

Approximation Functionals and Their Application

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Abstract—The purpose of this work is to obtain local estimates, and to calculate the constants of approximation of smooth functions. To achieve this goal, the trajectory of functionals, the concept of an approximation families of functionals, the tape of the trajectory of functionals and the width of the tape are considered. For this, the concepts of approximation functionals and their trajectories in the conjugate space are introduced, and the concepts of the trajectory tape and the tape width at a given point are discussed. The resulting overall evaluation of the approximation is applied to multidimensional cases, namely, to approximations by the Courant and Zlamal elements.

Keywords— finite element method, approximation functionals, evaluations of approximation

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1. Introduction

Modeling various phenomena requires constant development approximation apparatus. This is due not only to the emergence of new tasks, but also with the need to clarify solutions that were earlier assigned to tasks. Using approximations to solve various problems are considered in works [10], [15], [21], [23]-[27]. In this regard, there has been a relentless development of spline methods for decades (see [1], [7], [9], [11], [16]). A similar pace has a constant development and deepening of the finite element methods (see [2]-[6], [12], [18]-[20]). It is easy to see that in all cases, the basis is a certain approximation apparatus. To construct an approximation apparatus which is required for various phenomena, the approximation relations are applied (see [2]-[3], [5], [9], [19], [21]). The mentioned relations are considered as a system of equations for the unambiguous determination coordinate functions in some subdivision of their area definitions.

The first works in this direction were works by S.G. Mikhailin and J. Strang (see [2]-[3]). In what follows, the approximation ratios were considered for units of various domains up to arbitrary differentiable manifolds (see [9], [19]).

The approximation relations turned out to be a convenient tool for constructing a set of new approximations, called local ones. This, of course, includes almost all finite element approximations and splines. The use of approximation relations led to the construction of wavelet expansions of a new type, thanks to the systems of functionals generated by them that are biorthogonal to coordinate functions. All this can be done in the most general situations (nonuniform grids, cell subdivisions, differentiable manifolds, see [27]). They make it possible to construct a logically coherent theory of nesting and smoothness of spaces of local functions. Finally, the need to approximate functions depending on a point by a sufficiently arbitrary surfaces, led to the development of the theory of approximation functions

defined on a differentiable manifold (see [8] – [9], [13] – [14], [17], [22]). The entropy estimate on a manifold was considered at work [13]. The paper [17] was devoted to the local linear regression on the manifold. Approximations using moving projections were investigated in the article [22]. In works [8] – [9], [14], approximations of functions defined on a manifold and established global estimates of the rate of convergence of the mentioned approximations were built. Most of the well-known works (see [1] – [27]) are to one degree or another related to the issues of approximation. This applies to works on the method of finite elements, on the method of meshes and on the wavelet method. They illustrate the urgent need to develop a common approach to approximation that would be a unifying factor in research. It is desirable that this approach gives local estimates indicating the constants, and also contains the ability to take into account changes in the characteristics of the function being approximated. If the approximated function is characterized by transitions from fast change to slow change and vice versa, then local estimates of the approximation and acceptable estimates of the approximation constants are required to improve the quality of their applications. In some works, the approximation constants are calculated (see, for example, [2]), but in most cases the approximation constants are not provided with effective estimates (see [3] – [24], [26] – [27]).

The proposed approach serves to overcome this disadvantage. This approach is characterized by the introduction and use of new mathematical objects, namely, the approximation trajectory of functionals, the concept of a tape for a trajectory and the width of this tape.

The aim of the work is to develop a general approach to constructing an approximation apparatus suitable for the local adaptive approximation of functions whose characteristics change rapidly when passing from one region of variation of independent variables to another (i.e., for functions with rapidly changing characteristics). The proposed approach allows one to obtain estimates of local approximation with the calculation of constants in the general situation and to develop adaptive approximation schemes that take into account the rate of change of the approximated function.

The methods considered in this paper do not require (very laborious) processes for cutting the region into a finite number of parts followed by gluing the approximations obtained on each of the mentioned parts. Therefore the discussed method can be applied to functions defined on a differentiable manifold. The resulting overall evaluation of the approximation is applied to multidimensional cases, namely, to approximations by the Courant and Zlamal elements.

