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To our dear friend Mohammad Reza Darafsheh,
with affection and admiration

COMMUTATORS OF RELATIVE AND UNRELATIVE ELEMENTARY UNITARY GROUPS

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In the present paper, which is an outgrowth of our joint work with Anthony Bak and Roozbeh Hazrat on unitary commutator calculus [9,27,30,31], we find generators of the mixed commutator subgroups of relative elementary groups and obtain unrelativized versions of commutator formulas in the setting of Bak's unitary groups. It is a direct sequel of our papers [71, 76, 78, 79] and [77, 80], where similar results were obtained for $\mathrm{GL}(n, R)$ and for Chevalley groups over a commutative ring with 1, respectively. Namely, let (A, Λ) be any form ring and let $n \geqslant 3$. We consider Bak's hyperbolic unitary group $\mathrm{GU}(2n, A, \Lambda)$. Further, let (I, Γ) be a form ideal of (A, Λ). One can associate with the ideal (I, Γ) the corresponding true elementary subgroup $\mathrm{FU}(2n, I, \Gamma)$ and the relative elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$ of $\mathrm{GU}(2n, A, \Lambda)$. Let (J, Δ) be another form ideal of (A, Λ). In the present paper we prove an unexpected result that the nonobvious type of generators for $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$, as constructed in our previous papers with Hazrat, are redundant and can be expressed as products of the obvious generators, the elementary conjugates $Z_{ij}(\xi, c) = T_{ji}(c)T_{ij}(\xi)T_{ji}(-c)$, and the elementary commutators $Y_{ij}(a, b) = [T_{ij}(a), T_{ji}(b)]$, where $a \in (I, \Gamma)$, $b \in (J, \Delta)$, $c \in (A, \Lambda)$, and $\xi \in (I, \Gamma) \circ (J, \Delta)$. It follows that $[\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$. In fact, we establish much more precise generation results. In particular, even the elementary commutators $Y_{ij}(a, b)$ should be taken for one long root position and one short root position. Moreover, the $Y_{ij}(a, b)$ are central modulo $\mathrm{EU}(2n, (I, \Gamma) \circ (J, \Delta))$ and behave as symbols. This allows us to generalize and unify many previous results, including the multiple elementary commutator formula, and dramatically simplify their proofs.

Ключевые слова: Bak's unitary groups, elementary subgroups, congruence subgroups, standard commutator formula, unrelativized commutator formula, elementary generators, multiple commutator formula.

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Introduction

In a series of our joint papers with Anthony Bak and Roozbeh Hazrat [9,27, 30,31] we studied commutator formulas in Bak's unitary groups. In the present paper we generalize, refine and strengthen some of the main results of these works. Namely, we discover that the set of generators for the mixed commutator subgroup of relative elementary unitary groups listed in these papers can be substantially reduced and remove all commutativity conditions therein.[1] This allows us to prove unexpected unrelative versions of the commutator formulas, generalize multiple elementary commutator formulas, and more. These results both improve a great number of previous results, and path the way to several new unexpected applications.

Morally, the present paper is a direct sequel our papers [71, 76, 78, 79] and [77,80], where the same was done for $\mathrm{GL}(n, R)$ and for Chevalley groups over a commutative ring with 1, respectively. There, the proofs heavily relied on our previous works, in particular on [32, 33, 65, 74, 75] for $\mathrm{GL}(n, R)$ and on [28, 29] for Chevalley groups. Similarly, the present paper heavily hinges on the results of [9, 27, 30, 31].

0.1. The prior state of art. To enunciate the main results of the present paper, let us briefly recall the notation, which will be reviewed in somewhat more detail in §§1–4. Let (A, Λ) be a form ring, $n \geqslant 3$, and let $\mathrm{GU}(2n, A, \Lambda)$ be the hyperbolic Bak's unitary group. Below, $\mathrm{EU}(2n, A, \Lambda)$ denotes the [absolute] elementary unitary group, generated by the elementary root unipotents.

As usual, for a form ideal (I, Γ) of the form ring (A, Λ) we denote by

$$\mathrm{FU}(2n, I, \Gamma)$$

the unrelative elementary subgroup of level (I, Γ), and by

$$\mathrm{EU}(2n, I, \Gamma)$$

the relative elementary subgroup of level (I, Γ). By definition, $\mathrm{EU}(2n, I, \Gamma)$ is the normal closure of $\mathrm{FU}(2n, I, \Gamma)$ in $\mathrm{EU}(2n, A, \Lambda)$. Further, $\mathrm{GU}(2n, I, \Gamma)$ and $\mathrm{CU}(2n, I, \Gamma)$ denote the principal congruence subgroup and the full congruence subgroup of level (I, Γ), respectively.

We recapitulate two principal results of our joint papers with Roozbeh Hazrat, [27, 30, 31]. The first one is the birelative standard commutator formula, see [27, Theorems 1 and 2]. It is a very broad generalization of the commutator formulas for unitary groups, previously established by Anthony Bak, the first author, Leonid Vaserstein, Hong You, Günter Habdank, and others, see, for instance, [1, 2, 6, 9, 17, 18, 69].

[1] In particular, this solves [23, Problem 1] and [30, Problem 1].

Theorem 1. *Let R be a commutative ring, and (A,Λ) a form ring such that A is a quasifinite R-algebra. Further, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ) and let $n\geqslant 3$. Then the following commutator identity holds*

$$[\mathrm{GU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

When A is itself commutative, one even has

$$[\mathrm{CU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

Another crucial result is the description of a generating set for the mixed commutator subgroup $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ *as a group*, similar to the familiar generating set for relative elementary subgroups, see [9], Proposition 5.1 (compare with Lemma 3 below).

Recall that we denote by $T_{ij}(a)$ elementary unitary transvections. They come in two denominations, those of *short root type*, when $i\neq\pm j$, and those of *long root type*, when $i=-j$. The corresponding root subgroups are then parametrized by the ring A itself and by the form parameter Λ, respectively. To simplify the notation in the relative case, we introduce the following convention. For a form ideal (I,Γ), we write $a\in(I,\Gamma)$ to signalize that $a\in I$ if $i\neq\pm j$, and $a\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma$ if $i=-j$. Clearly, $a\in(I,\Gamma)$ means precisely that $T_{ij}(a)\in\mathrm{EU}(2n,I,\Gamma)$, see §§3,4 for details.

Further, we consider the elementary conjugates $Z_{ij}(a,c)$ and the elementary commutators $Y_{ij}(a,b)$, which are defined as follows:

$$Z_{ij}(a,c)=T_{ji}(c)T_{ij}(a)T_{ji}(-c),\quad Y_{ij}(a,b)=[T_{ij}(a),T_{ji}(b)],$$

In a slightly weaker form, the following result was stated as Theorem 9 of [31], and in precisely this form as Theorem 3B of [30]. Observe that there its proof depended on Theorem A, and thus ultimately, on localization methods.

Theorem 2. *Let R be a commutative ring, and (A,Λ) a form ring such that A is a quasifinite R-algebra. Let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ), and let $n\geqslant 3$. The relative commutator subgroup*

$$[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$$

is generated by the elements of the following three types:

- $Z_{ij}(\xi,c)$,
- $Y_{ij}(a,b)$,
- $[T_{ij}(a),Z_{ij}(b,c)]$,

where in all cases $a\in(I,\Gamma)$, $b\in(J,\Delta)$, $c\in(A,\Lambda)$, and $\xi\in(I,\Gamma)\circ(J,\Delta)$.

0.2. Statement of the principal result. The technical core of the present paper are Lemmas 6–12 that we prove in §§5–8. Together they imply that the above Theorem B can be *drastically* generalized and improved, as follows.

- We can lift the commutativity condition.
- The third type of generators are redundant.
- The second type of generators can be restricted to one long and one short root (and are subject to further relations, to be stated below).

The following result is the pinnacle of the present paper, other results are either preparation to its proof, or its easy consequence. For the general linear group $\mathrm{GL}(n,R)$ it was established in [76, Theorem 1]. For the Chevalley groups $G(\Phi,R)$ over commutative rings — and thus, in particular, for the usual symplectic group $\mathrm{Sp}(2n,R)$ and the split orthogonal group $\mathrm{SO}(2n,R)$ — it is essentially a conjunction of [77, Theorem 1.2] and [80, Theorem 1]. However, as explained below, in these special cases one can say somewhat more.

Theorem 1. *Let (A,Λ) be any associative form ring, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ) and let $n \geqslant 3$. Then the relative commutator subgroup $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ is generated by the elements of the following two types:*

- *$Z_{ij}(\xi,c)$ and $Z_{ij}(\xi,c)$,*
- *$Y_{ij}(a,b)$,*

where in all cases $a\in(I,\Gamma)$, $b\in(J,\Delta)$, $c\in(A,\Lambda)$, and $\xi\in(I,\Gamma)\circ(J,\Delta)$. Moreover, for the second type of generators it suffices to take one pair (h,k), $h\neq\pm k$, and one pair $(h,-h)$.

