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MARINE PHYSICS

Peculiarities of Formation the of Density Field in Mesoscale Eddies of the Lofoten Basin: Part 1

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Abstract—Variations in the Brunt–Väisälä frequency caused by mesoscale eddies of different polarity are analyzed. The Lofoten Basin in the Norwegian Sea is the study region. Two eddies are considered: the Lofoten eddy, an anticyclone located in the center of the basin, and a cyclone located southeast of the Lofoten eddy. The data of the GLORYS12V1 oceanic reanalysis for June 10, 2010, are used in the analysis. In the considered eddies, profiles, vertical sections of potential vorticity, and the Brunt–Väisälä frequency, as well as the associated characteristics of seawater, are investigated. Theoretical estimates of the Brunt–Väisälä frequency restructured by eddies of different polarity are proposed. It is shown that potential vorticity of anticyclones should have a limit. The properties of the considered eddies, which cannot be measured directly and cannot be found from the models, are estimated indirectly by joint analysis of field measurements and theoretical material. The article is divided into two parts. The first part presents the elements of the theory and derives and discusses the mathematical relations used in the second part needed to calculate the characteristics of eddies of the Lofoten Basin from field data.

Keywords: mesoscale eddies, potential vorticity, Brunt–Väisälä frequency, buoyancy frequency, Lofoten Basin

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1. INTRODUCTION

This article focuses on the influence of oceanic mesoscale eddies on variation of the Brunt–Väisälä (BV) frequency. The BV frequency (or buoyancy frequency) N in the ocean characterizes the stability of water mass stratification. It increases with increasing stratification of seawater and decreases in weakly stratified waters. The value of the BV frequency becomes negative at unstable stratification, when denser waters are located above less dense ones.

Vertical extension or contraction of liquid particles leads to the variations in the BV frequency. Based on the theory of quasi-geostrophic vortices, it is possible to obtain theoretical expressions for the BV frequency in the eddies. To verify the theoretical conclusions, two eddies are considered: a cyclone and an anticyclone, located in the Lofoten Basin of the Norwegian Sea. The goal of this work is to theoretically and practically assess the variations in the BV frequency associated with the influence of mesoscale ocean eddies on density stratification. Thus, the paper analyzes the variability of the BV frequency in the eddy core for eddies of different polarity, as well as the characteristics associated with these variations.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

2.1. Potential Vorticity

Let us assume that a mesoscale eddy formed in a stratified ocean at rest and that the eddy consists of an eddy core and surrounding fluid entrained in the motion. In the ocean at rest, in the absence of eddies, isopycnals are horizontal surfaces. The presence of eddies leads to deformation and displacement of these isopycnals, and those parts of the isopycnic surfaces that fall inside the eddy core (or are near the core) would undergo the strongest deformation.

It is known that cyclonic eddies bring isopycnals closer together, pull them together in the horizontal plane, pulling the near outer isopycnals into the eddy core, and this increases the BV frequency in the eddy core. At the same time, anticyclonic eddies push the isopycnals apart, pushing them out of the eddy core, which leads to a decrease in the BV frequency in the core and an increase in it below and above the eddy core. Thus, the evolution of isopycnic surfaces has a different effect on the BV frequency distributions in cyclones and anticyclones. These aspects are discussed in this article.

We assume that the total seawater pressure $P(x,y,z,t)$ consists of the hydrostatic pressure at rest $P_0(z)$ and deviations from the hydrostatic pressure at rest $p(x,y,z,t)$: $P(x,y,z,t) = P_0(z) + p(x,y,z,t)$. We similarly divide the density field into two terms $\rho(x,y,z,t) = \rho_0(z) + \rho(x,y,z,t)$. Functions $p(x,y,z,t)$ and $\rho(x,y,z,t)$ are related to water motion, and when the motion is zero, they are identically equal to zero. The coordinate axes are selected as follows: the x axis is directed to the east, the y axis is directed to the north, and the z axis is directed upwards; t is time. Since $P(x,y,z,t)$ and stream function $\psi(x,y,z,t)$ differ only by a constant multiplier in the quasi-geostrophic approximation $\rho_* f$ (f is the Coriolis parameter, which is assumed positive; ρ_* is the mean density of seawater over depth), in the future, we will consider both functions equivalent. At small Rossby numbers Ro < 1, the problem is reduced to a single equation for pressure $p(x,y,z,t)$, which is called equation of conservation of potential vorticity:

$$\frac{\partial}{\partial t}\left(\Delta_h p + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial p}{\partial z}\right) + \frac{1}{\rho_* f}\mathcal{J}_h\left(p, \Delta_h P + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial p}{\partial z}\right) = 0. \tag{1}$$

Here, $\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplacian, $N = N(z)$ is the BV frequency depending on the vertical coordinate, $\mathcal{J}_h(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial y}$ is the Jacobian with respect to horizontal coordinates.

