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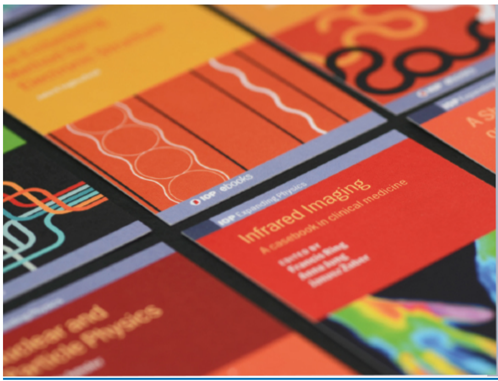

On an application of the Boundary control method to classical moment problems.

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Abstract. We establish relationships between the classical moments problems which are problems of a construction of a measure supported on a real line, on a half-line or on an interval from prescribed set of moments with the Boundary control approach to a dynamic inverse problem for a dynamical system with discrete time associated with Jacobi matrices. We show that the solution of corresponding truncated moment problems is equivalent to solving some generalized spectral problems.

1. Introduction

In [14] the authors put forward an approach to Hamburger, Stieltjes and Hausdorff moment problems based on their relationships with inverse problems for dynamical systems with discrete time associated with Jacobi matrices. In the present paper we utilize some ideas from [12] about de Branges spaces associated with such dynamical systems and extend and elaborate results obtained in [14]. We begin with introducing moment problems and spaces of polynomials associated with it, dynamical systems with discrete time associated with Jacobi matrices and de Branges spaces of analytic functions.

1.1. Classical moment problems.

For a given a sequence of numbers $s_0, s_1, s_2, \ldots$ called moments, a solution of a Hamburger moment problem [1, 18] is a Borel measure $d\rho(\lambda)$ on $\mathbb{R}$ such that

$$s_k = \int_{-\infty}^{\infty} \lambda^k \, d\rho(\lambda), \quad k = 0, 1, 2, \ldots \tag{1}$$

The measure is a solution to Stieltjes or Hausdorff moment problems provided $\operatorname{supp} d\rho \subset (0, +\infty)$ or $\operatorname{supp} d\rho \subset (0, 1)$ respectively; in these cases the moments are called Hamburger, Stieltjes or Hausdorff.

Following [18, 17] we denote by $C[X]$ the set of complex polynomials and by $C_N[X]$ the set of polynomials of order less than or equal to N. The moments $\{s_k\}_{k=0}^{\infty}$ determine on $C[X]$

the bilinear form by the rule: for $F, G \in C[X]$, $F(\lambda)=\sum_{n=0}^{N-1}\alpha_n\lambda^n$, $G(\lambda)=\sum_{n=0}^{N-1}\beta_n\lambda^n$, one defines

$$\langle F, G\rangle = \sum_{n,m=0}^{N-1} s_{n+m}\alpha_n\overline{\beta_m}. \tag{2}$$

Thus this quadratic form is determined by the following (semi-infinite) Hankel matrix:

$$S=\begin{pmatrix} s_0 & s_1 & s_2 & s_3 & \dots \\ s_1 & s_2 & s_3 & \dots & \dots \\ s_2 & s_3 & \dots & \dots & \dots \\ s_3 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3}$$

1.2. Initial boundary value problems associated with Jacobi matices.

An initial boundary value problem (IBVP) for an auxiliary dynamical system with discrete time for a Jacobi matrix is set up in the following way: for a given sequence of positive numbers $\{a_0, a_1, \dots\}$ (in what follows we assume $a_0 = 1$) and real numbers $\{b_1, b_2, \dots\}$, we denote by A the Jacobi operator, defined on l_2, which has a matrix form:

$$A=\begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{4}$$

All our considerations will be local, so when coefficients a_j, b_j are such that Jacobi matrix is in limit circle case, by A we can assume any self-adjoint extension. For $N \in \mathbb{N}$, by A^N we denote the $N \times N$ Jacobi matrix which is a block of (4) consisting of the intersection of first N columns with first N rows of A. We consider the dynamical system with discrete time associated with A^N:

$$\begin{cases} v_{n,t+1}+v_{n,t-1}-a_nv_{n+1,t}-a_{n-1}v_{n-1,t}-b_nv_{n,t}=0, \ t\in\mathbb{N}\cup\{0\}, \ n\in 1,\dots,N, \\ v_{n,-1}=v_{n,0}=0, \quad n=1,2,\dots,N+1, \\ v_{0,t}=f_t, \quad v_{N+1,t}=0, \quad t\in\mathbb{N}_0, \end{cases} \tag{5}$$

where $f=(f_0, f_1, \dots)$ is a *boundary control*. The solution to (5) is denoted by v^f. Note that (5) is a discrete analog of dynamical system with boundary control for a wave equation on an interval [3, 8].

The operator corresponding to a finite Jacobi matrix we also denote by $A^N : \mathbb{R}^N \mapsto \mathrm{R}^N$, it is given by

$$\begin{cases} (A\psi)_n &= a_n\psi_{n+1}+a_{n-1}\psi_{n-1}+b_n\psi_n, \quad 2\leqslant n\leqslant N-1, \\ (A\psi)_1 &= b_1\psi_1+a_1\psi_2, \quad n=1, \end{cases} \tag{6}$$

and the Dirichlet condition at the "right end":

$$\psi_{N+1}=0. \tag{7}$$

We also consider the dynamical system corresponding to a semi-infinite Jacobi matrix:

$$\begin{cases} u_{n,t+1}+u_{n,t-1}-a_nu_{n+1,t}-a_{n-1}u_{n-1,t}-b_nu_{n,t}=0, \ t\in\mathbb{N}\cup\{0\}, \ n\in 1,\dots,N, \\ u_{n,-1}=u_{n,0}=0, \quad n=1,2,\dots,N+1, \\ u_{0,t}=f_t, \quad t\in\mathbb{N}_0, \end{cases} \tag{8}$$

its solution is denoted by u^f.

