Baringhaus and Henze (2000), Henze and Meintanis (2005), Jansen van Rensburg and Swanepoel (2008), Litvinova (2004), Nikitin and Volkova (2010), Noughabi and Arghami (2011), Rank (1999) and Rao and Taufer (2006). The construction of tests based on characterizations is a relatively fresh idea which gradually becomes one of the main directions in goodness-of-fit testing.

In this paper we present new tests for exponentiality based on Arnold–Villasenor characterization. In Arnold and Villasenor (2013) Arnold and Villasenor stated the following hypothesis:

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Let $\mathcal{F}$ be the class of distributions whose densities have derivatives of all orders in the neighborhood of zero and let $X_1, X_2, \ldots, X_n$ be non-negative independent and identically distributed (i.i.d.) random variables with distribution function (d.f.) F from class $\mathcal{F}$. Then the random variables $\max(X_1, X_2, \ldots, X_k)$ and $\sum_{i=1}^{k} \frac{X_i}{i}$ are equally distributed if and only if F is exponential.

They were able to prove this hypothesis only for $k = 2$. Later Yanev and Chakraborty in Yanev and Chakraborty (2013) proved that this hypothesis was also true for $k = 3$. Related questions were addressed in Chakraborty and Yanev (2013) and Obradović (2014).

In Milošević and Obradović (2014) Milošević and Obradović proved the hypothesis for any k under the condition that the density has Maclaurin's expansion for $x > 0$. This condition was implicitly assumed in the proofs of particular cases $k = 2$ and $k = 3$.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. observations having the continuous d.f. F from the class $\mathcal{F}$. We are testing the composite hypothesis of exponentiality $H_0 : F(x)$ *belongs to exponential family of distributions* $\mathcal{E}(\lambda)$ *with the density* $f(x) = \lambda e^{-\lambda x}, x \geq 0$, where $\lambda > 0$ is an unknown parameter.

Let $F_n(t) = n^{-1} \sum_{i=1}^{n} \mathbf{I}\{X_i < t\}, t \in \mathbb{R}$, be the usual empirical d.f. based on the observations $X_1, X_2, \ldots, X_n$. In compliance with Arnold–Villasenor characterization for $t \geq 0$ we introduce the so-called V-empirical d.f.'s (see Janssen (1988), Korolyuk and Borovskikh (1994)) according to the formulae

$$H_n^{(k)}(t) = \frac{1}{n^k} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \mathbf{I}\big\{\max(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) < t\big\},$$

$$G_n^{(k)}(t) = \frac{1}{n^k k!} \sum_{i_1, \ldots, i_k = 1}^{n} \Bigg[\sum_{\pi(j_1, \ldots, j_k)} \mathbf{I}\Big\{\frac{X_{i_1}}{j_1} + \frac{X_{i_2}}{j_2} + \cdots + \frac{X_{i_k}}{j_k} < t\Big\}\Bigg],$$

where $\pi(j_1, \ldots, j_k)$ represents the set of all $k!$ permutations of natural numbers $1, 2, \ldots, k, k \geq 2$, while $\pi(j_1, \ldots, j_i^*, \ldots, j_k)$, which appears below, denotes the set of all $(k-1)!$ permutations of natural numbers $1, 2, \ldots, k$ excluding i.

It is well-known that the properties of V- and U-empirical d.f.'s are similar to the properties of usual empirical d.f.'s. In particular, Glivenko–Cantelli theorem is valid in this case (see Helmers et al. (1988), Janssen (1988)). Hence, according to Arnold–Villasenor characterization, the empirical d.f.'s $H_n^{(k)}$ and $G_n^{(k)}$ should be close for large n under H_0, and we can measure their proximity using appropriate test statistics.

Let us introduce two new sequences of statistics depending on natural $k > 1$ which are invariant with respect to the scale parameter λ:

$$I_n^{(k)} = \int_0^{\infty} \big(H_n^{(k)}(t) - G_n^{(k)}(t)\big)\, dF_n(t), \tag{1}$$

$$D_n^{(k)} = \sup_{t \geq 0} \mid H_n^{(k)}(t) - G_n^{(k)}(t) \mid, \tag{2}$$

where $k \geq 2$.

Large values of $I_n^{(k)}$ and $D_n^{(k)}$ are significant for rejection of null hypothesis. The sequence of statistics $I_n^{(k)}$ is not always consistent but nevertheless the consistency takes place for many common alternatives. At first glance the sequence of statistics of omega-square type

$$W_n^{(k)} = \int_0^{\infty} \big(H_n^{(k)}(t) - G_n^{(k)}(t)\big)^2\, dF_n(t),$$

could seem more adequate choice, but their asymptotic theory is very intricate and is currently underdeveloped. In the same time the statistics $I_n^{(k)}$ are usually asymptotically normal. As to the sequence $D_n^{(k)}$, it is consistent for any alternative.

In what follows we describe the limiting distributions and large deviations of both sequences of statistics under H_0, and calculate their local Bahadur efficiency under different alternatives. We also analyze the conditions of local asymptotic optimality of new statistics. In this regard we refer to the results from the theory of U- and V-statistics and the theory of Bahadur efficiency (Bahadur, 1971; DasGupta, 2008; Korolyuk and Borovskikh, 1994; Nikitin, 1995).

We have selected the Bahadur approach as a method of calculation of asymptotic efficiency for our tests because the Kolmogorov-type statistics $D_n^{(k)}$ are not asymptotically normal under null-hypothesis, and therefore the classical Pitman approach is not applicable. In case of integral statistic $I_n^{(k)}$, local Bahadur efficiency and Pitman efficiency coincide (Bahadur, 1960; Wieand, 1976).

We supplement our research with simulated powers which principally support the theoretical values of efficiency and present some examples of application to real data.