The horizontal velocity components $\vec{u} = (u,v)$ are determined from the geostrophic relations,

$$u = -\frac{1}{\rho_* f}\frac{\partial p}{\partial y}, \tag{2}$$

$$v = \frac{1}{\rho_* f}\frac{\partial p}{\partial x}, \tag{3}$$

and the density field is also related to the hydrostatic equations:

$$-\frac{1}{\rho_*}\frac{\partial p}{\partial z} - g\frac{\rho}{\rho_*} = 0. \tag{4}$$

When using Eqs. (1)–(4) together, we assume that their accuracies are the same, i.e., of order $O(\text{Ro})$.

Equation (1) describes conservation of the value of $q = \Delta_h p(x,y,z,t) + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial p(x,y,z,t)}{\partial z}$ for a moving liquid particle. In other words, this means that q is a Lagrangian invariant in quasi-geostrophic processes. This Lagrangian invariant q was first obtained by C.-G. Rossby and was called the "potential vortex" or "potential vorticity." In terms of the stream function, it can be written as $\sigma = \frac{q}{\rho_* f}$, i.e.,

$$\sigma = \Delta_h \psi(x,y,z,t) + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial \psi(x,y,z,t)}{\partial z}, \tag{5}$$

where, $\Delta_h \psi = \text{curl}_z \vec{u}$. The first term in (5) describes rotation of a liquid particle, the second describes the vertical compression–tension effect. According to its physical meaning, expression (5) is the law of conservation of angular momentum for a deformable liquid body. Potential vorticity σ has dimension c^{-1} that coincides with the dimension of the speed curl. Then, to simplify the problem, we assume that the BV frequency at rest is constant: $N_0 = \text{const}$, unless otherwise stated.

Potential vorticity σ at a constant BV frequency can be written as

$$\sigma = \text{curl}_z \vec{u} + \frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2} \tag{6}$$

or in other form as,

$$\sigma = \Delta_h \psi(x,y,z,t) + \frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial \tilde{z}^2} = \Delta\psi,$$

where $\tilde{z} = \frac{N_0}{f} z$ is extended vertical coordinate; Δ is three-dimensional Laplacian in the space with the extended vertical axis. Relation (6) can be considered the formulation of the problem of finding the stream function for an eddy with any shape of the core V, if we assume that the potential vorticity is uniform for all particles of the core, and outside the core the potential vorticity is zero:

$$\Delta\psi = \begin{cases} \sigma, \text{ if } (x,y,\tilde{z}) \in V \\ 0, \text{ if } (x,y,\tilde{z}) \notin V. \end{cases}$$

The core of eddy V may be deformed; in this case the boundary of the core ∂V would parametrically depend on time.

Since problem (6) is linear with respect to parameter σ, then terms $\text{curl}_z \vec{u}$ and $\frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2}$ are also linear with respect to σ. Even more, they must be of the same sign coinciding with the sign of σ. Thus, the absolute value of each of terms $\text{curl}_z \vec{u}$ and $\frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2}$ is limited by the value of $|\sigma|$, and their sum is also limited, i.e., if the absolute value of one of them increases, the second one should decrease and visa-versa. The shares of $\mathrm{curl}_z\vec{u}$ and $\frac{\partial^2\psi(x,y,\tilde{z},t)}{\partial\tilde{z}^2}$ in the potential vorticity depend on the dimensionless geometric parameters of the eddy core. The linear dimensions of the core (length, width, and height) correspond to two independent dimensionless parameters, which are the natural standard dimensionless geometric parameters of the core: horizontal extension of the core and its vertical flattening. We will rely on them in further considerations. The cores of a more complex structure leading to additional dimensionless parameters will not be considered here.

2.2. Density Field in Eddies and Brunt–Väisälä Frequency

Let us describe the variations in the density field associated with the seawater motion. Hydrostatic approximation (4) is well satisfied for mesoscale ocean phenomena, which is conveniently written in terms of the stream function:

$$\frac{\partial\psi(x,y,z,t)}{\partial z} = -\rho(x,y,z,t)\frac{g}{\rho_* f}. \tag{7}$$

Let us calculate a derivative from $\rho(x,y,z,t)$ with respect to z and express it through $\frac{\partial^2\psi(x,y,\tilde{z},t)}{\partial\tilde{z}^2}$. We obtain

$$\frac{\partial\rho(x,y,z,t)}{\partial z} = -\frac{\rho_* N_0^2}{gf}\frac{\partial^2\psi(x,y,\tilde{z},t)}{\partial\tilde{z}^2}. \tag{8}$$

The value of $\frac{g}{\rho_*}\frac{\partial\rho(x,y,z,t)}{\partial z}$ is related to the variation in the squared BV frequency:

$$-\frac{g}{\rho_*}\frac{\partial\rho(x,y,z,t)}{\partial z} = N^2(x,y,z,t) - N_0^2. \tag{9}$$

As a result, we obtain a new distribution of the BV frequency:

$$N^2(x,y,z,t) = N_0^2\left\{1 + \frac{1}{f}\frac{\partial^2\psi(x,y,\tilde{z},t)}{\partial\tilde{z}^2}\right\}. \tag{10}$$