1.3. De Branges spaces.
Here we provide the information on de Branges spaces in accordance with [16]. The entire function $E : \mathbb{C} \mapsto \mathbb{C}$ is called a *Hermite-Biehler function* if $|E(z)| > |E(\overline{z})|$ for $z \in \mathbb{C}_+$. We use the notation $F^{\#}(z) = \overline{F(\overline{z})}$. The *Hardy space* H_2 is defined by: $f \in H_2$ if f is holomorphic in $\mathbb{C}^+$ and $\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\,dx < \infty$. Then the *de Branges space* $B(E)$ consists of entire functions such that:

$$B(E) := \left\{ F : \mathbb{C} \mapsto \mathbb{C},\ F \text{ entire}, \int\limits_{\mathbb{R}} \left| \frac{F(\lambda)}{E(\lambda)} \right|^2 d\lambda < \infty,\ \frac{F}{E}, \frac{F^{\#}}{E} \in H_2 \right\}.$$

The space $B(E)$ with the scalar product

$$[F, G]_{B(E)} = \frac{1}{\pi} \int\limits_{\mathbb{R}} F(\lambda)\overline{G(\lambda)} \frac{d\lambda}{|E(\lambda)|^2}$$

is a Hilbert space. For any $z \in \mathbb{C}$ the *reproducing kernel* is introduced by the relation

$$J_z(\xi) := \frac{\overline{E(z)}E(\xi) - E(\overline{z})\overline{E(\overline{\xi})}}{2i(\overline{z} - \xi)}. \tag{9}$$

Then

$$F(z) = [J_z, F]_{B(E)} = \frac{1}{\pi} \int\limits_{\mathbb{R}} J_z(\lambda)\overline{G(\lambda)} \frac{d\lambda}{|E(\lambda)|^2}.$$

We observe that a Hermite-Biehler function $E(\lambda)$ defines J_z by (9).

In the second section we provide the results on solutions to (5) and (8) and introduce the operators of the Boundary control method according to [13]; using the ideas of [12] we introduce de Branges spaces corresponding to dynamical systems (8), (5) and give representation of reproducing kernel in the space of polynomials and Christoffel symbols [17] in dynamic terms. In the third section we outline the solution to a truncated moment problem following [14], more specifically, we reduce it to the generalized spectral problem for special matrices constructed from moments. In the last section we apply obtained results to the problem of uniqueness of solutions to moment problems.

2. IBVP for a dynamical system associated with Jacobi matrix and de Branges spaces.

We fix some positive integer T and denote by $\mathcal{F}^T$ the *outer space* of systems (5), (8), the space of controls: $\mathcal{F}^T := \mathbb{R}^T$, $f \in \mathcal{F}^T$, $f = (f_0, \dots, f_{T-1})$, we use the notation $\mathcal{F}^\infty = \mathbb{R}^\infty$ when control acts for all $t \geqslant 0$.

Definition 1. *For $f, g \in \mathcal{F}^\infty$ we define the convolution $c = f * g \in \mathcal{F}^\infty$ by the formula*

$$c_t = \sum_{s=0}^{t} f_s g_{t-s}, \quad t \in \mathbb{N} \cup \{0\}.$$

The input $\longmapsto$ output correspondences in systems (5), (8) are realized by a *response operators*: R_N^T, $R^T : \mathcal{F}^T \mapsto \mathbb{R}^T$ defined by rules

$$\left(R_N^T f\right)_t = v_{1,t}^f = \left(r^N * f_{\cdot-1}\right)_t, \quad t = 1, \dots, T,$$
$$\left(R^T f\right)_t = u_{1,t}^f = \left(r * f_{\cdot-1}\right)_t, \quad t = 1, \dots, T,$$

where $r^N = (r_0^N, r_1^N, \dots, r_{T-1}^N)$, $r = (r_0, r_1, \dots, r_{T-1})$ are *response vectors*, convolution kernels of response operators. These operators play the role of dynamic inverse data, corresponding inverse problems were considered in [11, 13]. By choosing the special control $f = \delta := (1, 0, 0, \dots)$, kernels of response operators can be determined as

$$\left(R_N^T \delta\right)_t = v_{1,t}^\delta = r_{t-1}^N, \quad \left(R^T \delta\right)_t = u_{1,t}^\delta = r_{t-1}.$$

Let $\phi_n(\lambda)$ be a solution to the following difference equation

$$\begin{cases} a_n \phi_{n+1} + a_{n-1}\phi_{n-1} + b_n \phi_n = \lambda \phi_n, \\ \phi_0 = 0, \ \phi_1 = 1. \end{cases} \tag{10}$$

Denote by $\{\lambda_k\}_{k=1}^N$ roots of the equation $\phi_{N+1}(\lambda) = 0$, it is known [1, 18] that they are real and distinct. We introduce vectors $\phi^n \in \mathbb{R}^N$ by the rule $\phi_i^n := \phi_i(\lambda_n)$, $n, i = 1, \dots, N$, and define numbers ρ_k by

$$(\phi^k, \phi^l) = \delta_{kl}\rho_k, \quad k, l = 1, \dots, N,$$

where $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^N$.

Definition 2. *The set of pairs*

$$\{\lambda_k, \rho_k\}_{k=1}^N$$

is called spectral data of the operator A^N*.*

Let $\mathcal{T}_k(2\lambda)$ be Chebyshev polynomials of the second kind, i.e. they satisfy

$$\begin{cases} \mathcal{T}_{t+1} + \mathcal{T}_{t-1} - \lambda \mathcal{T}_t = 0, \\ \mathcal{T}_0 = 0, \ \mathcal{T}_1 = 1. \end{cases} \tag{11}$$

The spectral function of A^N is introduced by the rule

$$\rho^N(\lambda) = \sum_{\{k \,|\, \lambda_k < \lambda\}} \frac{1}{\rho_k}, \tag{12}$$

The spectral function of A (non unique if A is limit circle at infinity) is denoted by $\rho(\lambda)$. In [11, 13] by the application of Fourier expansion method following representations for the solution v^f and components of response vector were established:

Proposition 1. *The solution to (5) and the kernel of* R_N^T *admit representations*

$$v_{n,t}^f = \int_{-\infty}^{\infty} \sum_{k=1}^{t} \mathcal{T}_k(\lambda) f_{t-k} \phi_n(\lambda)\, d\rho^N(\lambda), \tag{13}$$

$$r_{t-1}^N = \int_{-\infty}^{\infty} \mathcal{T}_t(\lambda)\, d\rho^N(\lambda), \quad t \in \mathbb{N}. \tag{14}$$