2. Approximation Relations

Let \mathbf{M} be a closed set in the Euclidean space R^n . Let $C(\mathbf{M})$ be the linear normed space of the continuous functions $w(\xi)$, $\xi \in \mathbf{M}$, with norm $\|w\|_{C(\mathbf{M})} = \max_{\xi \in \mathbf{M}} |w(\xi)|$. Consider an open domain Ω in the space R^n .

Discussing the linear functionals g in the normalized space $C(\bar{\Omega})$, we will consider their supports as closed sets. Typically, the action of a linear functional $g \in C^*(\bar{\Omega})$ to a function $w(\xi)$, $\xi \in \bar{\Omega}$, $w \in C(\bar{\Omega})$, is written as $\langle g, w(\cdot) \rangle$

or as $\langle g, w \rangle$, and in some cases --- in the form

$\langle g, w(\xi) \rangle$. By $w(\xi)|_{\xi \in D}$ (or $w|_D$) we denote the restriction of the function $w(\xi)$ to the set D , $D \subset \bar{\Omega}$. Let us agree to assume that $\langle g, w|_{\text{supp } g} \rangle = \langle g, w \rangle$. Let for each $t \in \bar{\Omega}$ in the space $C(\bar{\Omega})$ a linear functional $f(t)$

be defined such that $\text{supp } f(t) \subset \bar{\Omega}$. The set $f = \{f(t) | \forall t \in \bar{\Omega}\}$ is called a *trajectory in the space $C^*(\bar{\Omega})$* .

Let $[x, y]$ be the straight line segment connecting points x and y of the space R^n , i.e. $[x, y] = \{z | z = \theta x + (1 - \theta)y \ \forall \theta \in [0, 1]\}$. Each $t \in \bar{\Omega}$ we associate with the set $S_t = \{[t, \xi] | \xi \in \text{supp } f(t)\}$. The aggregate $B(f) = \{S_t | \forall t \in \bar{\Omega}\}$ is called by the *tape of the trajectory for f* , and the number

$$h_t(f) = \max_{\xi \in S_t} \|\xi - t\|_{R^n} \quad (1)$$

is called the *width of the tape $B(f)$ at the point t* . The tape $B(f)$ is called *regular tape* in the subregion $\varpi \subset \Omega$, if the condition

$$S_t \subset \varpi \quad \forall t \in \varpi \quad (2)$$

is fulfilled.

It is evident that the tape $B(f)$ is regular in ϖ , if $\varpi = \Omega$ and Ω is the convex region.

If $w \in C(\bar{\Omega})$ then we have

$$\langle f(t), w(\cdot) \rangle = \langle f(t), w(\xi)|_{\xi \in \text{supp } f(t)} \rangle$$

Therefore

$$|\langle f(t), w(\cdot) \rangle| \leq \|f(t)\|_{C^*(\overline{\text{supp } f(t)})} \|w\|_{C(\overline{\text{supp } f(t)})}. \quad (3)$$

Suppose that the property

$$\langle f(t), (\cdot - t)^\alpha \rangle = \delta_{0,\alpha} \quad \forall t \in \bar{\Omega}, \quad |\alpha| \leq m, \quad (4)$$

is fulfilled.

Relations (4) are called *approximation relations*, and functionals $f(t)$ are called *approximation functionals*.

Approximation relations (4) can be rewritten in the equivalent form

$$\langle f(t), \varphi_\alpha(\cdot) \rangle = \varphi_\alpha(t) \quad \forall t \in \bar{\Omega}, \quad |\alpha| \leq m, \quad (5)$$

where $\varphi_\alpha(t) = t^\alpha$.

By Taylor's formula for a function $u \in C^{m+1}(\bar{\Omega})$ we have

$$u(\xi) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} u^{(\alpha)}(t) (\xi - t)^\alpha + \mathbf{R}_m(\xi, t) \quad (6)$$

where $[t, \xi] \subset \Omega$,

$$\mathbf{R}_m(\xi, t) = (m+1) \times \int_0^1 (1-\tau)^m \sum_{|\alpha|=m+1} \frac{(\xi-t)^\alpha}{\alpha!} u^{(\alpha)}(t+\tau(\xi-t)) d\tau \quad (7)$$

If the subdomain $\varpi \subset \bar{\Omega}$ and $[t, \xi] \subset \varpi$, then

relation (7) implies the estimate

$$|\mathbf{R}_m(\xi, t)| \leq C_m^\varpi(u) \|\xi - t\|_{R^n}^{m+1}, \quad (8)$$

where

$$C_m^\varpi(u) = (m+1) \sum_{|\alpha|=m+1} \frac{\|u^{(\alpha)}\|_{C(\varpi)}}{\alpha!} \quad (9)$$