Equation (10) shows the new frequency $N^2(x,y,z,t)$ distribution associated with mesoscale motion with the stream function $\psi(x,y,\tilde{z},t)$ in a fluid with an initially constant frequency N_0^2. The positive values of the second term in the parentheses of expression (10), due to convergence of the isopycnals, is accompanied by an increase in the BV frequency. This is observed in cyclones. Conversely, the negative values observed in anticyclones should correspond to a decrease in the BV frequency; in this case, the isopycnals would move apart, and the eddy core would become more uniform in density. Taking into account the fact that the density distribution must be stable, it follows from (10) that the term $\frac{1}{f}\frac{\partial^2\psi}{\partial\tilde{z}^2}$ should have a lower limit of -1. In turn, expression $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ should linearly depend on the potential vorticity σ.

We consider the corresponding dimensionless coefficient ε:

$$\frac{\partial^2\psi}{\partial\tilde{z}^2} = \varepsilon\sigma, \tag{11}$$

which depends on the geometry of the core (depends on two dimensionless numbers – the horizontal extension of the core and vertical flattening); its values are within $0 < \varepsilon < 1$. Hence, the absolute value of potential vorticity of anticyclonic eddies is also bounded,

$$\frac{\varepsilon|\sigma_{\max}|}{f} = 1, \tag{12}$$

and the limiting potential vorticity $\sigma_{\max}$ is related to the geometry of the core by means of the proportionality coefficient ε. Variations in the core shape leads to the variation in the absolute value of the maximum permissible potential vorticity of anticyclone $\sigma_{\max}$. According to relations (5) and (11), we write relative vorticity through ε and σ:

$$\mathrm{curl}_z\vec{u} = (1-\varepsilon)\sigma. \tag{13}$$

It follows that the relative vorticity of anticyclone is also limited by

$$|\mathrm{curl}_z\vec{u}|_{\max} = \frac{1-\varepsilon}{\varepsilon}f. \tag{14}$$

Taking into account that the applied theory is valid for small Rossby numbers and the maximum possible Rossby number for anticyclones according to (14) is given by the relation $\mathrm{Ro}_{\max} = \frac{|\mathrm{curl}_z\vec{u}|_{\max}}{f} = \frac{1-\varepsilon}{\varepsilon}$, we should expect that this approach is valid for numbers $\varepsilon \sim 1$. There is no such restriction for cyclones. If the realistic values of the velocity curl in anticyclones do not reach critical values $|\mathrm{curl}_z\vec{u}|_{\max}$, then the Rossby number also would be smaller than $\mathrm{Ro}_{\max}$, and restriction $\varepsilon \sim 1$ could be abandoned. Lastly, the relative vorticity $\mathrm{curl}_z\vec{u}$ and $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ in the interior of a homogeneous eddy core are interrelated by

$$\frac{\partial^2 \psi}{\partial \tilde{z}^2} = \frac{\varepsilon}{1-\varepsilon} \mathrm{curl}_z \vec{u}, \quad (15)$$

which can be used together with (10) to estimate the variation in the BV frequency in the eddy core:

$$N^2 = N_0^2 \left\{ 1 + \frac{\varepsilon}{1-\varepsilon} \frac{\mathrm{curl}_z \vec{u}}{f} \right\} = N_0^2 \left\{ 1 + \mathrm{sgn}(\sigma) \frac{\varepsilon}{1-\varepsilon} \mathrm{Ro} \right\}, \quad (16)$$

Here, $\mathrm{Ro} = \frac{|\mathrm{curl}_z \vec{u}|}{f}$ is the Rossby number, which is considered a small parameter of the problem; it is calculated from a characteristic value of $\mathrm{curl}_z \vec{u}$ in the eddy core. As noted above, an additional limitation in the quasi-geostrophic approach is that $\mathrm{sgn}(\sigma) = -1$ in anticyclones, and the expression in curly braces (16) must remain positive. This leads to a restriction on parameter ε and the Rossby number

$$\varepsilon < \frac{1}{1+\mathrm{Ro}}, \quad (17)$$

the sense of which is that in anticyclones the share to the potential vorticity of term $\frac{\partial^2 \psi}{\partial \tilde{z}^2}$ associated with the vertical extension of the liquid particle should not be too large. Looking ahead, we note that in our case, the estimates of the Rossby number for both selected eddies turned out to be approximately the same $\mathrm{Ro} \cong 0.3$, and constraint (17) for anticyclones reduces to inequality $\varepsilon < 0.77$. Otherwise, the density stratification in the core of the anticyclonic eddy would be unstable.