The difference with Chevalley groups is that now we have to throw in elementary commutators for *two* roots, one long root and one short root. For Chevalley groups, one *long* root would suffice. Conversely, when 2 is invertible for types $\mathrm{B}_l,\mathrm{C}_l,\mathrm{F}_4$ and 3 is invertible for type G_2, one *short* root would suffice. For unitary groups, modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ we can still establish a cognate relation between short root type elementary commutators and long root type elementary commutators, Lemma 13. However, unlike Chevalley groups, for unitary groups the elements of long root subgroups are parametrized by the form parameter Λ, whereas the elements of short root subgroups are parametrized by the ring A itself. This means that now we could dispose of *some* short type elementary commutators, yet not all of them. In the opposite direction, the long type elementary commutators, one of whose arguments sits in the corresponding *minimal* ideal form parameter could be discarded — but not all of them! This can be done when one of the form parameters is either minimal, or as large as possible — not merely maximal! — see §9.

Observe that the proof of this theorem consists of two independent parts. The possibility to express the third type of generators as products of elementary conjugates and elementary commutators in $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ will be called the *first claim* of Theorem 1. The much more arduous bid that modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ all elementary commutators can be expressed in terms of such commutators in one short and two long positions, will be called the *second claim* of Theorem 1.

We mention another important trait. The published proofs of Theorem B heavily depended on some version of Theorem A, and thus, ultimately, on localization. The proof of Theorem 1 given below in §§5–7 is purely *elementary*[2] and thus works already at the level of *unitary Steinberg groups*, see [1, 2, 36]. The only reason why we do not state our results in this generality is to skip the discussion of *relative unitary Steinberg groups*. The details and technical facts are not readily available in the literature, and would noticeably increase the length of the present paper.

0.3. Unrelativization. Since both remaining types of generators listed in Theorem 1 already belong to the mixed commutator of the *unrelative* elementary subgroups $[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]$, we get the amazing equality in Theorem 2. Morally, it shows that the commutator of *relative* elementary subgroups $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ is smaller than one expects. Observe that it only depends on the [relatively] easy first claim of Theorem 1 whose proof is completed already in §5. For $\mathrm{GL}(n,R)$ the corresponding result is [71], Theorem 2 (for commutative rings, with a completely different proof), and [76], Theorem 1 (for arbitrary associative rings). For $\mathrm{Sp}(2n,R)$ and $\mathrm{SO}(2n,R)$ it is a special case of [77], Theorem 1.2.

Theorem 2. *Let (A,Λ) be any associative form ring, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ) and let $n\geqslant 3$. Then the mixed commutator subgroup $[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]$ is normal in $\mathrm{EU}(2n,A,\Lambda)$. Furthermore, we have the following commutator identity*

$$[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]=[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

In particular, in conjunction with Theorem A this shows that the birelative standard commutator formula also holds in the following unrelativized form. Again, for $\mathrm{GL}(n,R)$ this is [71], Theorem 1 and [76], Theorem 3, whereas for Chevalley groups it is [77], Theorem 1.3.

Theorem 3. *Let R be a commutative ring, and (A,Λ) a form ring such that A is a quasifinite R-algebra. Further, let (I,Γ) and (J,Δ) be two form ideals*

[2] In the technical sense that it does not invoke anything apart from the usual Steinberg relations.

of the form ring (A,Λ) *and let* $n\geqslant 3$*. Then we have an unrelative commutator identity*

$$[\mathrm{GU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)].$$

When A *is itself commutative, one even has*

$$[\mathrm{CU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)].$$

The following result is a unitary analog of the unrelative normality theorem proved for $\mathrm{GL}(n,R)$ by Bogdan Nica and ourselves, see [44, 71, 78]. It is an immediate corollary to our Theorem 3, if we set there $(I,\Gamma)=(J,\Delta)$.

Theorem 4. *Let* R *be a commutative ring, and* (A,Λ) *a form ring such that* A *is a quasifinite* R*-algebra. Further, let* (I,Γ) *be a form ideal of the form ring* (A,Λ) *and let* $n\geqslant 3$*. Then* $\mathrm{FU}(2n,I,\Gamma)$ *is normal in* $\mathrm{GU}(2n,I,\Gamma)$*.*

0.4. Elementary commutators. The proof of the second claim of Theorem 1 is the gist of the present paper, and proceeds as follows. First, in §6 we prove that the elementary commutators $Y_{ij}(a,b)$ are central in the absolute elementary group modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$. Recall that here

$$(I,\Gamma)\circ(J,\Delta)=(IJ+JI,{}^J\Gamma+{}^I\Delta+\Gamma_{\min}(IJ+JI))$$

denotes the symmetrized product of form ideals, see §2 for details.

Since by that time we already know that together with $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ these commutators generate $[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]$, this result can be stated as follows. For $\mathrm{GL}(n,R)$ and Chevalley groups this is [76, Theorem 2], and [80, Theorem 2], respectively.

Theorem 5. *Let* (A,Λ) *be any associative form ring, let* (I,Γ) *and* (J,Δ) *be two form ideals of the form ring* (A,Λ)*, and let* $n\geqslant 3$*. Then*

$$[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]$$

is central in $\mathrm{EU}(2n,A,\Lambda)$ *modulo* $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$*.*

In other words,

$$\big[[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)],\mathrm{EU}(2n,A,\Lambda)\big]\leqslant\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

In particular, it implies that the quotient

$$[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]/\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$$

is itself Abelian. This readily implies the additivity of the elementary commutator with respect to its arguments, and other similar useful properties, collected in Theorem 10, that are employed in the proofs of subsequent results.

However, the focal point of the present paper is §7, where we prove that modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ all elementary commutators of the same root type are equivalent. Moreover, for the short root type they are balanced with respect to the factors from R, both on the left and on the right. For the long root type, the balancing property is more complicated, and only occurs for the quadratic (=Jordan) multiplication. In the case of the usual symplectic group, where A is a commutative ring with trivial involution, it corresponds to the multiplication by squares, see [80, Theorem 5].

Theorem 6. *Let (A,Λ) be an associative form ring with 1, $n\geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ).*

- *Then for any $i\neq\pm j$, any $h\neq\pm l$ with $h,l\neq\pm i,\pm j$, and $a\in I$, $b\in J$, $c,d\in A$, the elementary commutator obeys the relation*

$$Y_{ij}(cad,b)\equiv Y_{hl}(a,dbc)\,(\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

- *Then for any $-n\leqslant i\leqslant n$, any $-n\leqslant k\leqslant n$, and $a\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $b\in\lambda^{(\varepsilon(i)-1)/2}\Delta$ $c\in A$, the elementary commutator obeys the relation*

$$\begin{aligned}&Y_{i,-i}(ca\overline{c},b)\\&\equiv Y_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a,-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)\,(\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.\end{aligned}$$

The calculation behind these congruences is the highlight of the whole theory. Inherently, it is precisely a birelative incarnation of a classical calculation that appeared dozens of times in the algebraic K-theory and the theory of algebraic groups since mid 60s, see §12 for a terse historical medley.

0.5. Further corollaries. As another illustration of the power of Theorem 1, we show that it allows us to [almost completely] lift commutativity conditions in some of the principal results of [27, 30, 31].

Under the additional assumptions such as quasifiniteness, the following result for any $n\geqslant 3$ is [31, Theorem 7]. From Theorem 1 we can derive that for $n\geqslant 4$ a similar result holds true for arbitrary associative form rings. For $\mathrm{GL}(n,R)$ such a generalization was already obtained in [76]. We believe this could be also done for $n=3$, see Problem 3, but in that case it would require formidable calculations.

Theorem 7. *Let (A,Λ) be any associative form ring with 1, let $n\geqslant 4$, and let $(I_i,\Gamma_i)\trianglelefteqslant R$, $i=1,\dots,m$, be form ideals of (A,Λ). Consider an arbitrary arrangement of brackets $[\![\ldots]\!]$ with the cut point s. Then one has*

$$\begin{aligned}&[\![\,\mathrm{EU}(2n,I_1,\Gamma_1),\mathrm{EU}(2n,I_2,\Gamma_2),\dots,\mathrm{EU}(2n,I_m,\Gamma_m)]\!]\\&=\big[\,\mathrm{EU}(2n,(I_1,\Gamma_1)\circ\ldots\circ(I_s,\Gamma_s)),\mathrm{EU}(2n,(I_{s+1},\Gamma_{s+1})\circ\ldots\circ(I_m,\Gamma_m))\big],\end{aligned}$$

where the bracketing of symmetrized products on the right-hand side coincides with the bracketing of the commutators on the left-hand side.

Under the additional assumption that the absolute standard commutator formulas are satisfied, the following result is [27, Theorem 3]. As we know from [9, 20, 21, 27], this condition is satisfied for quasifinite rings. But from the work of Victor Gerasimov [16] it follows that some commutativity or finiteness assumptions are necessary for the standard commutator formulas to be true. Now, we are in a position to prove the following result for *arbitrary* associative form rings.

