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PROPAGATION DYNAMICS OF A DIFFUSING SUBSTANCE ON THE SURFACE AND IN THE BULK OF WATER

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Summary. Analytical solutions of 2D and 3D boundary value problems for the diffusion equation in unbounded domains for the initial condition of special form are obtained. The characteristics of the pollution spot, where the concentration of the diffusing substance is higher than the threshold, is studied.

1 INTRODUCTION

Two and three dimensional problems of propagation of a diffusing substance on the surface and in the bulk of water are considered. Similar questions were discussed earlier in papers devoted to the solution of the heat equation in cylindrical and spherical domains with boundary conditions of the first, second, and mixed kinds [1]. In spite of the presence of extensive literature, the diversity of boundary and initial conditions due to new applied problems leads to boundary value problems that have not been considered earlier. In particular, the necessity to find bounded solutions for an unbounded domain sometimes turns out to be a non-trivial problem, and the existence of threshold values for the sought concentration function leads to the problem of finding the roots of an implicit function.

2 STATEMENT OF THE PROBLEM

Mathematic model of the diffusion substance on the surface and in the bulk of water at initial moment of time is a boundary value problem for the turbulent diffusion equation [2]. This equation should take into account both the advection (the fluid flow with respect to a chosen system of coordinates) and turbulent fluid flow. In assuming for incompressible fluid in the absence of sources (sinks) of the diffusing substance is not take into account influence of current (surface, wind). The medium is isotropic in horizontal plane and taking into account that a turbulent transport of substance is, as a rule, several orders of magnitude than molecular transport, we arrive to the equation of turbulent diffusion for function of concentration of the diffusing substance [3].

3 PROPAGATION OF A DIFFUSING SUBSTANCE ALONG THE SURFACE OF WATER

Propagation of a diffusing substance in the two-dimensional approximation under the condition that the effect of the (surface and/or drift) current is not taken into account, i.e., when $u = 0$, is described by the diffusion equation in which the unknown function is the concentration of the diffusing substance $c = c(x, y, t)$:

$$\frac{\partial c}{\partial t} = K \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right)$$

Here K — the horizontal coefficient of turbulent diffusion, which is assumed constant in what follows, The initial condition is $c(x, y, 0) = c_0(x, y), (x, y) \in D$, where $c_0(x, y)$ — concentration of diffusing substance in D (D — an area of pollution at initial moment of time $t = 0$) and the boundary conditions are $c(x, y, t) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$ and $c(0, 0, t) < \infty$.

For the initial condition of special form analytic solutions to the problems are obtained by the Fourier method followed by the expansion of an arbitrary function in terms of Bessel functions.

$$c(r, t) = \int\limits_0^\infty J_0(\lambda r) J_1(\lambda) e^{-\lambda^2 t} d\lambda.$$

When solving applied problems, it is of interest to study the domain of the surface (pollution spot) in which the concentration of the diffusing substance is greater than a certain value c_m. This threshold value is called the maximum permissible concentration. Fig. 1 represents the results of numerical solution of this equation by the Mathematica package for various values of $c_m (0 < c_m < 1)$.

4 PROPAGATION OF A DIFFUSING SUBSTANCE IN THE BULK OF WATER

In following problem diffusing substance is on the plane bottom of a basin. Make a coordinate system XOY on basin bottom combined the centre of coordinate with center of semisphere of small radius which a diffusing substance takes in initial moment. Diffusing process could be described by equation

$$\frac{\partial c}{\partial t} = K_x \frac{\partial^2 c}{\partial x^2} + K_y \frac{\partial^2 c}{\partial y^2} + K_z \frac{\partial^2 c}{\partial z^2}.$$

In assuming that water medium is transversally isotropic, that is $K = K_x = K_y$ $K_z \ll K$, after scaling of alternatives $x = \sqrt{K} x_1, y = \sqrt{K} y_1$, $z = \sqrt{K_z} z_1$ and arriving to