Remark 1. *The solution corresponding to semi-infinite Jacobi matrix* u^f *and entries of the kernel of* R^T *admit representations (13), (14) with* $d\rho^N$ *substituted by* $d\rho(\lambda)$*.*

The *inner space* of dynamical system (5) is $\mathcal{H}^N := \mathbb{R}^N$, $h \in \mathcal{H}^N$, $h = (h_1, \dots, h_N)$, $v^f_{\cdot,T} \in \mathcal{H}^N$ for all T. For the system (5) the *control operator* $W_N^T : \mathcal{F}^T \mapsto \mathcal{H}^N$ is defined by the rule

$$W_N^T f := v^f_{n,T}, \quad n = 1, \dots, N.$$

The set

$$\mathcal{U}^T := W_N^T \mathcal{F}^T = \left\{ v^f_{\cdot,T} | f \in \mathcal{F}^T \right\}$$

is called reachable. For the system (8) we have that $v^f_{\cdot,T} \in \mathcal{H}^T$, thus the control operator $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is introduced by

$$W^T f := u^f_{n,T}, \quad n = 1, \dots, T.$$

Everywhere below we substantially use the finiteness of the speed of wave propagation in systems (5), (8) which implies the following dependence of inverse data on coefficients $\{a_n, b_n\}$: for $M \in \mathbb{N}$, $M \leqslant N$, the element $v^f_{1,2M-1}$ depends on $\{a_1, \dots, a_{M-1}\}$, $\{b_1, \dots, b_M\}$. On observing this we can formulate the following

Remark 2. *Entries of the response vector* $(r_0^N, r_1^N, \dots, r_{2N-2}^N))$ *depend on* $\{a_0, \dots, a_{N-1}\}$, $\{b_1, \dots, b_N\}$, *and does not depend on the boundary condition at* $n = N + 1$, *the entries starting from* r^N_{2N-1} *does "feel" the boundary condition at* $n = N + 1$. *Moreover,*

$$u^f_{n,t} = v^f_{n,t}, \quad n \leqslant t \leqslant N, \quad \text{and} \quad W^N = W^N_N. \tag{15}$$

The *connecting operator* for the system (5) $C_N^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is defined via the quadratic form: for arbitrary $f, g \in \mathcal{F}^T$ we set

$$(C_N^T f, g)_{\mathcal{F}^T} = \left(v^f_{\cdot,T}, v^g_{\cdot,T} \right)_{\mathcal{H}^N} = (W_N^T f, W_N^T g)_{\mathcal{H}^N}.$$

For the system (8) the connecting operator $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced by the rule:

$$(C^T f, g)_{\mathcal{F}^T} = \left(u^f_{\cdot,T}, u^g_{\cdot,T} \right)_{\mathcal{H}^T} = (W^T f, W^T g)_{\mathcal{H}^N}.$$

In [11, 13] the following formulas were obtained:

Proposition 2. *Connecting operators for systems (5), (8) admit spectral representations*

$$\{C_N^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda)\mathcal{T}_{T-m}(\lambda)\, d\rho^N(\lambda), \quad l, m = 0, \dots, T-1, \tag{16}$$

$$\{C^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda)\mathcal{T}_{T-m}(\lambda)\, d\rho(\lambda), \quad l, m = 0, \dots, T-1,$$

and the following dynamic representation valid if $T \leqslant N$:

$$C^T = C_N^T = \begin{pmatrix} r_0 + r_2 + \ldots + r_{2T-2} & r_1 + \ldots + r_{2T-3} & \ldots & r_T + r_{T-2} & r_{T-1} \\ r_1 + r_3 + \ldots + r_{2T-3} & r_0 + \ldots + r_{2T-4} & \ldots & \ldots & r_{T-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{T-3} + r_{T-1} + r_{T+1} & \ldots & r_0 + r_2 + r_4 & r_1 + r_3 & r_2 \\ r_T + r_{T-2} & \ldots & r_1 + r_3 & r_0 + r_2 & r_1 \\ r_{T-1} & r_{T-2} & \ldots & r_1 & r_0 \end{pmatrix} \tag{17}$$

According to [2] the spectral measure $d\rho(\lambda)$ corresponding to operator A give rise to the Fourier transform $F: l^2 \mapsto L_2(\mathbb{R}, d\rho)$, defined by the rule:

$$(Fa)(\lambda) = \sum_{n=0}^{\infty} a_k \phi_k(\lambda), \quad a = (a_0, a_1, \dots) \in l^2, \tag{18}$$

where ϕ is a solution to (10). The inverse transform and Parseval identity reads:

$$a_k = \int_{-\infty}^{\infty} (Fa)(\lambda)\phi_k(\lambda)\, d\rho(\lambda),$$

$$\sum_{k=0}^{\infty} a_k b_k = \int_{-\infty}^{\infty} (Fa)(\lambda)(Fb)(\lambda)\, d\rho(\lambda). \tag{19}$$

Note [12] that for $\lambda \in \mathbb{C}$ we have the following representation for the Fourier transform of the solution to (5) at $t = T$:

$$\left(Fv^f_{\cdot,T}\right)(\lambda) = \sum_{k=1}^{T} \mathcal{T}_k(\lambda) f_{T-k}, \quad \lambda \in \mathbb{C}. \tag{20}$$

Now we assume that $T = N$ and introduce the linear manifold of Fourier images of states of dynamical system (5) at time $t = N$, i.e. the Fourier image of the reachable set:

$$B^N_J := F\mathcal{U}^N = \left\{ \left(Fu^f_{\cdot,N}\right)(\lambda) \mid f \in \mathcal{F}^N \right\} = \left\{ \left(Fv^f_{\cdot,N}\right)(\lambda) \mid f \in \mathcal{F}^N \right\}.$$