Relation (16) can be used to estimate the contribution of $\frac{\partial^2 \psi}{\partial \tilde{z}^2}$ based on the measured BV frequency profiles N^2 in the anticyclone core and unperturbed BV frequency profile N_0^2 outside the eddy. We express parameter ε from equality (16), taking into account the fact that $\mathrm{sgn}(\sigma) = -1$ in anticyclones:

$$\varepsilon = \frac{\delta}{\mathrm{Ro} + \delta}, \quad (18)$$

$$\text{where, } \delta = \frac{N^2 - N_0^2}{N_0^2}. \quad (19)$$

The physical sense of parameter δ is relative variations in the squared BV frequency due to the influence of the eddy on the density field. Anticyclones make the density profile more uniform; therefore, $N_0^2 > N^2$ everywhere on the levels of the eddy core; and therefore, in the case of an anticyclone, it is more convenient to use the absolute value of difference $|N^2 - N_0^2|$ in relation (19). Relation (16) was obtained under the assumption of the constant background BV frequency, N_0^2 and ε; hence, N^2 is also constant. Of course, in reality, all these characteristics depend on the vertical coordinate z.

Relation (10) makes it possible to estimate the extreme variations in the BV frequency caused by eddies. Since in cyclones the maximum possible value of the term associated with the vertical compression–expansion effect $\frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2}$ is equal to potential vorticity $\sigma > 0$, the maximum BV frequency in the cyclone is given by the relation $N^2 = N_{\max}^2 = N_0 \left(1 + \frac{\sigma}{f}\right)$. Similarly, for anticyclones, $\frac{\partial^2 \psi(x,y,\tilde{z},t)}{\partial \tilde{z}^2}$ cannot be smaller than $-|\sigma| < 0$.

Hence, the minimum BV frequency in anticyclones is $N^2 = N_{\min}^2 = N_0 \left(1 - \frac{|\sigma|}{f}\right)$ with an additional constraint on potential vorticity $|\sigma| \le f$. A complete or almost complete density homogeneity in the anticyclone core is possible under the condition $\sigma \approx -f$. These restrictions on the BV frequency were obtained under the assumption that the Rossby number $\mathrm{Ro} = \frac{|\mathrm{curl}_z \vec{u}|}{f}$ is small, which is taken into account in relation (16), which also shows the effect of the eddy core shape on the resulting BV frequency through the dimensionless parameter ε.

We note an important property: our theory of deformation of the BV frequency field is based on the fact that the eddy core can change its shape; rotation of the core particles can also change, but water with a different background stratification cannot enter the core. However, in reality, the eddy core can be fed by "foreign" water with its own density and potential vorticity distributions. In our approach, we do not take into account the variations in the BV frequency associated with this phenomenon.

2.3. Ellipsoidal Eddies

If we suggest an expression for $\psi(x,y,\tilde{z},t)$ from the theory of ellipsoidal vortices as a stream function in formula (10), then we would obtain the corresponding theoretical estimates of the variations in the BV frequency caused by the influence of oceanic eddies on the density field [4, 1, 2]. We present without derivation the expressions for $\mathrm{curl}_z \vec{u}$ and $\frac{\partial^2 \psi}{\partial \tilde{z}^2}$ needed for the points inside the eddy core:

$$\mathrm{curl}_z\vec{u} = \frac{1}{2}\sigma K\int_0^\infty \frac{(2\tilde{\mu}+\nu)\,d\tilde{\mu}}{(K^2+\tilde{\mu})^{\frac{1}{2}}(\tilde{\mu}^2+\nu\tilde{\mu}+1)^{\frac{3}{2}}}, \qquad (20)$$

$$\frac{\partial^2\psi}{\partial\tilde{z}^2} = \frac{1}{2}\sigma K\int_0^\infty \frac{d\tilde{\mu}}{(K^2+\tilde{\mu})^{\frac{3}{2}}(\tilde{\mu}^2+\nu\tilde{\mu}+1)^{\frac{1}{2}}}. \qquad (21)$$

As expected, both expressions for $\mathrm{curl}_z\vec{u}$ and $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ are linear with respect to the parameter of potential vorticity of the core σ; they do not depend on the coordinates of the considered point of the core (i.e., they are homogeneous over the volume of the core), but depend on two dimensionless parameters: parameter of horizontal extension $\nu = \frac{a}{b} + \frac{b}{a}$, which is symmetric with respect to the choice of large and small horizontal semiaxes of the ellipsoid a and b, and parameter of vertical flattening $K = \frac{\tilde{c}}{\sqrt{ab}} = \frac{N_0}{f}\frac{c}{\sqrt{ab}}$. Here, c is the vertical semiaxis of the ellipsoid, and $\tilde{c}$ is the vertical semiaxis stretched by $\frac{N_0}{f}$ times. Independence of (20) and (21) with respect to coordinates leads to the fact that, inside the eddy cores, the calculated BV frequency appears constant, increased or decreased compared to the background BV frequency, depending on the eddy's polarity.

We emphasize that during the evolution of an ellipsoidal eddy in an equally vortical barotropic flow, the product ab = const is conserved; the size of the vertical semiaxis remains unchanged; therefore, parameter K does not change when the eddy is deformed by a barotropic flow [3, 2]. However, parameter ν increases without limits when the eddy is extended horizontally. The minimum value $\nu = 2$ corresponds to a round eddy with equal semiaxes $a = b$.