2. Integral statistic $I_n^{(k)}$

Without loss of generality we can assume that $\lambda = 1$. The statistic $I_n^{(k)}$ is asymptotically equivalent to the V-statistic of degree $(k+1)$ with the centered kernel $\Psi_k(X_1, X_2, \ldots, X_{k+1})$ given by

$$\Psi_k(X_1, X_2, \ldots, X_{k+1}) = \frac{1}{k+1}\Bigg[\sum_{i=1}^{k+1} \mathbf{I}\big\{\max(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_{k+1}) < X_i\big\} \\ - \frac{1}{k!}\sum_{i=1}^{k+1} \sum_{\pi(j_1,\ldots,j_i^*,\ldots,j_{k+1})} \mathbf{I}\Big\{\frac{X_1}{j_1} + \ldots + \frac{X_{i-1}}{j_{i-1}} + \frac{X_{i+1}}{j_{i+1}} + \ldots + \frac{X_{k+1}}{j_{k+1}} < X_i\Big\}\Bigg].$$

It is well-known that non-degenerate U- and V-statistics are asymptotically normal (Hoeffding, 1948; Korolyuk and Borovskikh, 1994). To show that the kernel $\Psi_k(X_1, X_2, \ldots, X_{k+1})$ is non-degenerate, let us calculate its projection $\psi_k(s)$ under null hypothesis. For fixed $X_{k+1} = s$ this projection has the form:

$$\psi_k(s) = E\big(\Psi_k(X_1, X_2, \ldots, X_{k+1})|X_{k+1} = s\big) = \frac{1}{k+1}P\big(\max(X_1, \ldots, X_k) < s\big) \\ + \frac{k}{k+1}P\big(\max(s, X_2, \ldots, X_k) < X_1\big) - \frac{1}{(k+1)!}\sum_{\pi(j_1,\ldots,j_k)} P\Big(\frac{X_1}{j_1} + \cdots + \frac{X_k}{j_k} < s\Big) \\ - \frac{k}{(k+1)!}\sum_{\pi(j_1,\ldots,j_k)} P\Big(\frac{s}{j_1} + \frac{X_2}{j_2} + \cdots + \frac{X_k}{j_k} < X_1\Big).$$

It follows from Arnold and Villasenor's characterization that the first and the third term in the right hand side coincide, so they cancel out.

Next we calculate the second term:

$$\frac{k}{k+1}P\big(\max(s, X_2, \ldots, X_k) < X_1\big) = \frac{k}{k+1}\int_0^\infty \mathbf{I}\{s < t\}P(X_2 < t, \ldots, X_k < t)dF(t) \\ = \frac{k}{k+1}\int_s^\infty F^{k-1}(s)dF(s) = \frac{1}{k+1}\left(1 - F^k(s)\right),$$

where $F(x) = 1 - e^{-x}$. It remains to calculate the last term. Since

$$P\Big(\frac{s}{j_1} + \frac{X_2}{j_2} + \cdots + \frac{X_k}{j_k} < X_1\Big) = \int_0^\infty e^{-x_2}dx_2 \ldots \int_0^\infty e^{-x_k}dx_k \int_{\frac{s}{j_1}+\frac{x_2}{j_2}+\cdots+\frac{x_k}{j_k}}^\infty e^{-x_1}dx_1 \\ = \frac{1}{(k+1)}\Big(1 + \frac{1}{j_1}\Big)e^{-s/j_1},$$

after summing this expression over all permutations of indices $j_1, j_2, \ldots, j_k$ and some additional calculations, we get that the fourth term is $\frac{1}{(k+1)^2}\sum_{r=1}^k (1 + \frac{1}{r})e^{-s/r}$.

Finally we obtain the following expression for the projection ψ_k of the kernel Ψ_k:

$$\psi_k(s) = \frac{1 - (1 - e^{-s})^k}{k+1} - \frac{1}{(k+1)^2}\sum_{r=1}^k \Big(1 + \frac{1}{r}\Big)e^{-s/r}. \tag{3}$$

It is easy to show that $E(\psi_k(X_1)) = 0$. After some calculations we get that the variance of this projection is

$$\Delta_k^2 = \mathrm{Var}\big(\psi_k(X_1)\big) = \int_0^\infty \psi_k^2(s)e^{-s}ds = \frac{1}{(k+1)^3}\Bigg[\frac{-12k^4 - 38k^3 - 35k^2 - 11k}{4(k+1)^2(k+2)(2k+1)} \\ + 2k!\sum_{r=1}^k \frac{1}{\left(k+1+\frac{1}{r}\right)\left(k+\frac{1}{r}\right)\cdots\left(2+\frac{1}{r}\right)} + \frac{2}{k+1}\sum_{1\le i<j\le k}\frac{1}{i+j+ij}\Bigg]. \tag{4}$$

It is clear from (3) and (4) that the kernel Ψ_k is non-degenerate for any k.

In fact if the kernel is non-degenerate, we can consider instead of V-statistic $I_n^{(k)}$ the corresponding U-statistic with the same kernel which has very similar asymptotic properties but is considerably simpler for calculation.

2.1. Local Bahadur efficiency

Let $G(\cdot, \theta), \theta \ge 0$, be a family of d.f.'s with densities $g(\cdot, \theta)$, such that $G(\cdot, 0) \in \mathscr{E}(\lambda)$. The measure of Bahadur efficiency (BE) for any sequence $\{T_n\}$ of test statistics is the exact slope $c_T(\theta)$ describing the rate of exponential decrease for the attained level under the alternative d.f. $G(\cdot, \theta), \theta > 0$. According to Bahadur theory (Bahadur, 1971; Nikitin, 1995) the exact slopes may be found by using the following proposition.

Proposition. *Suppose that the following two conditions hold:*

(a) $T_n \xrightarrow{P_\theta} b(\theta), \quad \theta > 0,$

where $-\infty < b(\theta) < \infty$, *and* $\xrightarrow{P_\theta}$ *denotes convergence in probability under* $G(\cdot, \theta)$.

(b) $\lim_{n\to\infty} n^{-1} \ln P_{H_0}(T_n \geq t) = -h(t)$

for any t in an open interval I, on which h is continuous and $\{b(\theta), \theta > 0\} \subset I$. *Then* $c_T(\theta) = 2h(b(\theta))$.

The exact slopes always satisfy the inequality (Bahadur, 1971; Nikitin, 1995)

$$c_T(\theta) \leq 2K(\theta), \quad \theta > 0, \tag{5}$$

where $K(\theta)$ is the Kullback–Leibler divergence between the alternative H_1 and the null hypothesis H_0. In our case H_0 is composite, hence for any alternative density $g(x, \theta)$ one has

$$K(\theta) = \inf_{\lambda>0} \int_0^\infty \ln\left[g(x, \theta)/\lambda \exp(-\lambda x)\right] g(x, \theta)\, dx. \tag{6}$$

This quantity can be easily calculated as $\theta \to 0$ for particular alternatives. According to (5), the local BE of the sequence of statistics T_n is defined as

$$e^B(T) = \lim_{\theta\to 0} \frac{c_T(\theta)}{2K(\theta)}.$$

2.2. Integral statistic $I_n^{(2)}$

For $k = 2$ from (3) and (4) we get that the projection of the kernel $\Psi_2(X, Y, Z)$ is equal to

$$\psi_2(s) = \frac{4}{9}e^{-s} - \frac{1}{3}e^{-2s} - \frac{1}{6}e^{-s/2}, \tag{7}$$

and its variance is

$$\Delta_2^2 = \int_0^\infty \psi_2^2(s) e^{-s} ds = \frac{5}{13608} \approx 0.000367.$$