We equip B^N_J with the scalar product defined by the rule:

$$[F, G]_{B^N_J} = \left(C^N f, g\right)_{\mathcal{F}^N}, \quad F, G \in B^N_J, \tag{21}$$

$$F(\lambda) = \sum_{k=1}^{N} \mathcal{T}_k(\lambda) f_{N-k},\ G(\lambda) = \sum_{k=1}^{N} \mathcal{T}_k(\lambda) g_{N-k},\ f, g \in \mathcal{F}^N.$$

Evaluating (21) making use of (19) yields:

$$[F, G]_{B^N_J} = \left(v^f_{\cdot,N}, v^g_{\cdot,T}\right)_{\mathcal{H}^N} = \left(u^f_{\cdot,N}, u^g_{\cdot,T}\right)_{\mathcal{H}^N} = \int_{-\infty}^{\infty} (Fu^f_{\cdot,N})(\lambda)(Fu^g_{\cdot,N})(\lambda)\, d\rho(\lambda)$$

$$= \int_{-\infty}^{\infty} F(\lambda)\overline{G(\lambda)}\, d\rho(\lambda) = \int_{-\infty}^{\infty} F(\lambda)\overline{G(\lambda)}\, d\rho_N(\lambda). \tag{22}$$

On comparing (2) and (22) we see that:

$$[F, G]_{B^N_J} = \langle F, G \rangle = \sum_{n,m=0}^{N-1} s_{n+m} \alpha_n \overline{\beta_m} = \int_{-\infty}^{\infty} F(\lambda)\overline{G(\lambda)}\, d\rho(\lambda). \tag{23}$$

In [13] the authors proved the following

Theorem 1. *The vector $(r_0, r_1, r_2, \dots, r_{2N-2})$ is a response vector for the dynamical system (5) if and only if the matrix C^T (with $T = N$) defined by (16), (17) is positive definite.*

This theorem shows that (22) is a scalar product in B_J^N. But we can say even more [12]:

Theorem 2. *By dynamical system with discrete time (8) one can construct the de Branges space*

$$B_J^N := \left\{ \left(Fu^f_{\cdot,N}\right)(\lambda) \,|\, f \in \mathcal{F}^N \right\} = \left\{ \sum_{k=1}^{N} \mathcal{T}_k(\lambda) f_{N-k} \,|\, f \in \mathcal{F}^N \right\}.$$

As a set of functions it coincides with the space of Fourier images of states of dynamical system (8) at time N (the Fourier image of a reachable set) and is the set of polynomials with real coefficients of the order less or equal to $N-1$. The norm in B_J^N is defined via the connecting operator:

$$[F, G]_{B_J^N} := \left(C^N f, g\right)_{\mathcal{F}^N}, \quad F, G \in B_J^N,$$

where

$$F(\lambda) = \sum_{k=1}^{N} \mathcal{T}_k(\lambda) f_{N-k},\ G(\lambda) = \sum_{k=1}^{N} \mathcal{T}_k(\lambda) g_{N-k},\ f, g \in \mathcal{F}^N.$$

The reproducing kernel has a form

$$J_z(\lambda) = \sum_{k=1}^{N} \mathcal{T}_k(\lambda) j^z_{N-k},$$

where j_z is a solution to Krein-type equation

$$C^N j^z = \overline{\begin{pmatrix} \mathcal{T}_N(z) \\ \mathcal{T}_{N-1}(z) \\ \cdot \\ \mathcal{T}_1(z) \end{pmatrix}}.$$

Note [12, 14] that control j^z drives the system (8) to special state ϕ, that is:

$$\left(W^N j^z\right)_i = \left(W_N^N j^z\right)_i = \phi_i(z), \quad i = 1, \dots, N. \tag{24}$$

Thus the reproducing kernel in $C_N[X]$ (or in B_J^N) is given by

$$K_N(z, \lambda) = \left(\left(C^N\right)^{-1} \overline{\begin{pmatrix} \mathcal{T}_N(z) \\ \mathcal{T}_{N-1}(z) \\ \cdot \\ \mathcal{T}_1(z) \end{pmatrix}}, \begin{pmatrix} \mathcal{T}_N(\lambda) \\ \mathcal{T}_{N-1}(\lambda) \\ \cdot \\ \mathcal{T}_1(\lambda) \end{pmatrix} \right). \tag{25}$$

Remark 3. *The space of complex polynomials $C_N[X]$ with scalar product defined by matrix S is a de Branges space B_J^N where scalar product and reproducing kernel are given by (23) and (25).*

Definition 3. *The $n-th$ Christoffel function is defined by the rule*

$$\varkappa_n(\lambda) = \left(\sum_{k=1}^{N} \phi_k^2(\lambda) \right)^{-1}.$$

From (24) it immediately follows that

$$\varkappa_n(\lambda) = K_N(\lambda, \lambda) = \left(\left(C^N\right)^{-1} \overline{\begin{pmatrix} \mathcal{T}_N(\lambda) \\ \mathcal{T}_{N-1}(\lambda) \\ \cdot \\ \mathcal{T}_1(\lambda) \end{pmatrix}}, \begin{pmatrix} \mathcal{T}_N(\lambda) \\ \mathcal{T}_{N-1}(\lambda) \\ \cdot \\ \mathcal{T}_1(\lambda) \end{pmatrix} \right).$$

Different formulas for reproducing kernel and Christoffel functions are derived in [18, 17].