Applying $f(t)$ to relation (6), we obtain

$$\langle f(t), u(\cdot) \rangle = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} u^{(\alpha)}(t) \langle f(t), (\cdot - t)^\alpha \rangle + \langle f(t), \mathbf{R}_m(\cdot, t) \rangle. \quad (10)$$

Using assumption (4), from relation (10) we find

$$\langle f(t), u(\cdot) \rangle = u(t) + \langle f(t), \mathbf{R}_m(\cdot, t) \rangle \quad (11)$$

Assuming

$$\tilde{u}(t) = \langle f(t), u(\cdot) \rangle \quad (12)$$

we rewrite formula (11) in the form

$$\tilde{u}(t) - u(t) = \langle f(t), \mathbf{R}_m(\cdot, t) \rangle \quad (13)$$

By formula (3) we have

$$|\mathbf{R}_m(\cdot, t)| \leq \|f(t)\|_{C^*(\overline{\text{supp } f(t)})} \|\mathbf{R}_m(\cdot, t)\|_{C(\overline{\text{supp } f(t)})} \quad (14)$$

Consider the second factor on the right-hand side in inequality (14). We have

$$\|\mathbf{R}_m(\cdot, t)\|_{C(\overline{\text{supp } f(t)})} = \max_{\xi \in \overline{\text{supp } f(t)}} |\mathbf{R}_m(\xi, t)|.$$

If relation (3) is satisfied then by inequality (8) we find

$$\begin{aligned} \max_{\xi \in \overline{\text{supp } f(t)}} |\mathbf{R}_m(\xi, t)| &\leq \\ &\leq C_m^\varpi(u) \max_{\xi \in \overline{\text{supp } f(t)}} \|\xi - t\|_{R^n}^{m+1} \quad \forall t \in \varpi. \end{aligned} \quad (15)$$

Theorem 1. Let $f = \{f(t) | \forall t \in \bar{\Omega}\}$ be trajectory in space $C^*(\bar{\Omega})$. Let relation (4) and condition (3) be fulfilled. If $u \in C^{m+1}(\bar{\Omega})$ then the inequality

$$|\tilde{u}(t) - u(t)| \leq C_m^{\bar{\omega}}(u) \|f(t)\|_{C^*(\text{supp}f(t))} [h_t(f)]_{R^n}^{m+1} \quad \forall t \in \bar{\omega} \quad (16)$$

is right.

Here the constant $C_m^{\bar{\omega}}(u)$ is defined by formula (9), and the value $h_t(f)$ is indicated in relation (1).

Proof. From formulas (13) and (14) we obtain

$$|\tilde{u}(t) - u(t)| \leq \|f(t)\|_{C^*(\text{supp}f(t))} \|R_m(\cdot, t)\|_{C(\text{supp}f(t))}.$$

Using relation (15), we derive

$$|\tilde{u}(t) - u(t)| \leq \|f(t)\|_{C^*(\text{supp}f(t))} \times C_m^{\bar{\omega}}(u) \max_{\xi \in \text{supp}f(t)} \|\xi - t\|_{R^n}^{m+1} \quad \forall t \in \bar{\omega}. \quad (17)$$

Using notation (1) in formula (17), we obtain relation (16). This completes the proof.

3. Piecewise linear approximation

In this section we discuss the approach to the Courant approximations. In addition to the Cartesian coordinates of the point t in the Euclidean space R^n the barycentric coordinates are needed in what follows. Let the points x_1, x_2, \dots, x_{n+1} of the space R^n serve as vertices of the nondegenerate simplex T , which is an open set (thus the simplex T is homeomorphic to an open ball).

Consider $n + 1$ -component vector function (column) $\varphi(t) = (1, t^{(1)}, t^{(2)}, \dots, t^{(n)})$. In what follows, all considered vectors are discussed as column vectors (note that to save space we often represent a column vector as a row).

The barycentric coordinates of the point $t = (t^{(1)}, \dots, t^{(n)})$ in the space R^n is called an ordered collection of numbers

$\lambda^T(t) = (\lambda_1^T(t), \lambda_2^T(t), \dots, \lambda_{n+1}^T(t))$ defined by relations

$$\lambda_j^T(t) = \frac{\det(\varphi(x_1), \varphi(x_2), \dots, {}^{j'}\varphi(t), \dots, \varphi(x_{n+1}))}{\det(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n+1}))}. \quad (18)$$

Here the symbol ${}^{j'}$ means that the j -th column of the determinant is replaced by the column $\varphi(t)$, $j = 1, \dots, n + 1$.