The general solution to problem (6) has important properties: it is continuous together with the first derivatives in the entire space for any shape of the core. Hence, the property follows that the velocity field and the density field are continuous for any shape of the core and remain continuous under an arbitrary deformation of the eddy core. In the general case, the BV frequency formed by the eddy would not be constant inside the core, but would depend both on the coordinates and on the same dimensionless geometric parameters: parameter of horizontal extension and parameter of vertical flattening.

Since the terms in the sum $\mathrm{curl}_z\vec{u}$ and $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ from (20) and (21) result in a constant σ, and also considering that both of them are of the same sign, it is sufficient to consider one expression (the simpler one). In particular, it is easy to conclude from relation (21) that the absolute value of $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ monotonically decreases to zero with increasing parameter ν at any fixed parameter K. This means that the absolute value of $\mathrm{rot}_z\vec{u}$ at fixed K and increasing ν monotonically increases without exceeding $|\sigma|$. According to relation (10), parameter $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ is responsible for the stratification variations; therefore, it is important to find the conditions under which the absolute values of $\frac{\partial^2\psi}{\partial\tilde{z}^2}$ would be the largest. As follows from (21), at any fixed value of K, this parameter is maximum for eddy cores, which are round horizontally, i.e., when parameter $\nu = 2$. Hence, the maximum changes in the density field should be expected from round eddies in the horizontal plane.

Next, let us study the eddies that are round in the horizontal plain, depending on the value of K. To do this we substitute $\nu = 2$ in relations (20) and (21). As a result, the ratio for the curl of the velocity in the eddy core will be written as

$$\mathrm{curl}_z\vec{u} = \sigma K\int_0^\infty \frac{d\tilde{\mu}}{(1+\tilde{\mu})^2\sqrt{K^2+\tilde{\mu}}}. \qquad (22)$$

Integral (22) depends only on the vertical parameter of flattening. We will call an eddy "thick" if its parameter of flattening is $K \ge 1$, and "thin" if $K < 1$. Small values of K correspond to thin eddies, large values, to thick ones. The concepts of thin and thick eddies in this case are rather mathematical and need to be clarified, which will be done later. Integral (22) is calculated as

$$\mathrm{curl}_z\vec{u} = \begin{cases} \dfrac{\sigma K}{(1-K^2)^{\frac{3}{2}}}\left[\dfrac{\pi}{2} - K\sqrt{1-K^2} - \arctan\dfrac{K}{\sqrt{1-K^2}}\right] & \text{at} \quad 0 < K < 1 \\ \dfrac{2}{3}\sigma & \text{at} \quad K = 1 \\ \dfrac{\sigma K^2}{K^2-1} + \dfrac{\sigma K}{(K^2-1)^{\frac{3}{2}}}\ln(K - \sqrt{K^2-1}) & \text{at} \quad K > 1. \end{cases} \qquad (23)$$

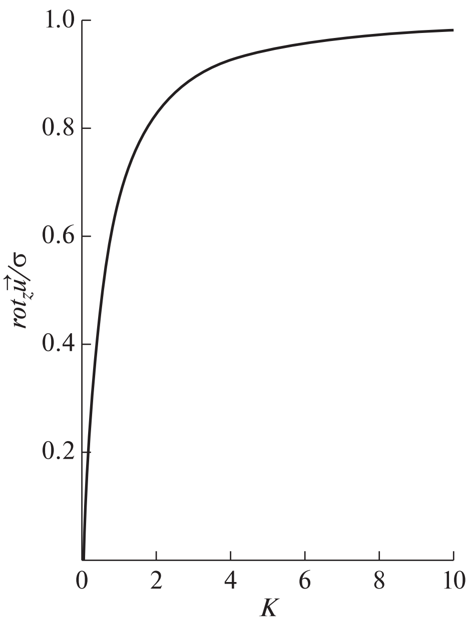

Fig. 1. Dimensionless parameter $\frac{\text{curl}_z\vec{u}}{\sigma}$ versus parameter of vertical flattening K for eddies with axial symmetry. Horizontal axis represents parameter K; vertical axis, $\frac{\text{curl}_z\vec{u}}{\sigma}$; $\frac{\text{curl}_z\vec{u}}{\sigma} = 0$ at $K = 0$ and $\frac{\text{curl}_z\vec{u}}{\sigma} \to 1$ at $K \to \infty$.

In the first equation (23), we can use the properties of inverse trigonometric functions $\arctan\frac{K}{\sqrt{1-K^2}} = \arcsin K = \frac{\pi}{2} - \arccos K$ and simplify expression $\frac{\pi}{2} - \arctan\frac{K}{\sqrt{1-K^2}} = \arccos K$.