Applying Hoeffding's theorem for U-statistics with non-degenerate kernels (see Hoeffding (1948), Korolyuk and Borovskikh (1994)), as $n \to \infty$, we obtain

$$\sqrt{n} I_n^{(2)} \xrightarrow{d} \mathcal{N}\left(0, \frac{5}{1512}\right).$$

Let us now find the logarithmic asymptotics of large deviations of the sequence of statistics $I_n^{(2)}$ under null hypothesis. The kernel Ψ_2 is centered, non-degenerate and bounded. Applying the results on large deviations of non-degenerate U- and V-statistics from Nikitin and Ponikarov (1999) (see also DasGupta (2008), Nikitin (2010)), we state the following theorem:

Theorem 1. *For* $a > 0$ *it holds*

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0}(I_n^{(2)} > a) = -f(a),$$

where the function f is analytic for sufficiently small $a > 0$, *moreover*

$$f(a) \sim \frac{a^2}{18\Delta_2^2} = \frac{756}{5}a^2 = 151.2a^2, \quad \text{as } a \to 0. \tag{8}$$

According to the law of large numbers for U- and V-statistics (Korolyuk and Borovskikh, 1994), the limit in probability under alternative H_1 is equal to

$$b_I^{(2)}(\theta) = P_\theta\left(\max(X, Y) < Z\right) - P_\theta\left(X + \frac{Y}{2} < Z\right).$$

It is easy to show (see also Nikitin and Peaucelle (2004)), that

$$b_I^{(2)}(\theta) \sim 3\theta \int_0^\infty \psi_2(s) h(s) ds, \quad \text{as } \theta \to 0, \tag{9}$$

where $h(x) = \frac{\partial}{\partial\theta} g_1(x, \theta)\mid_{\theta=0}$ and $\psi_2(s)$ is the projection from (7).

Table 1
Comparative table of local efficiencies for statistic $I_n^{(k)}$.

Alternative	eff $k=2$	eff $k=3$	$\max_k$ eff
Makeham	0.448	0.573	0.875 for $k=14$
Weibull	0.621	0.664	0.710 for $k=8$
Gamma	0.723	0.708	0.723 for $k=2$
EMNW(3)	0.694	0.799	0.885 for $k=6$

We present the following common alternatives against exponentiality which will be considered for all tests in this paper:

(i) Makeham distribution with the density

$$g_1(x,\theta) = \big(1+\theta(1-e^{-x})\big)\exp\big(-x-\theta(e^{-x}-1+x)\big), \theta > 0, \quad x \geq 0;$$

(ii) Weibull distribution with the density

$$g_2(x,\theta) = (1+\theta)x^{\theta}\exp(-x^{1+\theta}), \quad \theta > 0, x \geq 0;$$

(iii) gamma distribution with the density

$$g_3(x,\theta) = \frac{x^{\theta}}{\Gamma(\theta+1)}e^{-x}, \quad \theta > 0, x \geq 0;$$

(iv) exponential mixture with negative weights (EMNW(β)) (see Jevremović (1991))

$$g_4(x) = (1+\theta)e^{-x} - \theta\beta e^{-\beta x}, \quad x \geq 0, \theta \in \Big(0, \frac{1}{\beta-1}\Big].$$

Let us calculate the local Bahadur efficiencies for these alternatives.

For the Makeham alternative from (9) we get that

$$b_I^{(2)}(\theta) \sim 3\theta\int_0^{\infty}\Big(\frac{4}{9}e^{-s} - \frac{1}{3}e^{-2s} - \frac{1}{6}e^{-s/2}\Big)e^{-s}(2-2e^{-s}-s)ds$$
$$= \frac{\theta}{90} \approx 0.011\,\theta, \quad \text{as } \theta \to 0.$$

The local exact slope of the sequence $I_n^{(2)}$ as $\theta \to 0$ admits the representation

$$c_I^{(2)}(\theta) = \big(b_I^{(2)}(\theta)\big)^2/(9\Delta_2^2) \sim 0.037\theta^2.$$

From (6) the Kullback–Leibler divergence for Makeham distribution satisfies

$$K_{g_1}(\theta) \sim \frac{\theta^2}{24}, \quad \text{as } \theta \to 0. \tag{10}$$

Hence the local BE is

$$e^B(I^{(2)}) = \lim_{\theta\to 0}\frac{c_I^{(2)}(\theta)}{2K_{g_1}(\theta)} = 0.448.$$

The calculation for other alternatives is quite similar, therefore we omit it and we present local Bahadur efficiencies in Table 1.

2.3. Integral statistic $I_n^{(3)}$

For $k=3$ from (3) and (4) we get that the projection of the kernel $\Psi_3(X,Y,Z,W)$ is equal to

$$\psi_3(s) = \frac{5}{8}e^{-s} - \frac{3}{4}e^{-2s} + \frac{1}{4}e^{-3s} - \frac{3}{32}e^{-s/2} - \frac{1}{12}e^{-s/3}, \tag{11}$$

and its variance is

$$\Delta_3^2 = \int_0^{\infty}\psi_3^2(s)e^{-s}ds = \frac{14591}{30750720} \approx 0.000474.$$

As in the previous case, according to Hoeffding's theorem, as $n \to \infty$, the following convergence in distribution holds

$$\sqrt{n}I_n^{(3)} \xrightarrow{d} \mathcal{N}\Big(0, \frac{14591}{1921920}\Big).$$

Regarding the large deviation asymptotics of the sequence $I_n^{(3)}$ under the null hypothesis, we get exactly in the same manner as in the previous case:

Table 2
Simulated powers for statistics $I_n^{(k)}$ and $D_n^{(k)}$.