As it is known (see, for example, [4], p.55) the barycentric coordinates possess the properties

$$\lambda_j^T(x_i) = \delta_{i,j}, \quad \sum_{j=1}^{n+1} \lambda_j^T(t) = 1 \quad \forall t \in R^n,$$

$$0 < \lambda_j^T(t) < 1 \quad \forall j = 1, 2, \dots, n + 1 \iff t \in T. \quad (19)$$

Let the domain Ω have a piecewise linear boundary. Consider the simplicial subdivision \mathbf{T} . Lots of $T^0 =$

$\{x_i | i \in J\}$ vertices x_i of this subdivision is called a zero-dimensional skeleton; here J is a finite set of indices. Let

$x_j = (x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(n)})$ be a vector, composed of cartesian

coordinates of the vertex x_j . Vertices are also called nodes of the unit under consideration. By symbols T we denote an open n -dimensional simplex of the subdivision \mathbf{T} .

Select the vertex x_i of this subdivision and consider the closure of the union of simplices incident to a given vertex. The closure is called *the body of the barycentric star* for the vertex x_i . Let's denote it by Z_i . The barycentric star Z_i is the support of coordinate function $\omega_i(t)$ of R. Courant's approximation. The function $\omega_i(t)$ can be determined by the approximation relations

$$\sum_{x_i \in \bar{T}} \varphi(x_i) \omega_i(t) = \varphi(t), \quad t \in T, \quad \text{supp } \omega_j(t) \subset Z_j, \quad j \in J. \quad (20)$$

Since the simplex T is nondegenerate, the system $\{\varphi(x_i)\}_{x_i \in \bar{T}}$ is a linearly independent system of vectors.

Without loss of generality, we assume that the vertices x_i of the simplex T are numbered by $1, 2, \dots, n+1$. Then the solution to the system (20) can be represented by Cramer's formulas in the form

$$\omega_j(t) = \frac{\det(\varphi(x_1), \varphi(x_2), \dots, {}^{j'}\varphi(t), \dots, \varphi(x_{n+1}))}{\det(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n+1}))}. \quad (21)$$

Formulas (18) -- (21) show that $\omega_j(t)$ are barycentric coordinates of the point t relative to the vertices x_j of the simplex T . Thus

$$\omega_j(t) \equiv \lambda_j^T(t) \quad \forall j \in \{1, 2, \dots, n + 1\}, t \in T. \quad (22)$$

As a trajectory f in the space $C^*(\bar{\Omega})$ consider the family of functionals $f(t)$, $t \in \bar{\Omega}$, defined by the formula

$$\langle f(t), u \rangle = \sum_{x_i \in \bar{T}} u(x_i) \omega_i(t), \quad t \in \bar{T}, \quad T \in \mathbf{T}. \quad (23)$$

It follows from formulas (1) and (23) that the width of the tape of the considered trajectory f at the point $t \in \bar{T}$ satisfies the inequality

$$h_t(f) \leq \text{diam } T, \quad (24)$$

where $\text{diam } T = \sup_{\xi, t \in T} \|\xi - t\|_{R^n}$.

Noting that approximation relations (20) can be rewritten in the form

$$\sum_{x_i \in \bar{T}} x_i^\alpha \omega_i(t) = t^\alpha, \quad t \in T, \quad |\alpha| \leq 1, \quad (25)$$

$$\text{supp } \omega_j(t) \subset Z_j, \quad j \in J,$$

we use Theorem 1 for $m = 1$.

Theorem 2. Consider the trajectory f , defined by formula (23). For function u from the class $C^2(\bar{\Omega})$ construct an approximation $\tilde{u}(t)$ by the formula

$$\tilde{u}(t) = \sum_{x_i \in \bar{T}} u(x_i) \omega_i(t), \quad t \in \bar{T}, \quad T \in \mathbf{T}. \quad (26)$$

where the functions $\omega_j(t)$ are obtained from the approximation relations (20). Then the inequality

$$|\tilde{u}(t) - u(t)| \leq 2 \sum_{|\alpha|=2} \|u^{(\alpha)}\|_{C(\bar{T})} (\text{diam } T)^2 \quad (27)$$

is right.