Theorem 8. *Let (A,Λ) be any associative form ring and let $n\geqslant 3$. Then for any two comaximal form ideals (I,Γ) and (J,Δ) of the form ring (R,Λ), $I+J=A$, one has*

$$[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]=\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

Another bizarre corollary to Theorem 1 is the surjective stability of the quotients

$$[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]/\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),$$

again for *arbitrary* associative form rings, without any stability conditions, or commutativity conditions. This is a typical result in the style of Bak's paradigm "stability results without stability conditions," see [3] and also [4, 20, 21, 25, 26].

Theorem 9. *Let (A,Λ) be any associative form ring, let (I,Γ) and (J,Δ) be two form ideals of the form ring (A,Λ), and let $n\geqslant 3$. Then the stability map*

$$\begin{aligned}&[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]/\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))\\&\longrightarrow[\mathrm{FU}(2(n+1),I,\Gamma),\mathrm{FU}(2(n+1),J,\Delta)]/\,\mathrm{EU}(2(n+1),(I,\Gamma)\circ(J,\Delta))\end{aligned}$$

is surjective.

Indeed, in view of Theorems 1 and 5, as a normal subgroup of $\mathrm{EU}(2n,A,\Lambda)$, the group

$$[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$$

is generated by

$$[\mathrm{EU}(6,I,\Gamma),\mathrm{EU}(6,J,\Delta)].$$

An explicit calculation of these quotients presents itself as a natural next step. However, so far we were unable to resolve it, apart from some special cases, see a discussion in §12.

0.6. Organization of the paper. The rest of the paper is devoted to the proof of these results. In §§1–4 we recall the necessary definitions and collect requisite preliminary results. The next four sections §§5–8 are the technical core of the paper. Namely, in §5 we prove Theorem 5 and derive first consequences of it. In §6 we reduce the set of generators of $[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]$ to the first two types. In §7 we prove Theorem 6 and then in §8 establish another cognate result, relating *some* elementary commutators of short root type with *some* elementary commutators of long root type. This finishes the proof of Theorem 1 and its corollaries, and, in particular, also of Theorems 2–4. In §9 we establish the special cases of Theorem 7 pertaining to triple and quadruple commutators, and then in §10 derive Theorem 7 itself by an easy induction. In §11 we derive Theorem 8 and yet another corollary to our main results. Finally, in §12 we describe the general context, briefly review recent related publications, and state several further related open problems.

§1. Notation

Here we recall some basic notation that will be used throughout the present paper.

1.1. General linear group. Let, as above, A be an associative ring with 1. For natural m, n we denote by $M(m,n,A)$ the additive group of $m\times n$ matrices with entries in A. In particular $M(m,A)=M(m,m,A)$ is the ring of matrices of degree m over A. For a matrix $x\in M(m,n,A)$ we denote by x_{ij}, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant n$, its entry in the position (i,j). Let e be the identity matrix and e_{ij}, $1\leqslant i,j\leqslant m$, a standard matrix unit, i.e., the matrix that has 1 in the position (i,j) and zeros elsewhere.

As usual, $\mathrm{GL}(m,A)=M(m,A)^*$ denotes the general linear group of degree m over A. The group $\mathrm{GL}(m,A)$ acts on the free right A-module $V\cong A^m$ of rank m. Fix a base $e_1,\dots,e_m$ of the module V. We may think of elements $v\in V$ as columns with components in A. In particular, e_i is the column whose ith coordinate is 1, while all other coordinates are zeros.

Actually, in the present paper we are only interested in the case when $m=2n$ is even. We usually number the base as follows: $e_1,\dots,e_n,e_{-n},\dots,e_{-1}$. All other occurring geometric objects will be numbered accordingly. Thus, we write $v=(v_1,\dots,v_n,v_{-n},\dots,v_{-1})^t$, where $v_i\in A$, for vectors in $V\cong A^{2n}$.

The set of indices will be always ordered in conformity with this convention, $\Omega=\{1,\dots,n,-n,\dots,-1\}$. Clearly, $\Omega=\Omega^+\sqcup\Omega^-$, where $\Omega^+=\{1,\dots,n\}$ and $\Omega^-=\{-n,\dots,-1\}$. For an element $i\in\Omega$ we denote by $\varepsilon(i)$ the sign of Ω, i.e., $\varepsilon(i)=+1$ if $i\in\Omega^+$, and $\varepsilon(i)=-1$ if $i\in\Omega^-$.

1.2. Commutators. Let G be a group. For any $x,y\in G$, ${}^xy=xyx^{-1}$ and $y^x=x^{-1}yx$ denote the left conjugate and the right conjugate of y by x, respectively. As usual, $[x,y]=xyx^{-1}y^{-1}$ denotes the left-normed commutator of x and y. Throughout the present paper we repeatedly use the following commutator identities:

(C1) $[x, yz] = [x, y] \cdot {}^{y}[x, z]$,

(C1$^+$) an easy induction, with the use of identity (C1), shows that

$$\Big[x, \prod_{i=1}^{k} u_i\Big] = \prod_{i=1}^{k} {}^{\prod_{j=1}^{i-1} u_j}[x, u_i],$$

where by convention $\prod_{j=1}^{0} u_j = 1$,

(C2) $[xy, z] = {}^{x}[y, z] \cdot [x, z]$,

(C2$^+$) as in (C1$^+$), we have

$$\Big[\prod_{i=1}^{k} u_i, x\Big] = \prod_{i=1}^{k} {}^{\prod_{j=1}^{k-i} u_j}[u_{k-i+1}, x],$$

(C3) ${}^{x}[[x^{-1}, y], z] \cdot {}^{z}[[z^{-1}, x], y] \cdot {}^{y}[[y^{-1}, z], x] = 1$,

(C4) $[x, {}^{y}z] = {}^{y}[{}^{y^{-1}}x, z]$,

(C5) $[{}^{y}x, z] = {}^{y}[x, {}^{y^{-1}}z]$,

(C6) if H and K are subgroups of G, then $[H, K] = [K, H]$.

Especially important is (C3), the celebrated *Hall–Witt identity*. Sometimes it is used in the following form, known as the *three subgroup lemma*.

Lemma 1. *Let $F, H, L \trianglelefteq G$ be three normal subgroups of G. Then*

$$[[F, H], L] \leqslant [[F, L], H] \cdot [F, [H, L]].$$

§2. Form rings and form ideals

The notion of Λ-quadratic forms, quadratic modules, and generalized unitary groups over a form ring (A, Λ) were introduced by Anthony Bak in his Thesis, see [1, 2]. In this section, and in the next one, we *very briefly* review the most fundamental notation and results that will be constantly used in the sequel. We refer to [2,9,11,19–21,27,30,31,35,36,46,67] for details, proofs, and further references. In the final section we mention some further related recent works, and some generalizations.

2.1. Form rings. Let R be a commutative ring with 1, and A a (not necessarily commutative) R-algebra. An involution, denoted by $\overline{}$, is an anti-homomorphism of A of order 2. Namely, for $a, b \in A$, one has

$$\overline{a+b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b}\,\overline{a}, \quad \overline{\overline{a}} = a.$$

Fix an element $\lambda \in \mathrm{Cent}(A)$ such that $\lambda\overline{\lambda} = 1$. One may define two additive subgroups of A as follows:

$$\Lambda_{\min} = \{c - \lambda\overline{c} \mid c \in A\}, \quad \Lambda_{\max} = \{c \in A \mid c = -\lambda\overline{c}\}.$$

A *form parameter* Λ is an additive subgroup of A such that

(1) $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$,

(2) $c\Lambda\overline{c} \subseteq \Lambda$ for all $c \in A$.

The pair (A, Λ) is called a *form ring*.

2.2. Form ideals. Let $I \trianglelefteq A$ be a two-sided ideal of A. We assume I to be involution invariant, i.e., such that $\overline{I} = I$. Set

$$\Gamma_{\max}(I) = I \cap \Lambda, \quad \Gamma_{\min}(I) = \{a - \lambda\overline{a} \mid a \in I\} + \langle ac\overline{a} \mid a \in I, c \in \Lambda\rangle.$$

A *relative form parameter* Γ in (A, Λ) of level I is an additive group of I such that

(1) $\Gamma_{\min}(I) \subseteq \Gamma \subseteq \Gamma_{\max}(I)$,

(2) $c\Gamma\overline{c} \subseteq \Gamma$ for all $c \in A$.

The pair (I, Γ) is called a *form ideal*.

In the level calculations we will use sums and products of form ideals. Let (I, Γ) and (J, Δ) be two form ideals. Their sum is artlessly defined as $(I + J, \Gamma + \Delta)$, it is immediate to verify that this is indeed a form ideal.