We are generally interested in the range of variations $0 < K < 1$ in thin eddies. Figure 1 plotted from relations (23) shows the dependence of dimensionless parameter $\frac{\text{curl}_z\vec{u}}{\sigma}$ on the parameter of vertical flattening K for eddies with axial symmetry. Small values of $\frac{\text{curl}_z\vec{u}}{\sigma}$ and correspondingly large values of $\frac{1}{\sigma}\frac{\partial^2\psi}{\partial\tilde{z}^2} \cong 1$ correspond to thin eddies, in which the values of K are small. Since the maximum values of $\left|\frac{\partial^2\psi}{\partial\tilde{z}^2}\right|$ correspond to thin eddies, one should expect that extreme variations in the BV frequency would be observed precisely in the cores of thin eddies. This is easy to explain. It is easier to "push" isopycnal out of the core of anticyclones or "pull" isopycnals into the core of cyclones, provided that the vertical size of the cores is small.

In light of the previous discussion, inequality (17) acquires an additional physical interpretation: an anticyclone with a completely uniform density in the core cannot be too thin. Complete homogeneity occurs if $\varepsilon = \varepsilon_0 = \frac{1}{1+\text{Ro}}$.

2.4. Cores with "Complex Interiors"

Let us consider a more complex structure of the eddy core structure. We will consider oval cores, which still have three characteristic sizes: length a, horizontal width b, and vertical thickness c. Similarly to ellipsoidal eddies, these values can be used to form two dimensionless parameters: parameter of horizontal extension $\nu = \frac{a}{b} + \frac{b}{a}$ and parameter of vertical flattening K. Then, we will assume that the core is filled with liquid particles of different potential vorticity. By virtue of the conservation law (5), for each particle, we write the following relation:

$$\text{curl}_z\vec{u} + \frac{\partial^2\psi(x, y, \tilde{z}, t)}{\partial\tilde{z}^2} = \sigma. \quad (24)$$

The apparent simplicity of Eq. (24) is masked by the fact that the left side of this equation is written in the Eulerian approach, and the right side is a Lagrangian characteristic and is preserved only along the trajectory of the selected particle. For other particles, the σ values would be different. Only in the case of a uniform distribution of the potential vorticity in the core, when σ for all particles would be the same, this equation can be considered completely in the Eulerian approach, as it is written in the formulation of problem (6). In the case of an inhomogeneous smooth distribution of σ in the core, we can plot surfaces of equal potential vorticity in the core. Each particle must be on its surface, moving along it and moving along with this surface. At the same time, each particle, when moving, is on a surface of equal potential density and also does not leave it. Thus, the trajectory of each particle is the line of intersection of two surfaces: a surface of equal potential vorticity and equal potential density. The trajectories of motion of particles of the eddy core are limited by the size of the core and completely fill the volume of the core. The integral potential vorticity of the core is retained for any deformation of the considered surfaces and the core as a whole.

Let us average relation (6) over the volume of the core or over all trajectories inside the core (which in this case is the same). As a result, we obtain the relation between the averaged characteristics:

$$\langle\text{curl}_z\vec{u}\rangle + \left\langle\frac{\partial^2\psi}{\partial\tilde{z}^2}\right\rangle = \langle\sigma\rangle. \quad (25)$$

In this case $\langle\text{curl}_z\vec{u}\rangle$ and $\left\langle\frac{\partial^2\psi}{\partial\tilde{z}^2}\right\rangle$ conserve the dependence on the form of the core, while mean potential vorticity of the core $\langle\sigma\rangle$ is not related to the form of the core. Now, relation (25) can be interpreted as correlation in the Eulerian approach. The physical senses of terms $\frac{1}{\langle\sigma\rangle}\langle \mathrm{curl}_z\vec{u}\rangle$ and $\frac{1}{\langle\sigma\rangle}\left\langle\frac{\partial^2\psi}{\partial\tilde{z}^2}\right\rangle$ are the contributions of rotation and extension effects to the mean potential vorticity of the core. When the core is deformed, these contributions can change.

Let us introduce the notation of these contributions by means of parameter ε with a natural restriction $0 < \varepsilon < 1$:

$$\frac{1}{\langle\sigma\rangle}\left\langle\frac{\partial^2\psi}{\partial\tilde{z}^2}\right\rangle = \varepsilon, \quad \frac{1}{\langle\sigma\rangle}\langle \mathrm{curl}_z\vec{u}\rangle = 1-\varepsilon. \tag{26}$$

Here, we suppose that signs of $\langle \mathrm{curl}_z\vec{u}\rangle$ and $\left\langle\frac{\partial^2\psi}{\partial\tilde{z}^2}\right\rangle$ coincide with the sign of $\langle\sigma\rangle$. This property is valid for ellipsoidal eddies that are homogeneous in potential vorticity and due to the continuity of their properties would remain for more complex eddies both in shape and in core filling with weak variations in the properties of the eddy. Situations in which the filling of eddy would be complex (for example, cores with potential vorticity of particles of opposite signs, etc.), which can affect the sign of ε, will not be considered in this work. Note that the previously introduced parameter ε in relation (11) and the one just proposed in relation (26) have the same physical meaning: this is the part of the contribution of the effect of stretching of liquid particles to the potential vorticity of the eddy core; therefore, they have the same notation. The only difference is that in (11), a core homogeneous in potential vorticity is considered, and in (26) we consider a more complex core consisting of particles of different potential vorticity. This approach makes it possible to operate with core-averaged characteristics, similarly to the same parameters in the eddies in which their cores have equal vorticity.