Proof. Obviously, the conditions of Theorem 1 are satisfied for $m = 1$. By formula (16) we have

$$|\tilde{u}(t) - u(t)| \leq C_1^{\bar{\omega}}(u) \|f(t)\|_{C^*(\text{supp} f(t))} [h_t(f)]_{\bar{R}^n}^2 \quad \forall t \in \bar{\omega} \quad (28)$$

Let $\bar{\omega}$ be simplex T . According to formula (9), we have

$$C_1^T(u) \leq 2 \sum_{|\alpha|=2} \|u^{(\alpha)}\|_{C(\bar{T})}. \quad (29)$$

Let's turn to the estimate $\|f(t)\|_{C^*(\text{supp} f(t))}$.

From the ratio (23) we have

$$| \langle f(t), u \rangle | \leq \max_{x_i \in \bar{T}} |u(x_i)| \sum_{x_i \in \bar{T}} |\omega_i(t)|, \quad t \in T. \quad (30)$$

Using the non-negativity of the functions $\omega_i(t)$ and taking into account relation (25) for $\alpha = (0, 0, \dots, 0)$, we have

$$\sum_{x_i \in \bar{T}} |\omega_i(t)| = 1, \quad t \in T. \quad (31)$$

From (30) and (31) we obtain

$$\|f(t)\|_{C^*(\text{supp} f(t))} \leq 1. \quad (32)$$

Using formulas (24), (29), and (32) in relation (28), we have inequality (27). This completes the proof.

4. Zlamal's approximation

Similarly to the previous one, consider some general

Zlamal's approximation (see [4], p. 59).

Let $T \in \mathbf{T}$ be simplex with vertices x_1, x_2, \dots, x_{n+1} . Consider points $x_{ij} = \theta_{ij} x_i + (1 - \theta_{ij}) x_j$, $i < j$, where $\theta_{ij} \in (0, 1)$. It is clear to see that the polynomial $P^T(t)$ of the second order from n variables $t = (t_1, \dots, t_n)$,

$$P^T(t) = \sum_{i=1}^{n+1} \lambda_i^T(t) (2\lambda_i^T(t) - 1) v_i + \sum_{1 \leq i < j \leq n+1} \frac{\lambda_i^T(t) \lambda_j^T(t)}{\theta_{ij}(1 - \theta_{ij})} v_{ij} \quad (33)$$

is the solution of the interpolation problem

$$P^T(x_i) = v_i, \quad (34)$$

$$P^T(x_{ij}) = v_{ij}, \quad 1 \leq i < j \leq n+1. \quad (35)$$

Thus, we can assume that the numbers v_i are given at the vertices x_i , and the numbers v_{ij} are given at the selected points x_{ij} edges of simplex T , where $1 \leq i < j \leq n+1$.

Now consider the interpolation problem defined on simplicial subdivision \mathbf{T} . Let there be some numbering of all vertices x_i of the zero-dimensional skeleton T^0 , $i \in \{1, \dots, M\}$. We will associate indexing of edges l in the one-dimensional skeleton T^1 with this numbering. To do this, we use ordered pairs of natural numbers (j, k) , $j < k$, $j, k \in \{1, \dots, M\}$, and put $l_{jk} = [x_j, x_k]$. To each edge l_{jk} , $j < k$, we assign a number $\theta_{jk} \in (0, 1)$ and put $x_{jk} = \theta_{jk} x_j + (1 - \theta_{jk}) x_k$. Consider an interpolation problem

$$P(x_i) = v_i, \quad x_i \in T^0, \quad (36)$$

$$P(x_{jk}) = v_{jk}, \quad l_{jk} \in T^1, \quad (37)$$

where v_i and v_{jk} are given numbers, $i, j, k \in \{1, 2, \dots, M\}$, $j < k$.