Guided by analogy, one is tempted to set $(I, \Gamma)(J, \Delta) = (IJ, \Gamma\Delta)$. However, it is considerably harder to consistently define the product of two relative form parameters. The papers [17, 18, 20, 21] introduce the following definition

$$\Gamma\Delta = \Gamma_{\min}(IJ) + {}^J\Gamma + {}^I\Delta,$$

where

$${}^J\Gamma = \langle b\Gamma\overline{b} \mid b \in J\rangle, \qquad {}^I\Delta = \langle a\Delta\overline{a} \mid a \in I\rangle.$$

One can verify that this is indeed a relative form parameter of level IJ if $IJ = JI$.

However, in the present paper we do not wish to impose any such commutativity assumptions. Thus, we are forced to consider the symmetrized products

$$I \circ J = IJ + JI, \quad \Gamma \circ \Delta = \Gamma_{\min}(IJ + JI) + {}^J\Gamma + {}^I\Delta.$$

The notation $\Gamma\circ\Delta$ (as also $\Gamma\Delta$) is slightly misleading, because in fact it depends on I and J, not only on Γ and Δ. Thus, strictly speaking, one should talk of the symmetrized products of *form ideals*

$$(I, \Gamma) \circ (J, \Delta) = (IJ + JI, \Gamma_{\min}(IJ + JI) + {}^J\Gamma + {}^I\Delta).$$

Clearly, in the above notation one has

$$(I, \Gamma) \circ (J, \Delta) = (I, \Gamma)(J, \Delta) + (J, \Delta)(I, \Gamma).$$

§3. Unitary groups

In the present section we recall basic notation and facts related to Bak's generalized unitary groups.

3.1. Unitary group. For a form ring (A, Λ), one considers the *hyperbolic unitary group* $\mathrm{GU}(2n, A, \Lambda)$, see [9, §2]. This group is defined as follows.

One fixes a symmetry $\lambda \in \mathrm{Cent}(A)$, $\lambda\overline{\lambda} = 1$, and supplies the module $V = A^{2n}$ with the following λ-hermitian form $h : V \times V \longrightarrow A$,

$$h(u, v) = \overline{u}_1 v_{-1} + \ldots + \overline{u}_n v_{-n} + \lambda\overline{u}_{-n} v_n + \ldots + \lambda\overline{u}_{-1} v_1,$$

and the following Λ-quadratic form $q : V \longrightarrow A/\Lambda$,

$$q(u) = \overline{u}_1 u_{-1} + \ldots + \overline{u}_n u_{-n} \mod \Lambda.$$

In fact, both forms are engendered by a sesquilinear form f,

$$f(u, v) = \overline{u}_1 v_{-1} + \ldots + \overline{u}_n v_{-n}.$$

Now, $h = f + \lambda\overline{f}$, where $\overline{f}(u, v) = \overline{f(v, u)}$, and $q(v) = f(u, u) \mod \Lambda$.

By definition, the hyperbolic unitary group $\mathrm{GU}(2n, A, \Lambda)$ consists of all elements from $\mathrm{GL}(V) \cong \mathrm{GL}(2n, A)$ preserving the λ-Hermitian form h and the Λ-quadratic form q. In other words, $g \in \mathrm{GL}(2n, A)$ belongs to $\mathrm{GU}(2n, A, \Lambda)$ if and only if

$$h(gu, gv) = h(u, v) \quad \text{and} \quad q(gu) = q(u), \qquad \text{for all} \quad u, v \in V.$$

When the form parameter is neither maximal nor minimal, these groups are not algebraic. However, their internal structure is very similar to that of the usual classical groups. They are also oftentimes called general quadratic groups, or classical-like groups.

3.2. Unitary transvections. *Elementary unitary transvections* $T_{ij}(\xi)$ correspond to the pairs $i, j \in \Omega$ such that $i \neq j$. They come in two stocks. Namely, if, moreover, $i \neq -j$, then for any $c \in A$ we set

$$T_{ij}(c) = e + ce_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{c}e_{-j,-i}.$$

These elements are also often called *elementary short root unipotents*. On the other side for $j = -i$ and $c \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda$ we set

$$T_{i,-i}(c) = e + ce_{i,-i}.$$

These elements are also often called *elementary long root elements*.

Note that $\overline{\Lambda} = \overline{\lambda}\Lambda$. In fact, for any element $c \in \Lambda$ one has $\overline{c} = -\overline{\lambda}c$ and thus $\overline{\Lambda}$ coincides with the set of products $\overline{\lambda}c$, where $c \in \Lambda$. This means that in the above definition $c \in \overline{\Lambda}$ when $i \in \Omega^+$ and $c \in \Lambda$ when $i \in \Omega^-$.

Subgroups $X_{ij} = \{T_{ij}(c) \mid c \in A\}$, where $i \neq \pm j$, are called *short root subgroups*. Clearly, $X_{ij} = X_{-j,-i}$. Similarly, subgroups $X_{i,-i} = \{T_{ij}(c) \mid c \in \lambda^{-(\varepsilon(i)+1)/2}\Lambda\}$ are called *long root subgroups*.

The *elementary unitary group* $\mathrm{EU}(2n, A, \Lambda)$ is generated by elementary unitary transvections $T_{ij}(c)$, $i \neq \pm j$, $c \in A$, and $T_{i,-i}(c)$, $c \in \Lambda$, see [9, §3].

3.3. Steinberg relations. Elementary unitary transvections $T_{ij}(c)$ satisfy the following *elementary relations*, also known as *Steinberg relations*. These relations will be used throughout this paper.

(R1) $T_{ij}(c) = T_{-j,-i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{c})$,

(R2) $T_{ij}(c)T_{ij}(d) = T_{ij}(c+d)$,

(R3) $[T_{ij}(c), T_{hk}(d)] = e$, where $h \neq j, -i$ and $k \neq i, -j$,

(R4) $[T_{ij}(c), T_{jh}(d)] = T_{ih}(cd)$, where $i, h \neq \pm j$ and $i \neq \pm h$,

(R5) $[T_{ij}(c), T_{j,-i}(d)] = T_{i,-i}(cd - \lambda^{-\varepsilon(i)}\overline{d}\overline{c})$, where $i \neq \pm j$,

(R6) $[T_{i,-i}(c), T_{-i,j}(d)] = T_{ij}(cd)T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{d}ad)$, where $i \neq \pm j$.

Relation (R1) coordinates two natural parametrizations of the same short root subgroup $X_{ij} = X_{-j,-i}$. Relation (R2) expresses the additivity of the natural parametrizations. All other relations are various instances of the Chevalley commutator formula. Namely, (R3) corresponds to the case where the sum of two roots is not a root, whereas (R4) and (R5) correspond to the case of two short roots, whose sum is a short root, and a long root, respectively. Finally, (R6) is the Chevalley commutator formula for the case of a long root and a short root whose sum is a root. Observe that any two long roots are either opposite, or orthogonal, so that their sum is never a root.

§4. Relative subgroups

In this section we recall definitions and basic facts concerning relative subgroups. For the proofs of these results, see [9].

4.1. Relative subgroups. One associates with a form ideal (I, Γ) the following four relative subgroups.

• The subgroup $\mathrm{FU}(2n, I, \Gamma)$ generated by elementary unitary transvections of level (I, Γ),

$$\mathrm{FU}(2n, I, \Gamma) = \big\langle T_{ij}(a) \mid \ a \in I \text{ if } i \neq \pm j \text{ and } a \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma \text{ if } i = -j \big\rangle.$$

• The *relative elementary subgroup* $\mathrm{EU}(2n, I, \Gamma)$ of level (I, Γ), defined as the normal closure of $\mathrm{FU}(2n, I, \Gamma)$ in $\mathrm{EU}(2n, A, \Lambda)$,

$$\mathrm{EU}(2n, I, \Gamma) = \mathrm{FU}(2n, I, \Gamma)^{\mathrm{EU}(2n, A, \Lambda)}.$$

• The *principal congruence subgroup* $\mathrm{GU}(2n, I, \Gamma)$ of level (I, Γ) in $\mathrm{GU}(2n, A, \Lambda)$ consists of those $g \in \mathrm{GU}(2n, A, \Lambda)$, that are congruent to e modulo I and preserve $f(u, u)$ modulo Γ,

$$f(gu, gu) \in f(u, u) + \Gamma, \qquad u \in V.$$

• The full congruence subgroup $\mathrm{CU}(2n, I, \Gamma)$ of level (I, Γ), defined as

$$\mathrm{CU}(2n, I, \Gamma) = \{g \in \mathrm{GU}(2n, A, \Lambda) \mid [g, \mathrm{GU}(2n, A, \Lambda)] \subseteq \mathrm{GU}(2n, I, \Gamma)\}.$$

In some books, including [19], the group $\mathrm{CU}(2n, I, \Gamma)$ is defined differently. However, in many important situations these definitions yield the same group.

4.2. Some basic lemmas. We us collect several basic facts, concerning relative groups, which will be used in the sequel. The first one of them, see [9, Lemma 5.2], asserts that the relative elementary groups are $\mathrm{EU}(2n, A, \Lambda)$-perfect.