2.5. Influence of the Ocean Surface and Imaging Method

If the eddy core is far from the ocean surface and the bottom, then its influence due to the induced velocity propagates vertically up and down by approximately one of its vertical sizes. The same happens in the horizontal direction: notable induced velocities are limited by a zone approximately equal to its diameter from the peripheral edge of the eddy core. Therefore, an eddy, the upper boundary of the core of which is located deeper than the vertical size of the core, would not feel the presence of the sea surface. Similarly, if the distance from the lower boundary of the eddy core is greater than the vertical size of this core, the eddy would not feel the presence of the bottom. Therefore, the ocean can be considered vertically unlimited for such eddies.

Let us consider an example in which, within the framework of the quasi-geostrophic approximation, it is possible to accurately take into account the presence of a solid cover on the ocean surface. First, let us consider a vertically unbounded stratified rotating ocean and an ellipsoidal eddy with two horizontal and one vertical axes. Let the eddy center be located for definiteness at point (0, 0, 0). In this case, the horizontal coordinate plane (X, 0, Y) is the plane of symmetry of the eddy, and the vertical component of the velocity on this plane would be equal to zero. Due to this property, it is possible to place a solid cover instead of the plane of symmetry and, discarding the upper semiellipsoid, take into account only the lower part of the core: the lower semiellipsoid. As a result, we obtain a near-surface semi-ellipsoidal eddy, the evolution of which would completely repeat the behavior of the lower part of the whole ellipsoid. All properties assigned to the whole ellipsoid would be repeated for the near-surface semi-ellipsoidal eddy. Mathematically, the method that was applied here is the imaging method developed in the theory of gravitational potential. In practice, in our case, the image method means doubling of K.

The imaging method can also be applied in a more complex situation. Let the ocean surface correspond to depth $z = 0$. Let also the center of the ellipsoidal eddy be located at depth $z = z_0 < 0$ below the ocean surface. Moreover, $|z_0| > c$; i.e., the eddy core is completely submerged in water and does not come out to the surface. In our case, c is the vertical semiaxis of the ellipsoid. We will proceed as follows to maintain the vertical component of velocity equal to zero at $z = 0$. Let us extend the water layer above sea level $z = 0$. In this layer, we will place exactly the same fictitious eddy of the same sign at depth $z = -z_0$; let us call it a "mapped eddy.".Then, plane $z = 0$ would be a plane of symmetry for vertically unlimited water space and two identical the eddies: real and mapped. The vertical velocity on the plane of symmetry must be zero. Therefore, the problem of the behavior of the eddy under the sea surface becomes identical to the problem of the behavior of two identical eddies located on the same vertical in an unlimited water space. In this case, the resulting BV frequency in the cores of the eddies would no longer be constant. Indeed, the deformation of isopycnic surfaces would already depend on two eddies: the real one and the mapped one.

Let us consider a real eddy; let it be an anticyclone for definiteness. The isopycnals above surface $z = z_0$ would ascend and move apart with respect to this plane, as well as the isopycnals below this surface. If the mapped eddy does not exist, then the descent and expansion of the isopycnals would be symmetrical with respect to level $z = z_0$. The presence of the mapped eddy would additionally lower and expand the isopycnals inside the core of the real eddy. In this case, in the upper part of the core, above $z = z_0$, the effects of deformation of isopycnal surfaces would be oppositely directed and, therefore, here, theoretically, the BV frequency would be somewhat higher than in the absence of the mapped eddy. In addition, the BV frequency would no longer be constant over the volume of the upper half of the real eddy. In the lower part of the real eddy, the effects of deformation of isopycnals induced by both eddies would be added; therefore, the BV frequency here would be smaller than in the case if the mapped eddy was absent. The BV frequency would not be constant inside the bottom part of the real eddy either.

Thus, the presence of a solid cover on the free surface of the sea would lead to the next qualitative restructuring of the density field in the core of a real eddy. In the upper part of the eddy, the BV frequency should be higher than in the lower part. This asymmetry would be the greater, the closer the core of the real eddy to the sea surface. With an increase in the depth of a real eddy, the effects of asymmetry of location of isopycnals would decrease, and the BV frequency in the core would approach a constant that was predicted for an eddy in vertically unlimited ocean [2].

A similar asymmetry in the behavior of isopycnals would also be observed in cyclones whose cores are located below the sea surface. The BV frequency difference between cyclones and anticyclones would be as follows. In the upper half of the cyclone core (close to the ocean surface), the BV frequency would be lower than in the lower part, and would generally increase in comparison with the background value N_0.