3. Truncated moment problem. Recovering Dirichlet spectral data.

We observe the following: in the moment problem we are given the sequence of moments (1), and in the inverse dynamic problem for systems (5), (8) we are given a response vector [11, 13], whose spectral representation has a form (14). Thus the knowledge of moments $\{s_0, s_1, \ldots\}$ implies a possibility to calculate the response vector $\{r_0, r_1, \ldots\}$ by (14). Note that Chebyshev polynomials of the second kind $\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\lambda), \ldots \mathcal{T}_n(\lambda)\}$ (see (11)) are related to $\{1, \lambda, \lambda^{n-1}\}$ by the following formula

$$\begin{pmatrix} \mathcal{T}_1(\lambda) \\ \mathcal{T}_2(\lambda) \\ \ldots \\ \mathcal{T}_n(\lambda) \end{pmatrix} = \Lambda_n \begin{pmatrix} 1 \\ \lambda \\ \ldots \\ \lambda^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ a_{21} & 1 & \ldots & 0 \\ \ldots & & & \\ a_{n1} & a_{n2} & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \\ \ldots \\ \lambda^{n-1} \end{pmatrix}. \tag{26}$$

Proposition 3. *Entries of the matrix $\Lambda_n \in R^{n\times n}$ are given by*

$$\Lambda_n = a_{ij} = \begin{cases} 0, & \text{if } i > j, \\ 0, & \text{if } i+j \text{ is odd}, \\ C^j_{\frac{i+j}{2}} (-1)^{\frac{i+j}{2}+j}. \end{cases} \tag{27}$$

entries of the response vector are related to moments by the rule:

$$\begin{pmatrix} r_0 \\ r_1 \\ \ldots \\ r_{n-1} \end{pmatrix} = \Lambda_n \begin{pmatrix} s_0 \\ s_1 \\ \ldots \\ s_{n-1} \end{pmatrix}. \tag{28}$$

Definition 4. *By a solution of a truncated moment problem of order N we call a Borel measure $d\tilde{\rho}_N(\lambda)$ on $\mathbb{R}$ such that equalities (1) with this measure hold for $k = 0, 1, \ldots, 2N$.*

Remark 4. *Results from [11, 13] imply that from the finite set of moments $\{s_0, s_1, \ldots, s_{2N-2}\}$, or what is equivalent from $\{r_0, r_1, \ldots, r_{2N-2}\}$, it is possible to recover Jacobi matrix $A^N \in \mathbb{R}^{N\times N}$, whose elements can be thought of as a coefficients in dynamical system (5) with Dirichlet boundary condition at $n = N+1$, or $N \times N$ block in semi-infinite Jacobi matrix in (5).*

This theorem and formulas for the entries of Jacobi matrix obtained in [13] implies the following procedure of solving the truncated moment problem:

- 1) Calculate $(r_0, r_1, r_2, \ldots, r_{2N-2})$ from $\{s_0, s_1, \ldots, s_{2N-2}\}$ by using (28).
- 2) Recover $N \times N$ Jacobi matrix A^N using formulas for a_k, b_k from [13]
- 3) Recover spectral measure for finite Jacobi matrix A^N prescribing arbitrary selfadjoint condition at $n = N+1$. Or one can do
- 3') Extend Jacobi matrix A^N to *finite* Jacobi matrix A^M, $M > N$, prescribe arbitrary selfadjoint condition at $n = M+1$ and recover the spectral measure of A^M.
- 3") Extend Jacobi matrix A^N to *infinite* Jacobi matrix A, and recover the spectral measure of A.

Every measure obtained in 3), $3'$), $3''$) gives a solution to the truncated moment problem. Below we propose a different approach: we recover the spectral measure corresponding to Jacobi matrix directly from moments (from the operator C^N), without recovering the Jacoi matrix itself.

Agreement 1. *We assume that controls $f \in \mathcal{F}^N$, $f = (f_0, \ldots, f_{N-1})$ are extended: $f = (f_{-1}, f_0, \ldots, f_{N-1}, f_N)$, where $f_1 = f_N = 0$.*

We introduce the special space of controls $\mathcal{F}_0^N = \left\{f \in \mathcal{F}^T \,|\, f_0 = 0\right\}$ and the operator $D : \mathcal{F}^T \mapsto \mathcal{F}^T$ acting by

$$(Df)_t = f_{t+1} + f_{t-1}.$$

The following statements can be easily proved using arguments from [13]:

Proposition 4. *The operator W^N maps $\mathcal{F}^N$ isomorphically onto $\mathcal{H}^N$ and $\mathcal{F}_0^N$ maps isomorphically onto $\mathcal{H}^{N-1}$.*

Proposition 5. *On the set $\mathcal{F}_0^T$ the following relation holds:*

$$W^N Df = DW^N f, \quad f \in \mathcal{F}_0^N. \tag{29}$$

Taking $f, g \in \mathcal{F}_0^T$ we can evaluate the quadratic form, bearing in mind (29):

$$\left(C^N Df, g\right)_{\mathcal{F}^N} = \left(W^N Df, W^N g\right)_{\mathcal{H}^N} = \left(DW^N f, W^N g\right)_{\mathcal{H}^{N-1}} = \left(A^{N-1} v^f, v^g\right)_{\mathcal{H}^{N-1}}. \tag{30}$$

The last equality in (30) means that only A^{N-1} block from the whole matrix A^N is in use. Then it is possible to perform the spectral analysis of A^{N-1} using the classical variational approach, the controllability of the system (5) (see Proposition 4) and the representation (30), see also [5]. The spectral data of Jacobi matrix A^{N-1} with Dirichlet boundary condition at $n = N$ can be recovered by the following procedure:

1) The first eigenvalue is given by

$$\lambda_1^{N-1} = \min_{f \in \mathcal{F}_0^N,\, (C^N f, f)_{\mathcal{F}^N} = 1} \left(C^N Df, f\right)_{\mathcal{F}^N}. \tag{31}$$

2) Let f^1, be the minimizer of (31), then

$$\rho_1 = \left(C^N f^1, f^1\right)_{\mathcal{F}^N}.$$

3) The second eigenvalue is given by

$$\lambda_2^{N-1} = \min_{\substack{f \in \mathcal{F}_0^N, (C^N f, f)_{\mathcal{F}^N} = 1 \\ (C^N f, f_1)_{\mathcal{F}^N} = 0}} \left(C^N Df, f\right)_{\mathcal{F}^N}. \tag{32}$$

4) Let f^2, be the minimizer of (32), then

$$\rho_2 = \left(C^T f^2, f^2\right)_{\mathcal{F}^T}.$$

Continuing this procedure, one recovers the set $\{\lambda_k^{N-1}, \rho_k\}_{k=1}^{N-1}$ and construct the measure $d\rho^{N-1}(\lambda)$ by formula (12).