Solving problem (36) -- (37) in the class of piecewise polynomial functions of the second degree, similar to the previous one (see formulas (33) -- (35)) we arrive at a piecewise polynomial continuous functions $P(t)$,

$$P(t) = \sum_{i=1}^{n+1} \lambda_i^T(t) (2\lambda_i^T(t) - 1) v_i + \sum_{1 \leq i < j \leq n+1} \frac{\lambda_i^T(t) \lambda_j^T(t)}{\theta_{ij}(1 - \theta_{ij})} v_{ij}, \quad \forall t \in T \quad \forall T \in \mathbf{T}. \quad (38)$$

It is easy to distinguish the coordinate functions of this approximation. We fix the vertex $x_s \in T^0$ and in problem (36) -- (37) put $v_i = \delta_{is}$, $v_{jk} = 0$. As a result we get

$$P_s^T(t) = \lambda_s^T(t) (2\lambda_s^T(t) - 1) \quad \forall t \in T \quad \forall T \in \mathbf{T}. \quad (39)$$

Hence it is seen that the coordinate function corresponding to the vertex $x_s \in T^0$ has the form

$$\omega_s(t) = \begin{cases} P_s^T(t) & \forall t \in T \quad \forall T \in Z_s \\ 0 & \forall t \in T \quad \forall T \in \bar{\Omega} \setminus Z_s \end{cases}. \quad (40)$$

The coordinate function, corresponding to x_{pq} , $p < q$, can be found from the interpolation problem (37) -- (38), if $l_{pq} \in T^1$ and $v_i = 0$, $v_{jk} = \delta_{jp} \delta_{kq}$. We have

$$P_{pq}^T(t) = \frac{\lambda_p^T(t) \lambda_q^T(t)}{\theta_{pq}(1 - \theta_{pq})} \quad t \in \bar{T}, \quad x_{pq} \in \bar{T}. \quad (41)$$

From this we obtain the coordinate function

$$\omega_{pq}(t) = \begin{cases} P_{pq}^T(t) & \text{for } t \in \bar{T}, T \in Y_{pq} \\ 0 & \text{for } t \in \bar{\Omega} \setminus Y_{pq} \end{cases}, \quad (42)$$

where Y_{pq} is the union of all closed simplices, containing an edge l_{pq} , $Y_{pq} = \bigcup_{l_{pq} \in \bar{T}} \bar{T}$.

Lemma 1. *The next relations*

$$|\omega_s(t)| \leq 1, \quad (43)$$

$$0 \leq \omega_{pq}(t) \leq [\theta_{pq}(1 - \theta_{pq})]^{-1} \quad (44)$$

are right.

Proof. Taking into account properties (19) and inequality $\tau(2\tau - 1) \leq 1 \forall \tau \in [0, 1]$, by the relations (39) -- (40) we obtain (43). Inequality (44) follows from inequality (19) and formulas (41) -- (42).

Consider the trajectory f in the space $C^*(\bar{\Omega})$ given by the formula

$$\langle f(t), u \rangle = \sum_{x_i \in \bar{T}} u(x_i) \omega_i(t) + \sum_{x_{jk} \in \bar{T}} u(x_{jk}) \omega_{jk}(t). \quad (45)$$

Theorem 3. *If $t \in \bar{T}$, $T \in \mathbf{T}$, then the ratios*

$$\|f(t)\|_{C^*(\bar{T})} \leq G_T,$$

$$G_T = (n+1)\{1 + [\theta_{pq}(1 - \theta_{pq})]^{-1}n/2\}, \quad (46)$$

$$\text{supp } f(t) \subset \bar{T}, \quad h_t(f) \leq \text{diam } T, \quad t \in \bar{T} \quad (47),$$

are true.

Proof. Relations (47) follow in an obvious way from (45). Let us turn to the proof of relation (46). Using relations (43) -- (45), we derive

$$\begin{aligned} \langle f(t), u \rangle &\leq \max_{x_s \in \bar{T}} |u(x_s)| \sum_{x_i \in \bar{T}} |\omega_i(t)| + \\ &+ \max_{x_{pq} \in \bar{T}} |u(x_{pq})| \sum_{x_{jk} \in \bar{T}} |\omega_{jk}(t)| \quad \forall t \in T \quad \forall T \in \mathbf{T}. \end{aligned} \quad (48)$$

Taking into account (41) -- (42), we find

$$\begin{aligned} \sum_{x_{jk} \in \bar{T}} |\omega_{jk}(t)| &= \sum_{x_{jk} \in \bar{T}} \frac{\lambda_j^T(t) \lambda_k^T(t)}{\theta_{jk}(1 - \theta_{jk})} \leq \\ &\leq \max_{x_{pq} \in \bar{T}} [\theta_{pq}(1 - \theta_{pq})]^{-1} \sum_{x_{jk} \in \bar{T}} \lambda_j^T(t) \lambda_k^T(t). \end{aligned} \quad (49)$$