Close theoretical properties should be observed in the eddies near the bottom. In this case, the mapped eddy is located symmetrically to the bottom below its level. The mapping method can be applied to account for the simultaneous influence of the sea surface and the seafloor on the eddy behavior. To do this, one needs to construct a countable set of mappings symmetric to the bottom and sea surface. However, the small thickness of the eddy cores compared to the depth in the real ocean greatly simplifies the problem. It makes sense to study eddy models in a vertically boundless ocean if the core of the eddies is located far from the sea surface and far from the bottom, or to investigate the eddies whose cores are located near the sea surface (or the bottom). The simultaneous influence of the bottom and the sea surface on eddies is possible only in shallow water when the size of the eddy core is close to the depth of the sea.

3. A METHOD FOR CALCULATING POTENTIAL VERTICALITY FROM FIELD DATA

Our approach uses the potential vorticity introduced by Rossby, which in the general case is calculated by formula (5). We can calculate both terms on the right side of this ratio from the field data. As for the speed curl, this is a simple procedure, which we will not analyze in detail. Let us consider the second term.

We write the second term in terms of the density field with the same assumptions under which the previous relation is valid:

$$\frac{\partial \psi(x,y,z,t)}{\partial z} = -\frac{g}{\rho_* f}\left(\rho(x,y,z,t) - \rho_0(z)\right).$$

As a result, the second term, which was transferred to the dimensionless form by dividing by the Coriolis parameter, is reduced to analyze its relation to the density field and the BV frequency:

$$\frac{1}{f}\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial \psi(x,y,z,t)}{\partial z} = -\frac{g}{\rho_*}\frac{\partial}{\partial z}\left\{\frac{1}{N_0^2(z)}\left(\rho(x,y,z,t) - \rho_0(z)\right)\right\}. \quad (27)$$

Here, $\rho(x,y,z,t)$ is the current value of the total density at the considered point, $\rho_0(z)$ is water density at rest at the same depth (background), $N_0^2(z)$ is squared BV frequency at rest at the same depth, ρ_* is the mean density of seawater assumed as 1028 kg/m^3. Two useful relations follow from this expression:

(1) Transformation of the right-hand side of Eq. (27). As a result, we obtain the following equation:

$$\frac{1}{f}\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial \psi(x,y,z,t)}{\partial z} = \frac{N^2(x,y,z,t) - N_0^2(z)}{N_0^2(z)} - \frac{g}{\rho_*}\left(\rho(x,y,z,t) - \rho_0(z)\right)\frac{\partial}{\partial z}\left\{\frac{1}{N_0^2(z)}\right\}, \quad (28)$$

on the right-hand side of which all measured values are written. At the constant BV frequency $N_0^2(z) = \text{const}$, this relationship coincides with the previous equality (10). In the case of variable $N_0^2(z)$, we should calculate the first derivative of $N_0^2(z)$ from the field data or, which is the same, the second derivative of density $\rho_0(z)$, which will cause additional numerical errors. Looking ahead, we state that in the case of the eddies chosen for the analysis, the last term in the previous relation appears small; therefore, a simpler equation can be used with a good accuracy.

$$\frac{1}{f}\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial \psi(x,y,z,t)}{\partial z} = \frac{N^2(x,y,z,t) - N_0^2(z)}{N_0^2(z)}.$$

Details of the previous discussion will be provided in the second part of the work.

(2) Calculation of the mean value $\frac{1}{f}\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial\psi(x,y,z,t)}{\partial z}$ over the water layer between depths h_1 and h_2. We obtain the following relation:

$$\frac{1}{f}\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial\psi(x,y,z,t)}{\partial z}\Bigg|_{h_1}^{h_2} = -\frac{g}{\rho_*}\frac{1}{h_2-h_1} \times\left\{\frac{1}{N_0^2(h_2)}(\rho(x,y,h_2,t)-\rho_0(h_2)) - \frac{1}{N_0^2(h_1)}(\rho(x,y,h_1,t)-\rho_0(h_1))\right\}, \tag{29}$$

in which there are no derivatives of the BV frequency, and all the terms included in the right-hand side are calculated from the field data. Using relation (29) when calculating the profiles of potential vorticity with depth, we can estimate the contribution of σ to each of the terms. Our estimates show that the fraction of σ from the term $\frac{\partial}{\partial z}\frac{f^2}{N_0^2(z)}\frac{\partial\psi(x,y,z,t)}{\partial z}$ calculated for the variable BV frequency and the fraction of σ from the same term taken at the constant BV frequency in $\frac{\partial^2\psi(x,y,\tilde{z},t)}{\partial\tilde{z}^2}$ are qualitatively the same. Therefore, the qualitative conclusions obtained for the constant background BV frequency N_0 would remain valid for the BV variable N_0. This is also confirmed by calculations using formula (29) in the second part of the work.

4. CONCLUSIONS

This article describes theoretical foundations and suggest formulas for calculating all the necessary characteristics of the eddies to be used in the second part of the work. The main properties of the BV frequency field are shown, in particular, extremely possible variations in the BV frequency in cyclonic and anticyclonic eddies. It is shown that the BV frequency variations are most significant in the eddies with a thin core. The effect of the sea surface on the BV frequency in the eddy core is shown.

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