Remark 5. *The measure, constructed by the above procedure solves the truncated moment problem for the set of moments $\{s_0, s_1, \ldots, s_{2N-4}\}$.*

By f^k, $k = 1, \ldots, N$ we denote the control that drive system (5) to prescribed state (see (10)):

$$W^T f_k = \phi^k, \quad k = 1, \ldots, N.$$

Due to Proposition 4, such a control exists and is unique for every k. The remarkable fact that these controls as well as the spectrum of A^N can be found from Euler-Lagrange equations for the problem of the minimization of a functional $\left(C^N Df, f\right)_{\mathcal{F}^N}$ in $\mathcal{F}_0^N$ with the constrain

$\left(C^T f, f\right)_{\mathcal{F}^N} = 1$. Similar method of deriving equations which can be used for recovering of spectral data was used in [4]. We introduce the operator

$$B^N = \begin{pmatrix} c_{N,N+1}+c_{N,N-1} & c_{N,N}+c_{N,N-2} & \dots & c_{N,3}+c_{N,1} & c_{N,2} \\ c_{N-1,N+1}+c_{N-1,N-1} & \dots & \dots & c_{N-1,3}+c_{N-1,1} & c_{N-1,2} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ c_{1,N+1}+c_{1,N-1} & c_{1,N}+c_{1,N-2} & \dots & \dots & c_{1,2} \end{pmatrix}. \tag{33}$$

The following result was obtained in [14]:

Theorem 3. *The spectrum of A^N and (non-normalized) controls f_k, $k = 1, \dots, N$ are the spectrum and eigenvectors of the following generalized spectral problem:*

$$B^N f_k = \lambda_k C^N f_k, \quad k = 1, \dots, N. \tag{34}$$

Introduce the following Hankel matrices

$$S_m^N := \begin{pmatrix} s_{2N-2+m} & s_{2N-3+m} & \dots & s_{N-1+m} \\ s_{2N-3+m} & \dots & \dots & \dots \\ \cdot & \cdot & \dots & s_{1+m} \\ s_{N-1+m} & \dots & s_{1+m} & s_m \end{pmatrix}, \quad m = 0, 1, \dots,$$

the matrix $J_N \in \mathbb{R}^{N\times N}$:

$$J_N = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 1 & \dots & 0 \\ 1 & \dots & 0 & 0 \end{pmatrix}, \quad J_N J_N = I_N = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

and define

$$\widetilde{\Lambda}_N := J_N \Lambda_N J_N.$$

The remarkable fact is that the matrices B^N, C^N can be reduced to Hankel matrices by the same linear transformation:

Theorem 4. *The following relations hold:*

$$C^N = \widetilde{\Lambda}_N S_0^N \left(\widetilde{\Lambda}_N\right)^*,$$
$$B^N = \widetilde{\Lambda}_N S_1^N \left(\widetilde{\Lambda}_N\right)^*.$$

Then the generalized spectral problem (34) upon introducing the notation $g_k = \left(\widetilde{\Lambda}_N\right)^* f_k$ is equivalent to the following generalized spectral problem:

$$S_1^N g_k = \lambda_k S_0^N g_k. \tag{35}$$

Having found spectrum and non-nomalized controls from (34) one can recover the measure of A^N with Dirichlet boundary condition at $n = N + 1$ by the following procedure:

- 1) Normalize controls by choosing $\left(C^N f_k, f_k\right)_{\mathcal{F}^N} = 1$,
- 2) Observe that $W^N f^k = \alpha_k \phi^k$ for some $\alpha_k \in \mathbb{R}$, where the constant is defined by $\alpha_k = (Rf_k)_N$.
- 3) The norming coefficients are given by $\rho_k = \alpha_k^2$, $k = 1, \dots, N$.
- 4) Recover the measure by (12).

4. Existence and uniqueness for Hamburger, Stieltjes and Hausdorff moment problems.

We remind the reader that the moment problem is called *determinate* if it has only one solution, otherwise it is called *indeterminate*. It is well-known fact [18, 17] that the uniqueness of the solution to a moment problem is related to the index of the operator A. Here we provide well-known results on discrete version of Weyl limit point-circle theory, answering the question on the index of A, which will be subsequently used. By $\xi(\lambda)$ we denote the solution to the difference equation in (10) with Cauchy data $\xi_0=-1$, $\xi_1=0$.

Proposition 6. *The Jacobi operator A is limit circle at infinity (has index equal to one) if and only if one of the following occurs:*

1) $\phi(x),\xi(x)\in l^2$ for some $x\in\mathbb{R}$,
2) $\phi(x),\varphi'(x)\in l^2$ for some $x\in\mathbb{R}$,
3) $\xi(x),\xi'(x)\in l^2$ for some $x\in\mathbb{R}$.

In [14] the authors proved the following

Theorem 5. *The set of numbers $(s_0,s_1,s_2,\ldots)$ are moments of a spectral measure corresponding to the Jacobi operator A if and only if*

$$\text{the matrix } S_0^N \text{ is positive definite for all } N\in\mathbb{N}. \tag{36}$$

The Hamburger moment problem is indeterminate if and only if

$$\lim_{T\to\infty}\left(\left(C^T\right)^{-1}\Gamma_T,\Gamma_T\right)_{\mathcal{F}^T}<+\infty,\quad \lim_{T\to\infty}\left(\left(C^T\right)^{-1}\Delta_T,\Delta_T\right)_{\mathcal{F}^T}<+\infty, \tag{37}$$

where

$$\Gamma_T:=\begin{pmatrix}\mathcal{T}_T(0)\\ \mathcal{T}_{T-1}(0)\\ \cdots\\ \mathcal{T}_1(0)\end{pmatrix},\quad \Omega_T=\begin{pmatrix}\mathcal{T}_T'(0)\\ \mathcal{T}_{T-1}'(0)\\ \cdots\\ \mathcal{T}_1'(0)\end{pmatrix}. \tag{38}$$

Here we rewrite conditions (37) in more standard form:

Proposition 7. *Conditions in (37) are equivalent to*

$$\lim_{T\to\infty}\frac{\det S_2^{T-1}}{\det S_0^T}<+\infty,\quad \lim_{T\to\infty}\frac{\det S_0^{T-1,2}}{\det S_0^T}<+\infty, \tag{39}$$

$$\text{where}\quad S_0^{T-1,2}=\begin{pmatrix}s_0 & s_2 & \ldots & s_{T-2}\\ s_2 & s_4 & \ldots & s_{T-1}\\ \ldots & & & \\ s_{T-2} & s_{T-1} & \ldots & s_{2T-3}\end{pmatrix}.$$