Alternative	θ	k	$I_n^{(k)}$		$D_n^{(k)}$	
			$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.025$
Makeham	1.5	2	0.51	0.39	0.34	0.24
	1.5	3	0.59	0.48	0.41	0.29
	1.5	4	0.63	0.51	0.44	0.34
	0.5	2	0.19	0.12	0.13	0.08
	0.5	3	0.22	0.13	0.15	0.10
	0.5	4	0.22	0.14	0.15	0.10
	0.25	2	0.11	0.06	0.09	0.05
	0.25	3	0.12	0.07	0.09	0.05
	0.25	4	0.13	0.08	0.10	0.06
Weibull	0.5	2	1.00	0.99	0.88	0.80
	0.5	3	1.00	0.99	0.93	0.89
	0.5	4	1.00	1.00	0.95	0.91
	0.25	2	0.74	0.61	0.40	0.29
	0.25	3	0.77	0.66	0.47	0.35
	0.25	4	0.78	0.68	0.50	0.38
Gamma	0.5	2	0.86	0.76	0.45	0.32
	0.5	3	0.85	0.76	0.48	0.36
	0.5	4	0.84	0.75	0.48	0.36
	0.25	2	0.42	0.30	0.19	0.12
	0.25	3	0.42	0.30	0.20	0.13
	0.25	4	0.42	0.30	0.20	0.13
EMNW(3)	0.5	2	0.98	0.97	0.67	0.54
	0.5	3	0.98	0.97	0.68	0.55
	0.5	4	0.98	0.97	0.68	0.55
	0.25	2	0.45	0.33	0.22	0.14
	0.25	3	0.47	0.34	0.23	0.16
	0.25	4	0.47	0.34	0.23	0.16

Theorem 2. *For $a > 0$ it holds*

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0}(I_n^{(3)} > a) = -f(a),$$

where the function f is analytic for sufficiently small $a > 0$, moreover

$$f(a) \sim \frac{a^2}{32\Delta_3^2} = \frac{960960}{14591}a^2 = 65.86a^2, \quad \text{as } a \to 0. \tag{12}$$

In this case the limit in probability under alternative H_1 is equal to

$$b_I^{(3)}(\theta) = P_\theta\big(\max(X, Y, Z) < W\big) - P_\theta\Big(X + \frac{Y}{2} + \frac{Z}{3} < W\Big).$$

It is easy to show (Nikitin and Peaucelle, 2004) that $b_I^{(3)}(\theta) \sim 4\theta \int_0^\infty \psi_3(s)h(s)ds$, where again $h(x) = \frac{\partial}{\partial\theta} g_1(x, \theta)\mid_{\theta=0}$ and $\psi_3(s)$ is the projection from (11).

For the Makeham alternative we have

$$b_I^{(3)}(\theta) \sim 4\theta \int_0^\infty \Big(\frac{5}{8}e^{-s} - \frac{3}{4}e^{-2s} + \frac{1}{4}e^{-3s} - \frac{3}{32}e^{-s/2} - \frac{1}{12}e^{-s/3}\Big)e^{-s}(2 - 2e^{-s} - s)ds$$
$$= \frac{2}{105}\theta \approx 0.019\,\theta, \quad \text{as } \theta \to 0,$$

and the local exact slope of the sequence $I_n^{(3)}$ as $\theta \to 0$ admits the representation

$$c_I^{(3)}(\theta) = \big(b_I^{(3)}(\theta)\big)^2/(16\Delta_3^2) \sim 0.048\theta^2.$$

As previously stated, the Kullback–Leibler divergence satisfies the relation (10). Hence the local BE is equal to

$$e^B(I^{(3)}) = \lim_{\theta\to 0} \frac{c_I^{(3)}(\theta)}{2K_{g_1}(\theta)} \approx 0.573.$$

We again omit the calculations for other alternatives. In Table 1 we present the local Bahadur efficiencies against our four alternatives for $k = 2$ and $k = 3$, as well as the maximal (with respect to k) values we obtained using the MAPLE package.

In Table 2 we present the simulated powers for our four alternatives. The simulations have been performed for $n = 100$ with 10,000 replicates.

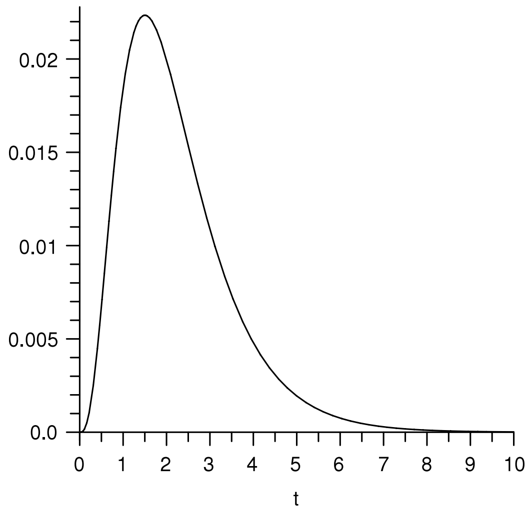

Fig. 1. Plot of the function $\delta_2^2(t)$.

3. Kolmogorov-type statistic $D_n^{(k)}$

In this section we consider the Kolmogorov-type statistic (2). For a fixed $t > 0$ the expression $H_n^{(k)}(t) - G_n^{(k)}(t)$ is the V-statistic with the following kernel:

$$\Xi_k(X_1, X_2, \dots, X_k; t) = \mathbf{I}\big\{\max(X_1, X_2, \dots, X_k) < t\big\} - \frac{1}{k!}\sum_{\pi(j_1,\dots,j_k)} \mathbf{I}\Big\{\frac{X_1}{j_1} + \frac{X_2}{j_2} + \cdots + \frac{X_k}{j_k} < t\Big\}.$$

Let $\xi_k(X_1; t)$ be the projection of $\Xi_k(X_1, X_2, \dots, X_k; t)$ on X_1. Then

$$\begin{aligned}\xi_k(s; t) &= E\big(\Xi_k(X_1, X_2, \dots, X_k; t)|X_1 = s\big)\\ &= P\big\{\max(s, X_2, \dots, X_k) < t\big\} - \frac{1}{k!}\sum_{\pi(j_1,\dots,j_k)} P\Big\{\frac{s}{j_1} + \frac{X_2}{j_2} + \cdots + \frac{X_k}{j_k} < t\Big\}\\ &= \mathbf{I}\{s < t\}(F(t))^{k-1} - \frac{1}{k}\sum_{j=1}^{k}\Big[\mathbf{I}\{s < jt\}\Big(1 - \sum_{\substack{i\neq j\\ i=1}}^{k}\Big(e^{-i\left(t-\frac{s}{j}\right)}\prod_{\substack{h\neq i,j\\ h=1}}^{k}\frac{h}{h-i}\Big)\Big)\Big],\end{aligned} \quad (13)$$

where $F(t)$ is d.f. of exponential distribution. The calculation of variance for this projection in terms of k is too complicated, therefore we calculate it only for particular cases.

3.1. Kolmogorov-type statistic $D_n^{(2)}$

For $k = 2$ from (13) we get that the projection of the family of kernels $\Xi_2(X, Y; t)$ is equal to

$$\xi_2(s; t) = \mathbf{I}\{s < t\}F(t) - \frac{1}{2}\mathbf{I}\{s < t\}F(2(t - s)) - \frac{1}{2}\mathbf{I}\{s < 2t\}F(t - s/2). \quad (14)$$

Now we calculate the variances of these projections $\delta_2^2(t)$ under H_0. Elementary calculations show that

$$\delta_2^2(t) = \frac{1}{3}e^{-t} - \frac{5}{4}e^{-2t} - \frac{1}{3}e^{-3t} - \frac{1}{12}e^{-4t} - \frac{2}{3}e^{-3t/2} + 2e^{-5t/2} + \frac{1}{2}te^{-2t},$$

and the plot of $\sigma_2^2(t)$ is given in Fig. 1.