Lemma 2. *Suppose either $n \geqslant 3$ or $n = 2$ and $I = \Lambda I + I\Lambda$. Then*

$$\mathrm{EU}(2n, I, \Gamma) = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, A, \Lambda)].$$

The next lemma gives generators of the relative elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$ as a subgroup. With this end, consider matrices

$$Z_{ij}(a, c) = {}^{T_{ji}(c)}T_{ij}(a) = T_{ji}(c)T_{ij}(a)T_{ji}(-c),$$

where $a \in I$, $c \in A$, if $i \neq \pm j$, and $a \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $c \in \lambda^{-(\varepsilon(j)+1)/2}\Lambda$, if $i = -j$. The following result is [9], Proposition 5.1.

Lemma 3. *Suppose $n \geqslant 3$. Then*

$$\mathrm{EU}(2n, I, \Gamma) = \langle Z_{ij}(a, c) \mid \ a \in I, c \in A \ \textit{if}\ i \neq \pm j, \ \textit{and} \\ a \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma, c \in \lambda^{-(\varepsilon(j)+1)/2}\Lambda, \ \textit{if}\ i = -j\rangle.$$

The following lemma was first established in [1], but remained unpublished. See [19] and [9], Lemma 4.4, for published proofs.

Lemma 4. *The groups*

$$\mathrm{GU}(2n, I, \Gamma) \ \textit{and} \ \mathrm{CU}(2n, I, \Gamma)$$

are normal in $\mathrm{GU}(2n, A, \Lambda)$.

In this form the following lemma was established in [31, Lemmas 7 and 8], see also [30, Lemma 1B] for a definitive exposition. Before that [27], Lemmas 21–23 only established weaker inclusions, with smaller left-hand sides, or larger right-hand sides.

Lemma 5. (A,Λ) *be an associative form ring with* 1, $n\geqslant 3$, *and let* (I,Γ) *and* (J,Δ) *be two form ideals of* (A,Λ). *Then*

$$\begin{aligned}\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))\leqslant&[\,\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)]\\ &\leqslant[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)]\\ &\leqslant[\,\mathrm{GU}(2n,I,\Gamma),\mathrm{GU}(2n,J,\Delta)]\\ &\leqslant\mathrm{GU}(2n,(I,\Gamma)\circ(J,\Delta)).\end{aligned}$$

§5. Unrelativization

Here we establish the first claim of Theorem 1, and thus also Theorems 2, 3 and 4. It immediately follows from the next two lemmas, the first of which addresses the case of short roots, while the second one pertain to the case of long roots.

Recall that for the easier case of the general linear group over *commutative* rings this result was first established in 2018 in our paper [77]. Then it was generalized to arbitrary associative rings in 2019, together with the second claim of Theorem 1, see [76]. The proof of the following results exploits the same ideas as the proof of [76, Lemma 4], but is noticeably more demanding from a technical viewpoint.

The following two lemmas address the case of short roots, where $i\neq\pm j$, and the case of long roots, where $i=-j$, respectively

Lemma 6. *Let* (A,Λ) *be an associative form ring with* 1, $n\geqslant 3$, *and let* (I,Γ), (J,Δ) *be form ideals of* (A,Λ). *Suppose that* $a\in I$, $b\in J$, $r\in A$, *and* $i\neq\pm j$. *Then*

$$[T_{ji}(a),Z_{ji}(b,r)]\in[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)].$$

Proof. For simplicity, we assume that $\varepsilon(i)=\varepsilon(j)$. Pick an $h\neq i,j$ with $\varepsilon(h)=\varepsilon(i)$. Then

$$x=[T_{ji}(a),Z_{ji}(b,r)]=T_{ji}(a)\cdot{}^{Z_{ji}(b,r)}T_{ji}(-a)=T_{ji}(a)\cdot{}^{Z_{ji}(b,r)}[T_{jh}(1),T_{hi}(-a)].$$

Expanding the conjugation by $Z_{ji}(b,r)$, we see that

$$\begin{aligned}x&=T_{ji}(a)[{}^{Z_{ji}(b,r)}T_{jh}(1),{}^{Z_{ji}(b,r)}T_{hi}(-a)]\\ &=T_{ji}(a)[T_{ih}(-rbr)T_{jh}(1-br),T_{hj}(-arbr)T_{hi}(-a(1-rb))]\\ &=T_{ji}(a)[yT_{jh}(1),T_{hi}(-a)z],\end{aligned}$$

where

$$\begin{aligned}y&=T_{ih}(-rbr)T_{jh}(-br)\in\mathrm{FU}(2n,J,\Delta),\\ z&=T_{hj}(-arbr)T_{hi}(arb)\in\mathrm{FU}(2n,(I,\Gamma)\circ(J,\Delta)).\end{aligned}$$

Since $T_{hi}(-a)\in \mathrm{FU}(2n,I,\Gamma)$, the second factor of the above commutator belongs to $\mathrm{FU}(2n,I,\Gamma)$. Thus,

$$[yT_{jh}(1),T_{hi}(-a)z]={}^{y}[T_{jh}(1),T_{hi}(-a)z]\cdot[y,T_{hi}(-a)z]. \tag{1}$$

Now, the first commutator on the right-hand side equals

$${}^{y}[T_{jh}(1),T_{hi}(-a)]\cdot{}^{yT_{hi}(-a)}[T_{jh}(1),z].$$

The second commutator in the last expression belongs to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$, and remains there after elementary conjugations, while the first commutator equals ${}^{y}T_{ij}(-a)$. But

$${}^{y}T_{ij}(-a)=[T_{ih}(-rbr)T_{jh}(-br),T_{ij}(-a)]\cdot T_{ij}(-a).$$

The first fact above lies in $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$, hence

$${}^{y}T_{ij}(-a)\in T_{ij}(-a)\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

On the other hand, the second commutator of (1) equals

$$[y,T_{hi}(-a)z] \quad = \quad [T_{ih}(-rbr)T_{jh}(-br),T_{hi}(-a)T_{hj}(-arbr)T_{hi}(arb)].$$

Expanding the commutator above by its second argument, we obtain

$$\begin{aligned}&[T_{ih}(-rbr)T_{jh}(-br),T_{hi}(-a)T_{hj}(-arbr)T_{hi}(arb)]\\ =\;&[T_{ih}(-rbr)T_{jh}(-br),T_{hi}(-a)]\\ &\qquad\qquad{}^{T_{hi}(-a)}[T_{ih}(-rbr)T_{jh}(-br),T_{hj}(-arbr)T_{hi}(arb)].\end{aligned}$$

The second factor above belongs to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$. And the first factor above equals

$$\begin{aligned}&{}^{T_{ih}(-rbr)}[T_{jh}(-br),T_{hi}(-a)]\cdot[T_{ih}(-rbr),T_{hi}(-a)]\\ &\qquad={}^{T_{ih}(-rbr)}T_{ji}(bra)\cdot[T_{ih}(-rbr),T_{hi}(-a)]\\ &\qquad\qquad\in[T_{ih}(-rbr),T_{hi}(-a)]\cdot\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).\end{aligned}$$

Summarising the above, we see that

$$x\in[T_{ih}(-rbr),T_{hi}(-a)]\cdot\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$$

which finishes the proof. □

Lemma 7. *Let (A,Λ) be an associative form ring with 1, $n\geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ). Suppose that $a\in\Gamma$, $b\in\Delta$, and $r\in\Lambda$. Then*

$$[T_{-i,i}(a),Z_{-i,i}(b,r)]\in[\mathrm{FU}(2n,I,\Gamma),\mathrm{FU}(2n,J,\Delta)].$$

Proof. Without loss of generality, we may assume that $i > 0$. Pick an $h > 0$ with $h \neq i$. Then

$$\begin{aligned} x = [T_{-i,i}(a), Z_{-i,i}(b,r)] &= T_{-i,i}(a) \cdot {}^{Z_{-i,i}(b,r)}T_{-i,i}(-a) \\ &= T_{-i,i}(a) \cdot {}^{Z_{-i,i}(b,r)}\big(T_{hi}(-a) \cdot [T_{h,-h}(a), T_{-h,i}(1)]\big). \end{aligned}$$

Thus,

$$\begin{aligned} x &= T_{-i,i}(a) \cdot \big({}^{Z_{-i,i}(b,r)}T_{hi}(-a) \cdot [T_{h,-h}(a), {}^{Z_{-i,i}(b,r)}T_{-h,i}(1)]\big) \\ &= T_{-i,i}(a) \cdot T_{h,i}(-a(1-br)) \cdot T_{i,-h}(\lambda rbr\overline{a}) \\ &\quad \times \big[T_{h,-h}(a), T_{-h,i}(1-rb) \cdot T_{i,h}(\lambda rbr)\big]. \end{aligned}$$

Using the additivity of root unipotents, we can rewrite this as

$$\begin{aligned} x = T_{-i,i}(a)T_{h,i}(-a) &\cdot T_{h,i}(-abr)T_{i,-h}(\lambda rbr\overline{a}) \\ &\times \big[T_{h,-h}(a), T_{-h,i}(1)T_{-h,i}(-rb) \cdot T_{i,h}(\lambda rbr)\big]. \end{aligned}$$