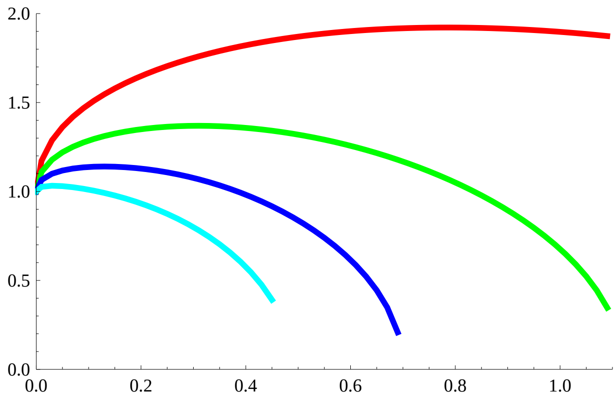

Figure 1: The radius of the pollution spot as a function of time for $c_m = 0.1$ (red line), $c_m = 0.2$ (green line), $c_m = 0.3$ (blue line) and $c_m = 0.4$ (light blue line).

spherical coordinates equation takes the form

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial c}{\partial \theta}$$

with boundary conditions $c(r,\theta,t) \to 0$ as $r \to \infty$, $c(0,\theta,t) < \infty$ and $\frac{\partial c}{\partial \theta} = 0$ for $\theta = \frac{\pi}{2}$ and initial conditions $c(r,\theta,0) = f(r,\theta)$.

General solution of a boundary value problem takes the form

$$c(r,\theta,t) = \sum_{n=0}^{\infty} \int_0^{+\infty} A_{\lambda,n} j_{2n}(\lambda r) P_{2n}(\cos\theta) e^{-\lambda^2 t} d\lambda,$$

where constants $A_{\lambda,n}$ are obtained from the initial condition. If we take as initial condition $c(r,\theta,0) = 1$ for $1, 0 \le r \le 1$ and $c(r,\theta,0) = 0$ for $r > 1$ and we come to following problem solution

$$c(r,t) = \frac{2}{\pi r} \int\limits_0^{\infty} \left(\frac{\sin\lambda}{\lambda} - \cos\lambda \right) \frac{e^{-\lambda^2 t}}{\lambda} \sin \lambda r d\lambda. \tag{1}$$

Analytical solutions to the problems are obtained by the Fourier method followed by the expansion of an arbitrary function in terms of Bessel functions and Legendre polynomials.

The above threshold range of concentrations of the diffusing substance is analyzed. Propagation of the diffusing substance along the free surface and at the bottom of a basin is considered. The analytic solutions constructed are compared with the numerical solutions of a boundary value problem obtained by the software package Mathematica. The size of the pollution spot as a function of time, as well as the effect of geometric and physical parameters used on the spot size, is analyzed. Asymptotic relations for the lifetime of the spot with concentration higher maximum permissible concentration are constructed for two-dimension problem $T = (4c_m)^{-1}$ and three-dimension problem $T = (6^2 c_m^2 \pi)^{-1/3}$ [4].

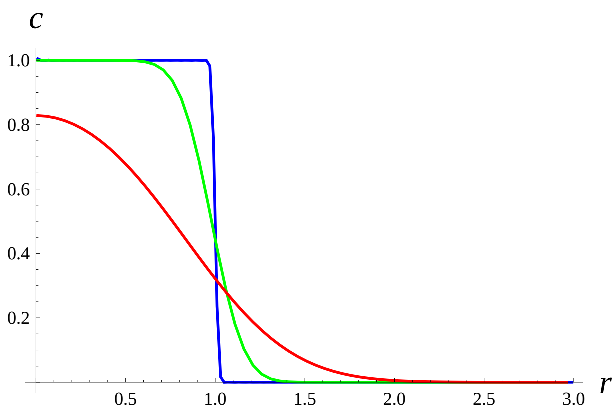

Figure 2: The function $c(r,t)$ at time points $t = 0.0001$ (blue line), $t = 0.01$ (green line), and $t = 0.1$ (red line).

In Fig. 2, the value of concentration function $c(r,t)$ are calculated by formula (1) for various values of time. As time increases, the contribution of large λ decreases, and the required accuracy of calculation is achieved for small values of the upper limit of the integral (1).

5 CONCLUSIONS

We have obtained an analytic solution to boundary value problems for the diffusion equation in unbounded domains with the initial condition of special form. We have found a domain of the diffusing substance with concentration higher than a threshold value.The solution of this problem has an important applied value in the problem of environmental protection in emergency situations on ships.

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