Indeed, bearing in mind (26) and relations

$$C^T=\tilde\Lambda_T S_0^T\tilde\Lambda_T^*,\quad \tilde\Lambda_T=J_T\Lambda_T J_T,$$

we pass to

$$\left(\left(C^T\right)^{-1}\Gamma_T,\Gamma_T\right)_{\mathcal{F}^T}=\left(\left(\tilde S_0^T\right)^{-1}e_1,e_1\right)_{\mathcal{F}^T}, \tag{40}$$

$$\left(\left(C^T\right)^{-1}\Delta_T,\Delta_T\right)_{\mathcal{F}^T}=\left(\left(\tilde S_0^T\right)^{-1}e_2,e_2\right)_{\mathcal{F}^T}. \tag{41}$$

The right hand side of above equalities can be computed using the following formula for a bilinear form of inverse matrix. Namely, for the matrix $D = (d_{ij})_{i,j=1}^n$ and vectors $h = (h_1, \dots, h_n)$, $c = (c_1, \dots, c_n)$ we have that

$$(D^{-1}b, c) = \frac{\det \begin{pmatrix} 0 & h_1 & \dots & h_n \\ c_1 & d_{11} & \dots & d_{1n} \\ \dots \\ c_n & d_{n1} & \dots & d_{nn} \end{pmatrix}}{\det \begin{pmatrix} d_{11} & \dots & d_{1n} \\ \dots \\ d_{n1} & \dots & d_{nn} \end{pmatrix}}. \tag{42}$$

Then applying (42) to (40), (41), we get (39).

The following result concerning the Stieltjes problem was formulated in [14], where the authors obtained expressions for the mass M_∞ and the length L_∞ of a string in terms of operators of the Boundary control method, associated with the dynamical system (8):

Theorem 6. *The set of numbers $(s_0, s_1, s_2, \dots)$ are moments of a spectral measure, supported on $(0, +\infty)$, corresponding to Jacobi operator A*

matrices S_0^N and S_1^N are positive definite for all $N \in \mathbb{N}$.

The Stieltjes moment problem is indeterminate if and only if the following relations hold:

$$M_\infty = \lim_{T\to\infty} \left(\left(C^T\right)^{-1} \Gamma_T, \Gamma_T \right)_{\mathcal{F}^T} < +\infty, \quad L_\infty = \lim_{K\to\infty} \frac{\left(\left(C^K\right)^{-1} \left(R^K\right)^* \Gamma_K, e_1 \right)}{\left(\left(C^K\right)^{-1} \Gamma_K, e_1 \right)} < +\infty. \tag{43}$$

Notice that the the necessity of (43) is a subtle result, see (see [18, 17]) for details. Here we reformulate (43):

Proposition 8. *Conditions in (43) are equivalent to*

$$\lim_{T\to\infty} \frac{\det S_2^{T-1}}{\det S_0^T} < +\infty, \quad \lim_{T\to\infty} \frac{\det S_{0,0}^T}{\det S_1^{T-1}} < +\infty, \tag{44}$$

$$\text{where } S_{0,0}^T = \begin{pmatrix} 0 & s_0 & \dots & s_{T-2} \\ s_0 & s_1 & \dots & s_{T-1} \\ \dots \\ s_{T-1} & s_T & \dots & s_{2T-2} \end{pmatrix}.$$

Indeed, we notice that the operator R^T has a form of a Toeplitz matrix

$$R^T = \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_0 & 0 & \dots & 0 \\ \dots \\ r_{T-2} & r_{T-3} & \dots & 0 \end{pmatrix}.$$

Bearing in mind (26) and relations

$$C^T = \tilde{\Lambda}_T S_0^T \tilde{\Lambda}_T^*, \quad \tilde{\Lambda}_T = J_T \Lambda_T J_T,$$

we pass to

$$\left(\left(C^T\right)^{-1}\Gamma_T, e_1\right)_{\mathcal{F}^T} = \left(\left(\tilde{S}_0^T\right)^{-1} e_1, e_T\right)_{\mathcal{F}^T},$$
$$\left(\left(C^T\right)^{-1}(R^T)^*\Gamma_T, e_1\right)_{\mathcal{F}^T} = \left(\left(\tilde{S}_0^T\right)^{-1} g, e_T\right)_{\mathcal{F}^T}.$$

where $g = \Lambda_T^{-1} J_T (R^T)^* \Gamma_T = (0, s_0, s_1, \dots, s_{T-2})$. Using (42) we obtain (44).

Remark 6. *Condition* $\xi(0) \in l^2$ *is equivalent to*

$$\left(\left(C^T\right)^{-1}\left(R^T\right)^*\Gamma_T, \left(R^T\right)^*\Gamma_T\right)_{\mathcal{F}^T} = \left(\left(\tilde{S}_0^T\right)^{-1} g, g\right)_{\mathcal{F}^T} < \infty,$$

where $g = (0, s_0, s_1, \dots, s_{T-2})$.

The following stetement was proved in [18], but we give a new proof in order to show that it is a direct consequence of the spectral problem (34).