Clearly,

$$T_{h,i}(-abr)T_{i,-h}(\lambda rbr\overline{a}) \in \mathrm{EU}(2n, (I,\Gamma) \circ (J,\Delta)).$$

On the other hand, the commutator in the last expression equals

$$\begin{aligned} &\big[T_{h,-h}(a), T_{-h,i}(1)T_{-h,i}(-rb) \cdot T_{i,h}(\lambda rbr)\big] \\ &\qquad = \big[T_{h,-h}(a), T_{-h,i}(1)\big] \cdot {}^{T_{-h,i}(1)}\big[T_{h,-h}(a), T_{-h,i}(-rb) \cdot T_{i,h}(\lambda rbr)\big] \\ &\qquad = T_{h,i}(a)T_{-i,i}(-a) \cdot {}^{T_{-h,i}(1)}\big[T_{h,-h}(a), T_{-h,i}(-rb) \cdot T_{i,h}(\lambda rbr)\big]. \end{aligned}$$

Again, clearly

$$\big[T_{h,-h}(a), T_{-h,i}(-rb) \cdot T_{i,h}(\lambda rbr)\big] \in [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

On the other hand, the previous factors assemble to a left $T_{-i,i}(a)T_{h,i}(-a)$ conjugate of an element of $\mathrm{EU}(2n, (I,\Gamma) \circ (J,\Delta))$, which is contained in

$$[\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

This proves Lemma 7. □

Combined with Theorem 2, these results imply the first claim of Theorem 1.

§6. Elementary commutators modulo $\mathrm{EU}(2n, (I,\Gamma) \circ (J,\Delta))$

Now we embark on the proof of the second claim of Theorem 1. Our first major goal is to prove that the commutator $[\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)]$ is central in $\mathrm{EU}(2n, A, \Lambda)$, modulo $\mathrm{EU}(2n, (I,\Gamma) \circ (J,\Delta))$. Namely, here we establish Theorem 5 and derive some corollaries to it. We prove the congruence in Theorem 5 separately for short root positions, and then for long root positions.

Lemma 8. *Let (A,Λ) be an associative form ring with* 1, $n \geqslant 3$, *and let* (I,Γ), (J,Δ) *be form ideals of* (A,Λ). *For any* $i \neq \pm j$, *any* $a \in I$, $b \in J$, *and any* $x \in \mathrm{EU}(2n,A,\Lambda)$, *one has*

$$ {}^{x}Y_{ij}(a,b) \equiv Y_{ij}(a,b) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}\,. $$

Proof. Consider the elementary conjugate ${}^{x}Y_{ij}(a,b)$. We argue by induction on the length of $x \in \mathrm{EU}(2n,A,\Lambda)$ in elementary generators. Let $x = yT_{kl}(c)$, where $y \in \mathrm{EU}(2n,A,\Lambda)$ is shorter than x.

We start with the case where $k \neq \pm l$.

- If $k,l \neq \pm i, \pm j$, then $T_{kl}(c)$ commutes with $z = Y_{ij}(a,b)$ and can be discarded.
- On the other hand, for any $h \neq \pm i, \pm j$ direct computations show that

$$
\begin{aligned}
[T_{ih}(c),z] &= T_{ih}(-abc - ababc)T_{jh}(-babc),\\
[T_{jh}(c),z] &= T_{ih}(abac)T_{jh}(bac),\\
[T_{hi}(c),z] &= T_{hi}(cab)T_{hj}(-caba),\\
[T_{hj}(c),z] &= T_{hi}(cbab)T_{hj}(-cba - cbaba),
\end{aligned}
$$

Similarly, one has

$$
\begin{aligned}
[T_{-i,h}(c),z] &= [T_{-h,i}(-\lambda^{(\varepsilon(h)+\varepsilon(i))/2}c),z]\\
&= T_{-h,i}(-\lambda^{(\varepsilon(h)+\varepsilon(i))/2}cab)T_{-h,j}(-\lambda^{(\varepsilon(h)+\varepsilon(i))/2}caba),\\
[T_{-j,h}(c),z] &= [T_{-h,j}(-\lambda^{(\varepsilon(h)+\varepsilon(j))/2}c),z]\\
&= T_{-h,i}(-\lambda^{(\varepsilon(h)+\varepsilon(j))/2}cbab)\\
&\quad\; T_{-h,j}(-\lambda^{(\varepsilon(h)+\varepsilon(j))/2}cba - \lambda^{(\varepsilon(h)+\varepsilon(j))/2}cbaba),\\
[T_{h,-i}(c),z] &= [T_{i,-h}(-\lambda^{-(\varepsilon(i)-\varepsilon(h))/2}c),z]\\
&= T_{i,-h}(-\lambda^{-(\varepsilon(i)-\varepsilon(h))/2}abac)T_{j,-h}(-\lambda^{-(\varepsilon(i)-\varepsilon(h))/2}bac),\\
[T_{h,-j}(c),z] &= [T_{j,-h}(-\lambda^{-(\varepsilon(j)-\varepsilon(h))/2}c),z]\\
&= T_{i,-h}(-\lambda^{(-(\varepsilon(j)-\varepsilon(h))/2}abac)T_{j,-h}(-\lambda^{-(\varepsilon(j)-\varepsilon(h))/2}bac)
\end{aligned}
$$

All factors on the right-hand side belong already to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

If $(k,l) = (\pm i, \pm j)$ or $(\pm j, \pm i)$, then we take an index $h \neq \pm i, \pm j$ and rewrite $T_{kl}(c)$ as $[T_{k,h}(c), T_{h,l}(1)]$ and apply the previous items to get the same congruence modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

It remains to consider the case where $k = -l$.

- If $k \neq \pm i, \pm j$, then $T_{k,-k}(c)$ commutes with z and can be discarded.

• Otherwise, we have

$$\begin{aligned}[T_{i,-i}(c),z] =&T_{i,-i}(c-(1+ab+abab)c\overline{(1+ab+abab)})\\ &T_{j,-j}(-\lambda^{(\varepsilon(i)-\varepsilon(j))/2}babc\overline{bab})\\ &T_{i,-j}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}(1+ab+abab)c\overline{(bab)}),\\ [T_{j,-j}(c),z] =&T_{j,-j}(c-(1-ba)c\overline{(1-ba)})T_{i,-i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}abac\overline{aba})\\ &T_{i,-j}(abac(1-\overline{ba})),\\ [T_{-i,i}(c),z] =&[T_{-i,i}(c),[T_{ij}(a),T_{ji}(b)]]\\ =&[T_{-i,i}(c),[T_{-j,-i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}a),T_{-i,-j}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}b)]],\\ [T_{-j,j}(c),z] =&[T_{-j,j}(c),[T_{ij}(a),T_{ji}(b)]]\\ =&[T_{-j,-j}(c),[T_{-j,-i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}a),T_{-i,-j}(\lambda^{(\varepsilon(i)-\varepsilon(j))/2}b)]].\end{aligned}$$

The last two cases reduce to the first two. Hence all factors on the right belong to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

We have shown that for $i\neq\pm j$,

$$ {}^{x}z\equiv {}^{y}z \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}\,. \qquad \square$$

Lemma 9. *Let (A,Λ) be an associative form ring with 1, $n\geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ). For any $a\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $b\in\lambda^{(\varepsilon(i)-1)/2}\Delta$, and any $x\in\mathrm{EU}(2n,A,\Lambda)$, one has*

$$ {}^{x}Y_{i,-i}(a,b)\equiv Y_{i,-i}(a,b) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}\,.$$

Proof. We argue by induction on the length of $x\in\mathrm{EU}(2n,A,\Lambda)$ in elementary generators as we did in the previous lemma. Let $x=yT_{kl}(c)$, where $y\in\mathrm{EU}(2n,A,\Lambda)$ is shorter than x.

We start with the case where $k=-l$. Denote $Y_{i,-i}(a,b)=[T_{i,-i}(a),T_{-i,i}(b)]$ by z.