By (19) we obtain

$$\sum_{x_{jk} \in \bar{T}} \lambda_j^T(t) \lambda_k^T(t) \leq n(n+1)/2$$

so from (49) we find

$$\sum_{x_{jk} \in \bar{T}} |\omega_{jk}(t)| \leq \max_{x_{pq} \in \bar{T}} [\theta_{pq}(1 - \theta_{pq})]^{-1} n(n+1)/2. \quad (50)$$

Using (48) -- (50), we have

$$\begin{aligned} \langle f(t), u \rangle &\leq \max_{t \in \bar{T}} |u(t)| \{(n+1) + \max_{x_{pq} \in \bar{T}} [\theta_{pq}(1 - \\ &- \theta_{pq})]^{-1} n(n+1)/2\} \quad \forall t \in T \quad \forall T \in \mathbf{T}. \end{aligned} \quad (51)$$

Inequality (46) follows from relation (51). This completes the proof.

Corollary 1. *If $\theta_{pq} = 1/2$ then inequality (46) is simplified:*

$$\|f(t)\|_{C^*(\bar{T})} \leq (n+1)(1+2n). \quad (52)$$

Theorem 4. *Suppose a function u belongs to the class $C^3(\bar{\Omega})$. Let the function $\tilde{u}(t)$ be defined by the formula*

$$\begin{aligned} \tilde{u}(t) &= \sum_{x_i \in \bar{T}} u(x_i) \omega_i(t) + \\ &+ \sum_{x_{jk} \in \bar{T}} u(x_{jk}) \omega_{jk}(t) \quad \forall t \in T \quad \forall T \in \mathbf{T}, \end{aligned} \quad (53)$$

where $\omega_i(t)$ and $\omega_{jk}(t)$ are defined by formulas (39) -- (42). Then the inequality

$$\begin{aligned} |\tilde{u}(t) - u(t)| &\leq \\ &\leq G_T \sum_{|\alpha|=3} \|u^{(\alpha)}\|_{C(\bar{T})} (\text{diam } T)^3 \quad \forall t \in T \quad \forall T \in \mathbf{T}, \end{aligned} \quad (54)$$

is correct.

Formula (54) is obtained in the same way as formula (27) (see Theorem 2).

From (36) -- (42) for any polynomial $P(t)$ of the second degree we have the identity

$$\begin{aligned} P(t) &\equiv \sum_{x_i \in \bar{T}} P(x_i) \omega_i(t) + \\ &+ \sum_{x_{jk} \in \bar{T}} P(x_{jk}) \omega_{jk}(t) \quad \forall t \in T \quad \forall T \in \mathbf{T}. \end{aligned} \quad (55)$$

Relation (55) is equivalent to the system of identities

$$\begin{aligned} t^\alpha &\equiv \sum_{x_i \in \bar{T}} x_i^\alpha \omega_i(t) + \\ &+ \sum_{x_{jk} \in \bar{T}} x_{jk}^\alpha \omega_{jk}(t) \end{aligned} \quad (56)$$

$$\forall t \in T \quad \forall T \in \mathbf{T} \quad \forall |\alpha| \leq 2. \quad (57)$$

By adding information about the support location of the coordinate functions, we obtain:

$$\text{supp } \omega_s = Z_s, \quad \text{supp } \omega_{pq} = Y_{pq}. \quad (58)$$

Formulas (56) -- (58) are the approximation relations for Zlamal's approximation.

5. Conclusion

In this paper the approach to approximation of the functions defined on the plane region is discussed. The proposed approach allows one to obtain local estimates with the indication of constants. It allows one to take into account changes in the characteristics of the function being approximated (i.e., it allows developing adaptive algorithms). Let us describe some prospects for the development of the proposed approach. Very important objects, often encountered in practice, are functions, the domain of which is a differentiable manifold (n -dimensional sphere, torus, pretzel, projective plane). The problems of mathematical physics about the definition of functions with such domain of definition are encountered in the construction of buildings and structures, as well as in the study of the fundamental properties of matter (using colliders and other devices). The approach proposed in this paper can be applied to rapidly changing functions defined on a manifold. The resulting approximation apparatus is necessary to construct and estimate the rate of convergence

of methods for solving the above-mentioned problems of mathematical physics. Thus, the development of the proposed approach for rapidly changing functions given on a manifold is a prospect for the development of this approach.

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