Hence our family of kernels $\Xi_2(X, Y; t)$ is non-degenerate as defined in Nikitin (2010) and besides

$$\delta_2^2 = \sup_{t\geq 0}\delta_2^2(t) = 0.02234.$$

Limiting distribution of the statistic $D_n^{(2)}$ is unknown. Using the methods of Silverman (1983), one can show that the U-empirical process

$$\eta_n^{(2)}(t) = \sqrt{n}\left(H_n^{(2)}(t) - G_n^{(2)}(t)\right), \quad t \geq 0,$$

weakly converges in $D(0, \infty)$ as $n \to \infty$ to certain centered Gaussian process $\eta^{(2)}(t)$ with calculable covariance. Then the sequence of statistics $\sqrt{n}D_n^{(2)}$ converges in distribution to the random variable $\sup_{t\geq 0}|\eta^{(2)}(t)|$ but it is currently impossible

Table 3
Critical values for $D_n^{(k)}$, $(n = 100)$.

k	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
2	0.09	0.10	0.12
3	0.12	0.13	0.16

to find explicitly its distribution. Hence it is reasonable to determine the critical values for statistics $D_n^{(2)}$ by simulation. Therefore in Table 3 we give the critical values for Kolmogorov-type statistic $D_n^{(k)}$ for $k = 2$ and $k = 3$ obtained via simulation using 10,000 replicates.

The family of kernels $\{\Xi_2(X, Y; t)\}, t \geq 0$, is centered and bounded in the sense described in Nikitin (2010). Applying the large deviation theorem for the supremum of the family of non-degenerate U- and V-statistics from Nikitin (2010), we get the following result.

Theorem 3. *For $a > 0$ it holds*

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0}(D_n^{(2)} > a) = -f_2(a),$$

where the function f_2 is continuous for sufficiently small $a > 0$, moreover

$$f_2(a) = (8\delta_2^2)^{-1} a^2 (1 + o(1)) \sim 5.595a^2, \quad \text{as } a \to 0.$$

3.1.1. Local Bahadur efficiency of the statistic $D_n^{(2)}$

According to Glivenko–Cantelli theorem for V-statistics (Janssen, 1988) the limit in probability under the alternative for statistics $D_n^{(2)}$ is equal to

$$b_D^{(2)}(\theta) = \sup_{t\geq 0} |b_D^{(2)}(t, \theta)| = \sup_{t\geq 0} \left| P_\theta\big(\max(X, Y) < t\big) - P_\theta\left(X + \frac{Y}{2} < t\right)\right|.$$

Assuming the regularity of the alternative d.f., we can deduce

$$b_D^{(2)}(t, \theta) \sim 2\theta \int_0^\infty \xi_2(s; t) h(s) ds, \quad \text{as } \theta \to 0, \tag{15}$$

where again $h(x) = \frac{\partial}{\partial\theta} g(x, \theta)\,|_{\theta=0}$ and $\xi_2(s; t)$ is the projection from (14).

We now proceed with calculation of local Bahadur efficiencies for our four alternatives.

For Makeham alternative from (15) we get that

$$\begin{aligned} b_D^{(2)}(t, \theta) &\sim \theta\bigg(2\int_0^t F(t)e^{-s}(2 - 2e^{-s} - s)ds - \int_0^t F\big(2(t - s)\big)e^{-s}(2 - 2e^{-s} - s)ds \\ &\quad - \int_0^{2t} F(t - s/2)e^{-s}(2 - 2e^{-s} - s)ds\bigg) \\ &= \theta\left(\frac{2}{3}e^{-t} + (1 - 2t)e^{-2t} - 2e^{-3t} + \frac{1}{3}e^{-4t}\right), \quad \text{as } \theta \to 0, \end{aligned}$$

and the plot of the function $b_2(t)$, the coefficient next to θ in the expression above, is given in Fig. 2.

Thus we have that

$$\sup_{t>0} b_D^{(2)}(t, \theta) = b_D^{(2)}(1.908, \theta) \sim 0.03055\,\theta, \quad \text{as } \theta \to 0.$$

The local exact slope of the sequence $D_n^{(2)}$ as $\theta \to 0$ satisfies

$$c_D^{(2)}(\theta) = \big(b_D^{(2)}(\theta)\big)^2/(4\delta_2^2) \sim 0.0104\,\theta^2.$$

Using $K_{g_1}(\theta)$ from (10), we get that the local BE is equal to

$$e^B(D^{(2)}) = \lim_{\theta\to 0} \frac{c_D^{(2)}(\theta)}{2K_{g_1}(\theta)} \approx 0.125.$$

For other alternatives the calculations are similar. Therefore we omit them and present their local Bahadur efficiencies in Table 4.

We see that the efficiencies are very low, considerably lower than in case of other tests of exponentiality based on characterizations with the exception, apparently, of Nikitin (1996). Probably this is related to intrinsic properties of Arnold–Villasenor characterization. Furthermore, the Kolmogorov-type tests usually demonstrate lower efficiencies than the integral tests (see Henze and Meintanis (2005), Litvinova (2004), Nikitin (1995), Nikitin (1996), Nikitin and Volkova (2010), etc.).

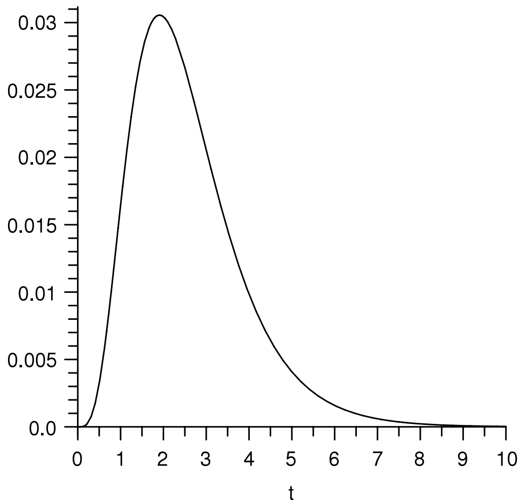

Fig. 2. Plot of the function $b_2(t)$, Makeham alternative.

Table 4
Local Bahadur efficiency for the statistic $D_n^{(2)}$.