• If $(k,l)=(-i,i)$, then

$$[T_{-i,i}(c),z]=[T_{-i,i}(c),[T_{i,-i}(a),T_{-i,i}(b)]]=[T_{-i,i}(c),Z_{-i,i}(b,a)].$$

The same computation as in Lemma 7 shows that

$$[T_{-i,i}(c),z]\in\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

• If $(k,l)=(i,-i)$, then

$$\begin{aligned}[T_{i,-i}(c),z]&=[T_{i,-i}(c),[T_{i,-i}(a),T_{-i,i}(b)]]\\ &=[T_{i,-i}(c),[T_{-i,i}(b),T_{i,-i}(a)]^{-1}]\\ &=[T_{-i,i}(b),T_{i,-i}(a)]^{-1}[[T_{-i,i}(b),T_{i,-i}(a)],T_{i,-i}(c)][T_{-i,i}(b),T_{i,-i}(a)].\end{aligned}$$

Now the inner factor $[[T_{-i,i}(b), T_{i,-i}(a)], T_{i,-i}(c)]$ falls into the previous case, hence belongs to $\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))$. But then the same applies also to its conjugate

$$[T_{-i,i}(b), T_{i,-i}(a)]^{-1}\cdot\big[[T_{-i,i}(b), T_{i,-i}(a)], T_{i,-i}(c)\big]\cdot[T_{-i,i}(b), T_{i,-i}(a)].$$

• If $k=i$ and $j\neq\pm k$, then

$$\begin{aligned}[T_{i,j}(c), z] &= [T_{i,j}(c), [T_{i,-i}(a), T_{-i,i}(b)]]\\ &= T_{i,j}(-abc-ababc)T_{-i,j}(-babc)\cdot T_{-j,j}\big(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2}\overline{(bc)}a(bc)\\ &\quad -\lambda^{\varepsilon(j)}(\overline{(abc)}b(abc)+\overline{(babc)}a(babc))\big).\end{aligned}$$

Since $a\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma$ and $b\in\lambda^{(\varepsilon(i)-1)/2}\Delta$, it follows that the right side belongs to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

• If $k=-i$ and $j\neq\pm k$, then

$$\begin{aligned}[T_{-i,j}(c), z] &= [T_{-i,j}(c), [T_{i,-i}(a), T_{-i,i}(b)]]\\ &= [T_{-i,i}(b), T_{i,-i}(a)][T_{-i,j}(c), [T_{-i,i}(b), T_{i,-i}(a)]]^{-1}[T_{-i,i}(b), T_{i,-i}(a)]^{-1}.\end{aligned}$$

By the previous case,

$$[T_{-i,j}(c), [T_{-i,i}(b), T_{i,-i}(a)]]\in\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)).$$

As above, the normality of $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$ then implies that the whole right side belongs to $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

• Finally, the case where $l=\pm i$ and $k\neq\pm i$ reduces to the case of $k=\pm i$ via relation (R1).

We have shown that

$${}^{x}z\equiv{}^{y}z\,(\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

By induction we get

$${}^{x}z\equiv z\,(\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

□

In particular, these results immediately imply the following additivity property of the elementary commutators with respect to their arguments.

Theorem 10. *Let R be an associative ring with* 1, $n\geqslant 3$, *and let* (I,Γ), (J,Δ) *be form ideals of R. Then for any* $i\neq j$, *and any* $a,a_1,a_2\in(I,\Gamma)$,

$b, b_1, b_2 \in (J, \Delta)$ *one has*

$$\begin{aligned} Y_{ij}(a_1+a_2,b) &\equiv Y_{ij}(a_1,b)\cdot Y_{ij}(a_2,b) \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,,\\ Y_{ij}(a,b_1+b_2) &\equiv Y_{ij}(a,b_1)\cdot Y_{ij}(a,b_2) \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,,\\ Y_{ij}(a,b)^{-1} &\equiv Y_{ij}(-a,b)\equiv Y_{ij}(a,-b) \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,,\\ Y_{ij}(ab_1,b_2) &\equiv Y_{ij}(a_1,a_2b)\equiv e \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,,\\ Y_{i,-i}(\overline{b_1}ab_1,b_2) &\equiv Y_{i,-i}(a_1,\overline{a_2}ba_2)\equiv e \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \end{aligned}$$

Proof. The first item can be derived from Lemma 8 for $i \neq \pm j$ and Lemma 9 for $i = -j$ as follows. By definition,

$$Y_{ij}(a_1+a_2,b) = [T_{ij}(a_1+a_2), T_{ji}(b)] = [T_{ij}(a_1)T_{ij}(a_2), T_{ji}(b)],$$

and it only remains to apply the multiplicativity of commutators in the first factor, and then apply Lemma 8 and Lemma 9 respectively. The second item is similar, and the third item follows. The last two items are obvious from the definition. □

§7. Rolling over elementary commutators

Now we pass to the final, and most difficult part of the proof of Theorem 1, rolling an elementary commutator over to a different position. Since we assume that $n \geqslant 3$, the case of *short* root type elementary commutators is easy. It is settled by essentially the same calculation as for the general linear group $\mathrm{GL}(n,R)$, $n \geqslant 3$, see [76, 78]. But for the case of *long* root type elementary commutators we have to imitate the proof of [80, Theorems 4 and 5], for $\mathrm{Sp}(4,R)$. In the presence of a nontrivial involution, noncommutativity, and nontrivial form parameters, this is quite a challenge. In §12 we make some observations, to put this calculation in historical context.

Lemma 10. *Let (A,Λ) be an associative form ring with* 1, $n \geqslant 3$, *and let (I,Γ), (J,Δ) be form ideals of (A,Λ). Then for any $i \neq \pm j$, any $h \neq \pm l$, and any $a \in I$, $b \in J$, $c_1, c_2 \in A$, one has*

$$Y_{ij}(c_1ac_2, b) \equiv Y_{hl}(a, c_2bc_1) \,(\operatorname{mod}\, \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

Proof. Take any $h \neq \pm i, \pm j$, and rewrite the elementary commutator $z = Y_{ij}(c_1ac_2, b)$ on the left-hand side of the above congruence as follows

$$\begin{aligned} z &= [T_{ij}(c_1ac_2), T_{ji}(b)] = T_{ij}(c_1ac_2)\cdot {}^{T_{ji}(b)}T_{ij}(-c_1ac_2)\\ &= T_{ij}(c_1ac_2)\cdot {}^{T_{ji}(b)}[T_{hj}(ac_2), T_{ih}(c_1)]. \end{aligned}$$

Expanding the conjugation by $T_{ji}(b)$, we see that

$$\begin{aligned} z &= T_{ij}(c_1ac_2)\cdot[{}^{T_{ji}(b)}T_{hj}(ac_2),{}^{T_{ji}(b)}T_{ih}(c_1)] \\ &= T_{ij}(c_1ac_2)\cdot\big[[T_{ji}(b),T_{hj}(ac_2)]T_{hj}(ac_2),T_{ih}(c_1)[T_{ih}(-c_1),T_{ji}(b)]\big] \\ &= T_{ij}(c_1ac_2)\cdot\big[T_{hi}(-ac_2b)T_{hj}(ac_2),T_{ih}(c_1)T_{jh}(bc_1)\big]. \end{aligned}$$

Now, the first factor $T_{hi}(-ac_2b)$ of the first argument in this last commutator already belongs to the group $\mathrm{FU}(2n,(I,\Gamma)\circ(J,\Delta))$. Thus, as above,

$$z\equiv T_{ij}(c_1ac_2)\cdot\big[T_{hj}(ac_2),T_{ih}(c_1)T_{jh}(bc_1)\big]\ (\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

Using the multiplicativity of the commutator with respect to the second argument, cancelling the first two factors of the resulting expression, and then applying Lemma 8, we see that

$$\begin{aligned} z &\equiv {}^{T_{ih}(c_1)}[T_{hj}(ac_2),T_{jh}(bc_1)] \\ &\equiv [T_{hj}(ac_2),T_{jh}(bc_1)]\ (\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \end{aligned}$$

On the other hand, choosing another index $l\neq\pm j,\pm h$ and rewriting the commutator $[T_{hj}(ac_2),T_{jh}(bc_1)]$ on the right-hand side of the last congruence as

$$[T_{hj}(ac_2),T_{jh}(bc_1)]=[[T_{hl}(a),T_{lj}(c_2)],T_{jh}(bc_1)],$$

by the same argument we get the congruence

$$z\equiv[T_{hj}(ac_2),T_{jh}(bc_1)]\equiv[T_{hl}(a),T_{lh}(c_2bc_1)]\ (\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

Obviously, for $n\geqslant 3$ we can pass from any position (i,j), $i\neq j$, to any other such position (k,m), $k\neq\pm m$, by a sequence of at most three such elementary moves. □

Lemma 11. *Let (A,Λ) be an associative form ring with 1, $n\geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ). Then for any $-n\leqslant i\leqslant n$, any $-n\leqslant k\leqslant n$, and any $a\in\lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $b\in\lambda^{(\varepsilon(i)-1)/2}\Delta$, $c\in A$, one has*

$$\begin{aligned} &Y_{i,-i}(ca\overline{c},b) \\ &\equiv Y_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a,-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)\cdot Y_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca,\lambda^{(\varepsilon(k)+\varepsilon(i))/2}\overline{\overline{c}b}) \\ &\qquad (\mathrm{mod}\ \mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \end{aligned}$$

Proof. Rewrite the elementary commutator $z=Y_{i,-i}(ca\overline{c},b)$ on the left-hand side of the above congruence as follows

$$\begin{aligned} z &= T_{i,-i}(ca\overline{c})\cdot{}^{T_{-i,i}(b)}\,T_{i,-i}(-ca\overline{c}) \\ &= T_{i,-i}(ca\overline{c})\cdot{}^{T_{-i,i}(b)}\,\big([T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a),T_{i,k}(c)]T_{k,-i}(-a\overline{c})\big). \end{aligned}$$