Proposition 9. *Let* $\{s_k\}_{k=0}^\infty$ *be a set of Stieltjes moments, we set* $\{h_m\}_{m=0}^\infty = \{s_0, 0, s_1, 0, s_2, 0, s_3, 0, \dots\}$. *Then* $\{h_m\}_{m=0}^\infty$ *corresponds to a determinate Hamburger moment problem if and only if* $\{s_k\}_{k=0}^\infty$ *corresponds to a determinate Stieltjes moment problem.*

We use the spectral problem (35) with two matrices for Stiltjes moment problem:

$$\tilde{S}_0^T = \begin{pmatrix} s_0 & s_1 & \dots & s_{T-1} \\ s_1 & s_2 & \dots & s_T \\ \dots & & & \\ s_{T-1} & s_T & \dots & s_{2T-2} \end{pmatrix}, \quad \tilde{S}_1^T = \begin{pmatrix} s_1 & s_2 & \dots & s_T \\ s_2 & s_3 & \dots & s_{T+1} \\ \dots & & & \\ s_T & s_{T+1} & \dots & s_{2T-1} \end{pmatrix} \tag{45}$$

and two matrices for Hamburger moment problem:

$$\tilde{H}_0^{2T} = \begin{pmatrix} s_0 & 0 & s_1 & \dots & 0 \\ 0 & s_1 & 0 & \dots & s_T \\ s_1 & 0 & s_2 & \dots & 0 \\ \dots & & & & \\ 0 & s_T & 0 & \dots & s_{2T-1} \end{pmatrix}, \quad \tilde{H}_1^{2T} = \begin{pmatrix} 0 & s_1 & 0 & \dots & s_T \\ s_1 & 0 & s_2 & \dots & 0 \\ 0 & s_2 & 0 & \dots & s_{T+1} \\ \dots & & & & \\ s_T & 0 & s_{T+1} & \dots & 0 \end{pmatrix} \tag{46}$$

Simple observation $\det \tilde{H}_0^{2T} = \det \tilde{S}_0^T \det \tilde{S}_1^T$ allows us to conclude that the set $\{h_m\}_{m=0}^\infty = \{s_0, 0, s_1, 0, s_2, 0, s_3, 0, \dots\}$ indeed corresponds to Hamburger moment problem. Even more, if we look for eigenvalues of (35) with Hamburger matrices (46), we see that

$$0 = \det(\lambda \tilde{H}_0^{2T} - \tilde{H}_1^{2T}) = \det \begin{pmatrix} \lambda s_0 & -s_1 & \lambda s_1 & \dots & -s_T \\ -s_1 & \lambda s_1 & -s_2 & \dots & \lambda s_T \\ \lambda s_1 & -s_2 & \lambda s_2 & \dots & -s_{T+1} \\ \dots & & & & \\ -s_T & \lambda s_T & -s_{T+1} & \dots & \lambda s_{2T-1} \end{pmatrix} =$$
$$= \det \begin{pmatrix} \lambda^2 s_0 - s_1 & 0 & \lambda^2 s_1 - s_2 & \dots & 0 \\ 0 & s_1 & 0 & \dots & \lambda^2 s_T - s_{T+1} \\ \lambda^2 s_1 - s_2 & 0 & \lambda^2 s_2 - s_3 & \dots & 0 \\ \dots & & & & \\ 0 & s_T & 0 & \dots & s_{2T-1} \end{pmatrix} = \det(\lambda^2 S_0^T - S_1^T) \det S_1^T.$$

The later means that if $0 < \mu$ is an eigenvalue for (35) with Stieltjes matrices (45), then $\pm\sqrt{\mu}-$ are two eigenvalues for (35) with Hamburger matrices (46) and vise-versa.

To prove that Hamburger and Stieltjes problems are determinate simultaneously we note that

$$\frac{\det H_2^{2T-1}}{\det H_0^{2T}} = \frac{\det S_1^T \det S_2^{T-1}}{\det S_0^T \det S_1^T} = \frac{\det S_2^{T-1}}{\det S_0^T}$$

and

$$\left(\left(\tilde{H}_0^{2T}\right)^{-1} g, g\right)_{\mathcal{F}^T} = \frac{1}{\det H_0^{2T}} \det \begin{pmatrix} 0 & 0 & s_0 & \dots & s_{T-1} \\ 0 & s_0 & 0 & \dots & 0 \\ s_0 & 0 & s_1 & \dots & s_T \\ \dots \\ s_{T-1} & 0 & s_T & \dots & s_{T-1} \end{pmatrix} = \frac{\det S_0^T \det S_{0,0}^{T+1}}{\det S_0^T \det S_1^T} = \frac{\det S_{0,0}^{T+1}}{\det S_1^T}.$$

Using Theorem 7 for Stieltjes problem and Theroem 5 and Remark 5 for Hamburger we complete the proof.

As we saw, the reproducing kernel $K^N(z, \lambda)$ has a form (25). Using (42) and carrying out similar transformations, we obtain the following

Remark 7. *The reproducing kernel admits the following representation:*

$$K^N(z, \lambda) = \frac{1}{\det S_0^T} \det \begin{pmatrix} 0 & 1 & z & \dots & z^{N-1} \\ 1 & s_0 & s_1 & \dots & s_{T-1} \\ \lambda & s_1 & s_2 & \dots & s_T \\ \dots \\ \lambda^{N-1} & s_{T-1} & s_T & \dots & s_{2T-2} \end{pmatrix}$$

In [14] the authors used (35) to prove the following

Theorem 7. *The set of numbers $(s_0, s_1, s_2, \dots)$ are moments of a spectral measure, supported on $(0, 1)$, corresponding to operator A if and only if the condition*

$$S_0^N \geqslant S_1^N > 0 \quad \textit{holds for all } N \in \mathbb{N}$$

The Hausdorff moment problem is determinate.

Conclusion

In the present paper we considered the dynamical system (5) associated with Jacobi matrix with the Dirichlet boundary condition at the "right end". In the third section it is shown how to use the Boundary control method to recover the measure $d\rho_N$ of the operator A^N (6), (7) from the finite set of moments. After taking the limit $d\rho^N(\lambda) \mapsto d\rho^*(\lambda)$, where convergence is understood in the week sense, we have two options: when A is in limit point case at infinity, $d\rho^*(\lambda)$ is a unique solution to a moment problem, but when A is in limit circle case, $d\rho^*(\lambda)$ gives certain distinguished solution (week limit of measures corresponding to operators with Dirichlet condition). At the same time it is well-known [1, 18, 17] that the answer in Hamburger moment problem is a spectral measures of any self-adjoint extension of Jacobi operator A.

The prospective problem is to use dynamic inverse data $\{r_0, r_1, \dots\}$ of the dynamical system (8) associated with A, which is obtained from the set of moments $\{s_0, s_1, \dots\}$ by (28), for the construction of the dynamic model of self-adjoint extensions of operator A in the spirit of [9, 10].

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