Alternative	Efficiency
Makeham	0.125
Weibull	0.092
Gamma	0.093
EMNW(3)	0.149

3.2. Kolmogorov-type statistic $D_n^{(3)}$

For $k = 3$ from (13) we get that the projection of the family of kernels $\Xi_3(X, Y, Z; t)$ is equal to

$$\xi_3(s;t) = \mathbf{I}\{x < t\}\Big[F^2(t) - F(2(t-x)) + \frac{2}{3}F(3(t-x))\Big] - \mathbf{I}\{x < 2t\}\Big[\frac{1}{2}F(t-x/2) - \frac{1}{6}F(3(t-x/2))\Big] - \mathbf{I}\{x < 3t\}\Big[\frac{2}{3}F(t-x/3) - \frac{1}{3}F(2(t-x/3))\Big]. \quad (16)$$

Now we calculate the variances of these projections $\delta_3^2(t)$ under H_0. We get that

$$\delta_3^2(t) = \frac{8}{15}e^{-t} + \Big(\frac{1}{2}t - \frac{1}{24}\Big)e^{-2t} + \Big(\frac{41}{9} - \frac{4}{3}t\Big)e^{-3t} - \frac{179}{210}e^{-4t} + \frac{113}{210}e^{-5t} - \frac{419}{2520}e^{-6t} - \frac{14}{15}e^{-3t/2} + \frac{122}{35}e^{-5t/2} - \frac{2}{3}e^{-7t/2} - \frac{2}{3}e^{-9t/2} - \frac{5}{7}e^{-5t/3} - \frac{5}{2}e^{-7t/3} + \frac{10}{7}e^{-8t/3} - 4e^{-10t/3} - 2e^{-11t/3} + 2e^{-13t/3},$$

and the plot of this function is given in Fig. 3.

Hence our family of kernels $\Xi_3(X, Y, Z; t)$ is non-degenerate in the sense described in Nikitin (2010) and

$$\delta_3^2 = \sup_{t \geq 0} \delta_3^2(t) = 0.02241.$$

Using the same reasoning as in the case $D_n^{(2)}$ we conclude that it is impossible to find explicitly the limiting distribution of the statistic $D_n^{(3)}$. The family of kernels $\{\Xi_3(X, Y, Z; t)\}$, $t \geq 0$, is centered and bounded in the sense given in Nikitin (2010). Applying the large deviation theorem for the supremum of the family of non-degenerate U- and V-statistics from Nikitin (2010), we get the following result.

Theorem 4. *For $a > 0$ it holds*

$$\lim_{n\to\infty} n^{-1} \ln P_{H_0}(D_n^{(3)} > a) = -f_3(a),$$

where the function f is continuous for sufficiently small $a > 0$, moreover

$$f_3(a) = (18\delta_3^2)^{-1}a^2(1 + o(1)) \sim 2.479a^2, \quad \text{as } a \to 0.$$

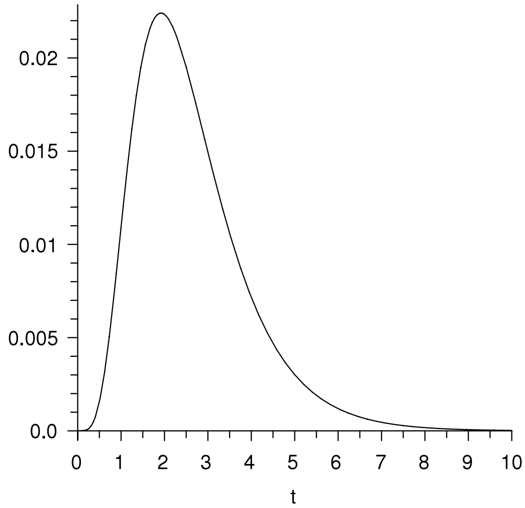

Fig. 3. Plot of the function $\delta_3^2(t)$.

3.2.1. Local Bahadur efficiency of the statistic $D_n^{(3)}$

In this case the limit in probability under the alternative, according to Glivenko–Cantelli theorem for V-statistics (Janssen, 1988), is equal to

$$b_D^{(3)}(\theta) = \sup_{t\geq 0} |b_D^{(3)}(t,\theta)| = \sup_{t\geq 0} \Big|P_\theta\big(\max(X,Y,Z) < t\big) - P_\theta\Big(X + \frac{Y}{2} + \frac{Z}{3} < t\Big)\Big|.$$

It is not difficult to show that $b_D(t,\theta)$ for regular alternatives satisfies the relation

$$b_D^{(3)}(t,\theta) \sim 3\theta \int_0^\infty \xi_3(s;t)h(s)ds, \tag{17}$$

where $h(x) = \frac{\partial}{\partial\theta} g(x,\theta)\,|_{\theta=0}$, and $\xi_3(s;t)$ is the projection from (16).

As in the previous sections we first calculate local BE for Makeham alternative. From (17) we get that

$$\begin{aligned} b_D^{(3)}(t,\theta) \sim \theta\bigg(&\int_0^t \Big[F^2(t) - F(2(t-s)) + \frac{2}{3}F(3(t-s))\Big]e^{-s}(2 - 2e^{-s} - s)ds \\ &- \int_0^{2t}\Big[\frac{1}{2}F(t-s/2) - \frac{1}{6}F(3(t-s/2))\Big]e^{-s}(2-2e^{-s}-s)ds \\ &- \int_0^{3t}\Big[\frac{2}{3}F(t-s/3) - \frac{1}{3}F(2(t-s/3))\Big]e^{-s}(2-2e^{-s}-s)ds\bigg) \\ = \theta\Big(&\frac{8}{5}e^{-t} + \Big(\frac{9}{2} - 6t\Big)e^{-2t} - 8e^{-3t} + 2e^{-4t} - \frac{1}{10}e^{-6t}\Big), \quad \text{as } \theta \to 0, \end{aligned}$$

and the plot of the function $b_3(t)$, the coefficient next to θ in the expression above, is given in Fig. 4.

Therefore we get that

$$\sup_{t>0} b_D^{(3)}(t,\theta) = b_D^{(3)}(2.087,\theta) \sim 0.0602\,\theta.$$

The local exact slope of the sequence $D_n^{(3)}$ as $\theta \to 0$ satisfies

$$c_D^{(3)}(\theta) = \big(b_D^{(3)}(\theta)\big)^2/(9\delta_3^2) \sim 0.018\,\theta^2, \tag{18}$$

and the local BE is equal to $e^B(D^{(3)}) = 0.216$. Omitting again the detailed calculations, we present in Table 5 the values of local Bahadur efficiency for our alternatives.

We see that these efficiencies are slightly better than in the previous case, but still rather low. The simulated powers for our four alternatives are presented in Table 2. Again the simulations have been performed for $n = 100$ with 10,000 replicates.

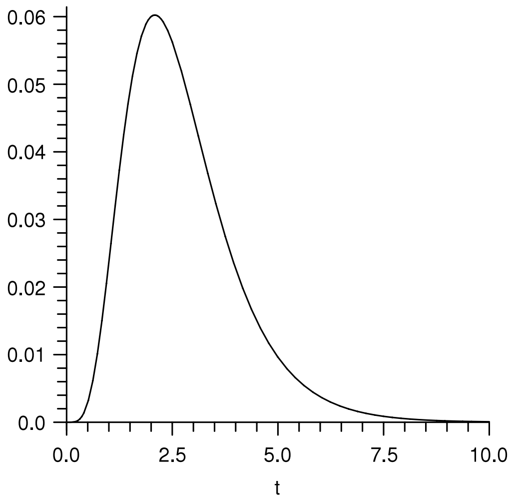

Fig. 4. Plot of the function $b_3(t)$, Makeham alternative.

Table 5
Local Bahadur efficiency for statistic $D_n^{(3)}$.

Alternative	Efficiency
Makeham	0.216
Weibull	0.152
Gamma	0.138
EMNW(3)	0.230

4. Application to real data

In this section we apply our tests to two well-known real data examples.

The first data set represents inter-occurrence times of fatal accidents to British-registered passenger aircraft, 1946–63, measured in number of days and listed in the order of their occurrence in time (see Pyke (1965)):

20 106 14 78 94 20 21 136 56 232 89 33 181 424 14
430 155 205 117 253 86 260 213 58 276 263 246 341 1105 50 136.

Applying our tests to these data, we get the following values of test statistics $I_n^{(k)}$ and $D_n^{(k)}$, as well as the corresponding p-values:

Statistic	$I_n^{(2)}$	$I_n^{(3)}$	$D_n^{(2)}$	$D_n^{(3)}$
Value	0.02	0.02	0.13	0.18
p-value	0.37	0.32	0.28	0.31

so we conclude that the tests do not reject exponentiality.

The second data set represents failure times for right rear breaks on D9G-66A Caterpillar tractors (see Barlow and Campo (1975)):

56 83 104 116 244 305 429 452 453 503 552 614 661 673 683 685 753 763 806
834 838 862 897 904 981 1007 1008 1049 1060 1107 1125 1141 1153 1154 1193
1201 1253 1313 1329 1347 1454 1464 1490 1491 1532 1549 1568 1574 1586 1599
1608 1723 1769 1795 1927 1957 2005 2010 2016 2022 2037 2065 2096 2139 2150
2156 2160 2190 2210 2220 2248 2285 2325 2337 2351 2437 2454 2546 2565 2584
2624 2675 2701 2755 2877 2879 2922 2986 3092 3160 3185 3191 3439 3617 3685
3756 3826 3995 4007 4159 4300 4487 5074 5579 5623 6869 7739.

Applying our tests to these data, we get that all p-values are less than 10^{-2}. Therefore we conclude that all our tests strongly reject the exponentiality of these data.

Table 6
Most favorable alternatives for $I_n^{(k)}$.

	Alternative density $g(x,\theta)$ as $\theta \to +0,\ x \geq 0$
$k=2$	$g(x,\theta) = e^{-x}\big(1 + \frac{\theta}{3}(\frac{4}{3}e^{-x} - e^{-2x} - \frac{1}{2}e^{-x/2})\big)$
$k=3$	$g(x,\theta) = e^{-x}\big(1+\frac{\theta}{4}(\frac{5}{2}e^{-x} - 3e^{-2x} + e^{-3x} - \frac{3}{8}e^{-x/2} - \frac{1}{3}e^{-x/3})\big)$

5. Conditions of local asymptotic optimality

The efficiencies of our tests for standard alternatives are far from maximal ones. Nevertheless, there exist special alternatives (we call them *most favorable*) for which our sequences of statistics $I_n^{(k)}$ and $D_n^{(k)}$ are locally asymptotically optimal (LAO) in Bahadur sense (see general theory in Nikitin (1995, Ch.6)). In this section we describe the local structure of such alternatives, for which the given statistic has maximal possible local efficiency, so that the relation

$$c_T(\theta) \sim 2K(\theta), \quad \text{as } \theta \to 0,$$

holds (see Bahadur (1971), Nikitin (1995), Nikitin and Tchirina (1996), Nikitin and Peaucelle (2004)). Such alternatives form the so-called domain of LAO for the given sequence of statistics $\{T_n\}$.

Denote by $\mathcal{G}$ the class of densities $g(\cdot\,,\theta)$ with the d.f.'s $G(\cdot\,,\theta)$. Define the functions

$$H(x) = \frac{\partial}{\partial\theta}G(x,\theta)\mid_{\theta=0}, \qquad h(x) = \frac{\partial}{\partial\theta}g(x,\theta)\mid_{\theta=0},$$

assuming that the derivatives exist. Suppose also that for G from $\mathcal{G}$ the following stronger regularity conditions hold:

$$h(x) = H'(x), \quad x \geq 0, \qquad \int_0^\infty h^2(x)e^x dx < \infty,$$
$$\frac{\partial}{\partial\theta}\int_0^\infty xg(x,\theta)dx\mid_{\theta=0} = \int_0^\infty xh(x)dx.$$

It is easy to show, see also Nikitin and Tchirina (1996), that under these conditions

$$2K(\theta) \sim \left[\int_0^\infty h^2(x)e^x dx - \Big(\int_0^\infty xh(x)dx\Big)^2\right]\theta^2, \quad \text{as } \theta \to 0.$$

It can be shown that for the statistic (1) holds

$$b_I^{(k)}(\theta) \sim (k+1)\theta\int_0^\infty \psi_k(x)h(x)dx, \quad \text{as } \theta \to 0.$$

Let us introduce the auxiliary function

$$h_0(x) = h(x) - (x-1)\exp(-x)\int_0^\infty uh(u)du. \tag{19}$$

It is straightforward that

$$\int_0^\infty h^2(x)e^x dx - \Big(\int_0^\infty xh(x)dx\Big)^2 = \int_0^\infty h_0^2(x)e^x dx, \tag{20}$$
$$\int_0^\infty \psi_k(x)h(x)dx = \int_0^\infty \psi_k(x)h_0(x)dx.$$

Consequently the local BE takes the form

$$e^B(I_n^{(k)}) = \lim_{\theta\to 0}\frac{\big(b_I^{(k)}(\theta)\big)^2}{2(k+1)^2\Delta_k^2K(\theta)}$$
$$= \Big(\int_0^\infty \psi_k(x)h_0(x)dx\Big)^2 \Big/ \Big(\int_0^\infty \psi_k^2(x)e^{-x}dx \cdot \int_0^\infty h_0^2(x)e^x dx\Big).$$

The local Bahadur asymptotic optimality means that the expression on the right-hand side is equal to 1. It follows from Cauchy–Schwarz inequality (see also Nikitin and Peaucelle (2004)) that this is satisfied if $h_0(x) = C_1e^{-x}\psi(x)$ for some constant $C_1 > 0$, so that $h(x) = e^{-x}(C_1\psi(x) + C_2(x-1))$ for some constants $C_1 > 0$ and C_2. Such distributions constitute the LAO domain in the class $\mathcal{G}$.

The simplest examples of such alternative densities $g(x,\theta)$ for small $\theta > 0$ are given in Table 6.

Table 7
Most favorable alternatives for $D_n^{(k)}$.

	Alternative densities $g(x,\theta)$ as $\theta \to +0$, $x \geq 0$
$k=2$	$g(x,\theta) = e^{-x}\Big[1+\theta\cdot\mathbf{I}\{x<t_0\}\big(1-e^{-t_0}\big) - \frac{1}{2}\theta\cdot\Big(\mathbf{I}\{x<t_0\}\big(1-e^{-2(t_0-x)}\big)+\mathbf{I}\{x<2t_0\}\big(1-e^{-(t_0-x/2)}\big)\Big)\Big]$
$k=3$	$g(x,\theta) = e^{-x}\Big[1+\theta\cdot\mathbf{I}\{x<t_1\}\Big(\big(1-e^{-t_1}\big)^2+e^{-2(t_1-x)}-\frac{2}{3}e^{-3(t_1-x)}-\frac{1}{3}\Big)-\frac{1}{3}\theta\cdot\mathbf{I}\{x<2t_1\}\Big(1-\frac{3}{2}e^{-(t_1-x/2)}+\frac{1}{2}e^{-3(t_1-x/2)}\Big)$ $-\frac{1}{3}\theta\cdot\mathbf{I}\{x<3t_1\}\Big(1-2e^{-(t_1-x/3)}+e^{-2(t_1-x/3)}\Big)\Big]$

Let us now consider the Kolmogorov-type statistic (2). It can be shown that

$$b_D^{(k)}(\theta) \sim k\theta\int_0^\infty \xi_k(x;t)h(x)dx, \quad \text{as } \theta\to 0.$$

For $h_0(x)$ defined in (19), besides (20), also holds

$$\int_0^\infty \xi_k(x;t)h(x)dx = \int_0^\infty \xi_k(x;t)h_0(x)dx.$$

In this case the efficiency is equal to

$$e^B(D_n^{(k)}) = \lim_{\theta\to 0}\frac{\big(b_D^{(k)}(\theta)\big)^2}{\sup_{t\geq 0}\big(2k^2\delta_k^2(t)\big)K(\theta)} = \sup_{t\geq 0}\Big(\int_0^\infty \xi_k(x;t)h_0(x)dx\Big)^2\Big/\sup_{t\geq 0}\Big(\int_0^\infty \xi_k^2(x;t)e^{-x}dx\cdot\int_0^\infty h_0^2e^x dx\Big).$$

From Cauchy–Schwarz inequality we obtain that efficiency is equal to 1 if $h(x) = e^{-x}\big(C_1\xi_k(x;t_0)+C_2(x-1)\big)$ for $t_0 = \text{argmax}_{t\geq 0}\delta_k^2(t)$ and some constants $C_1 > 0$ and C_2. The alternative densities having such function $h(x)$ form the domain of LAO in the corresponding class.

The simplest examples are given in Table 7. To facilitate the presentation, we denote:

$$t_0 = \underset{t\geq 0}{\text{argmax}}\Big(\frac{1}{3}e^{-t}-\frac{5}{4}e^{-2t}-\frac{1}{3}e^{-3t}-\frac{1}{12}e^{-4t}-\frac{2}{3}e^{-3t/2}+2e^{-5t/2}+\frac{1}{2}te^{-2t}\Big)\approx 1.502;$$

$$t_1 = \underset{t\geq 0}{\text{argmax}}\Big[\frac{8}{15}e^{-t}+\Big(\frac{1}{2}t-\frac{1}{24}\Big)e^{-2t}+\Big(\frac{41}{9}-\frac{4}{3}t\Big)e^{-3t}-\frac{179}{210}e^{-4t}+\frac{113}{210}e^{-5t} - \frac{419}{2520}e^{-6t}-\frac{14}{15}e^{-3t/2}+\frac{122}{35}e^{-5t/2}-\frac{2}{3}e^{-7t/2}-\frac{2}{3}e^{-9t/2}-\frac{5}{7}e^{-5t/3}-\frac{5}{2}e^{-7t/3} + \frac{10}{7}e^{-8t/3}-4e^{-10t/3}-2e^{-11t/3}+2e^{-13t/3}\Big]\approx 1.919.$$

6. Discussion

In this paper we have proposed two families of asymptotic tests of exponentiality based on recent characterization of exponentiality by Arnold and Villasenor (2013). The integral test statistics $I_n^{(k)}$ are asymptotically normal and have reasonably simple form which can be easily computed for small k. They are consistent for many common alternatives and have local Bahadur efficiency around 0.5–0.7. There exist also special (most favorable) alternatives described in Section 5 for which the integral statistics are locally asymptotically optimal in this sense.

We also obtained via simulation the power of new integral statistics for chosen alternatives. For each statistic we calculated the powers for $\theta = 0.5$ and $\theta = 0.25$. In case of Makeham distribution we added the case $\theta = 1.5$ to demonstrate that reasonable powers are obtained for more distant alternatives.

In theory, the ordering of tests by power is linked more closely to Hodges–Lehmann efficiency (see Nikitin (1995)), and should not necessarily coincide with the ordering by local Bahadur efficiency. Nevertheless, we observe tolerable correspondence of test quality according to both criteria with somewhat less satisfactory consent in case of Weibull distribution. In whole we can recommend new integral tests of exponentiality as additional and auxiliary tests of exponentiality, especially when one is trying to reject exponentiality in a specific example using a "battery" of statistical tests.

In the case of Kolmogorov type tests the values of local Bahadur efficiency turned out to be rather low for common alternatives, and the simulated powers (which are slightly more optimistic) do not change somewhat disadvantageous regard to new tests of exponentiality of supremum type. Probably it is closely related to intrinsic properties of Arnold–Villasenor characterization. However, even these tests, in virtue of their consistency, can be of some use in statistical research, especially when the (unknown) alternative is close to the most favorable one.

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