Expanding the conjugation by $T_{-i,i}(b)$, we see that

$$\begin{aligned} z &= T_{i,-i}(ca\overline{c}) \\ &\times \left[{}^{T_{-i,i}(b)}T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), {}^{T_{-i,i}(b)}T_{i,k}(c)\right] \cdot {}^{T_{-i,i}(b)}\, T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca). \end{aligned}$$

Clearly, the last factor

$$y = {}^{T_{-i,i}(b)}\, T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)$$

can be rewritten as

$$[T_{-i,i}(b), T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)] \cdot T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)$$

which gives us the following congruence

$$y \equiv T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca) \pmod{\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))}\,.$$

On the other hand, the commutator

$$u = \left[{}^{T_{-i,i}(b)}T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), {}^{T_{-i,i}(b)}T_{i,k}(c)\right]$$

in the expression of u equals

$$\left[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-i,k}(bc)T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)T_{i,k}(c)\right].$$

Expanding this last expression, we get

$$\begin{aligned} u &= [T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-i,k}(bc)] \\ &\times {}^{x}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)] \\ &\times {}^{v}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{i,k}(c)], \end{aligned}$$

where

$$x = T_{-i,k}(bc), \quad v = T_{-i,k}(bc)T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc).$$

It is easy to see that

$$[T_{k,-k}(-\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-i,k}(bc)] \in \mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta)),$$

so we can drop it. Further, by Lemma 9, modulo $\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))$ the second factor can be simplified as follows

$$\begin{aligned} &{}^{x}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)] \\ &\equiv [T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)] \\ &\qquad\qquad (\mathrm{mod}\ \mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta)).) \end{aligned}$$

Summarising the above, we get

$$z \equiv T_{i,-i}(ca\overline{c}) \cdot [T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)] \cdot {}^{v}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{i,k}(c)] \cdot T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}.$$

Thus, to finish the proof it suffices to show that

$$x' = T_{i,-i}(ca\overline{c}) \cdot {}^{v}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{i,k}(c)] \cdot T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca) \equiv Y_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca, \lambda^{(\varepsilon(i)+\varepsilon(k))/2}\overline{c}\overline{b}) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}.$$

Clearly, the second factor of x'

$${}^{v}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{i,k}(c)]$$

can be rewritten as

$${}^{v}[T_{k,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}a), T_{i,k}(c)] = {}^{v}(T_{i,-i}(-ca\overline{c}) \cdot T_{i,-k}(-\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)).$$

Therefore we obtain

$$x' = {}^{T_{i,-k}(-\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)}[T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca)T_{i,-i}(ca\overline{c}), v]$$

Expanding this last commutator with respect to its first and second arguments, we express it as the product of elementary conjugates of the four following commutators.

- $[T_{i,-i}(ca\overline{c}), T_{-i,k}(bc)]$,
- $[T_{i,-i}(ca\overline{c}), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)]$,
- $[T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca), T_{-i,k}(bc)] = Y_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca, \lambda^{(\varepsilon(k)+\varepsilon(i))/2}\overline{c}\overline{b})$,
- $[T_{i,-k}(\lambda^{(\varepsilon(i)-\varepsilon(k))/2}ca), T_{-k,k}(-\lambda^{(\varepsilon(k)-\varepsilon(i))/2}\overline{c}bc)]$.

A direct computation convinces us that each of these commutators except for the third one belongs to the elementary subgroup $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$. This finishes the proof of the lemma, and thus also of Theorem 1. □

Lemma 12. *Let (A,Λ) be an associative form ring with 1, $n \geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ). Then for any $-n \leqslant i \leqslant n$, any $-n \leqslant k \leqslant n$ with $i \neq \pm k$ and $\varepsilon(i) = \varepsilon(k)$, and any $a \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma$, $b \in \lambda^{(\varepsilon(i)-1)/2}\Delta$, one has*

$$Y_{i,-i}(a,b) \equiv Y_{k,-k}(a,b) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}.$$

Proof. Pick an integer l with $-n \leqslant l \leqslant n$. Applying Lemma 11 with $c = 1$, we get

$$\begin{aligned} &Y_{i,-i}(a,b) \\ &\quad \equiv Y_{l,-l}(\lambda^{(\varepsilon(i)-\varepsilon(l))/2}a, -\lambda^{(\varepsilon(l)-\varepsilon(i))/2}b) \cdot Y_{i,-l}(\lambda^{(\varepsilon(i)-\varepsilon(l))/2}a, \lambda^{(\varepsilon(l)+\varepsilon(i))/2}\overline{b}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\operatorname{mod} \operatorname{EU}(2n,(I,\Gamma)\circ(J,\Delta))) \end{aligned}$$

and

$$\begin{aligned} &Y_{k,-k}(a,b) \\ &\quad \equiv Y_{l,-l}(\lambda^{(\varepsilon(k)-\varepsilon(l))/2}a, -\lambda^{(\varepsilon(l)-\varepsilon(k))/2}b) \cdot Y_{k,-l}(\lambda^{(\varepsilon(k)-\varepsilon(l))/2}a, \lambda^{(\varepsilon(l)+\varepsilon(k))/2}\overline{b}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\operatorname{mod} \operatorname{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \end{aligned}$$

By Lemma 10, we obtain

$$\begin{aligned} Y_{i,-l}(\lambda^{(\varepsilon(i)-\varepsilon(l))/2}a, \lambda^{(\varepsilon(l)+\varepsilon(i))/2}\overline{b}) &\equiv Y_{k,-l}(\lambda^{(\varepsilon(k)-\varepsilon(l))/2}a, \lambda^{(\varepsilon(l)+\varepsilon(k))/2}\overline{b}) \\ &\qquad (\operatorname{mod} \operatorname{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \end{aligned}$$

Therefore we conclude that

$$Y_{i,-i}(a,b) \equiv Y_{k,-k}(a,b)\,(\operatorname{mod} \operatorname{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,. \qquad \square$$

§8. Mat[ch]ing elementary commutators of different root lengths

In this section we prove a congruence relating elementary commutators of long root type with those of short root type. In the case where one of the relative form parameters is as small as possible (=minimal), this congruence can be used to eliminate long root type elementary commutators. On the other hand, when one of the relative form parameters is as large as possible (=equals the corresponding ideal), one can abandon short root type elementary commutators.

Lemma 13. *Let (A,Λ) be an associative form ring with 1, $n \geqslant 3$, and let (I,Γ), (J,Δ) be form ideals of (A,Λ). Then for any $-n \leqslant i \leqslant n$, any $-n \leqslant k \leqslant n$ with $i \neq \pm k$, and $a \in I$, $b \in \lambda^{(\varepsilon(i)-1)/2}\Delta$, one has*

$$\big[T_{i,-i}(a-\lambda^{\varepsilon(-i)}\overline{a}), T_{-i,i}(b)\big] \equiv [T_{i,k}(a), T_{k,i}(b)]\,(\operatorname{mod} \operatorname{EU}(2n,(I,\Gamma)\circ(J,\Delta)))\,.$$

Proof. Pick an index $k \neq \pm i$, and rewrite the elementary commutator

$$z = \big[T_{i,-i}(a-\lambda^{\varepsilon(-i)}\overline{a}), T_{-i,i}(b)\big]$$

on the left-hand side as

$$z = \big[[T_{k,-i}(-1), T_{i,k}(a)], T_{-i,i}(b)\big] = \big[{}^{T_{k,-i}(-1)}T_{i,k}(a)\cdot T_{i,k}(-a), T_{-i,i}(b)\big].$$

Using the multiplicativity of the commutator with respect to the first argument, we see that

$$z = {}^{T_{k,-i}(-1)T_{i,k}(a)T_{k,-i}(1)}[T_{i,k}(-a), T_{-i,i}(b)] \cdot \big[{}^{T_{k,-i}(-1)}T_{i,k}(a), T_{-i,i}(b)\big].$$

The first factor belongs to $\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))$, so we leave it out. Thus, z is congruent modulo this subgroup to

$$\begin{aligned}
&\big[{}^{T_{k,-i}(-1)}T_{i,k}(a), T_{-i,i}(b)\big] \\
&= {}^{T_{k,-i}(-1)}\big[T_{i,k}(a), {}^{T_{k,-i}(1)}T_{-i,i}(b)\big] \\
&= {}^{T_{k,-i}(-1)}\big[T_{i,k}(a), [T_{k,-i}(1), T_{-i,i}(b)]T_{-i,i}(b)\big] \\
&= {}^{T_{k,-i}(-1)}\big[T_{i,k}(a), T_{k,i}(b)T_{k,-k}(\lambda^{(\varepsilon(-i)-\varepsilon(k))/2}(b))T_{-i,i}(b)\big].
\end{aligned}$$

Expanding this last commutator with respect to the second argument, we see that the second and the third factors belong to $\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))$, so that we can leave them out. Now we have

$$z \equiv {}^{T_{k,-i}(-1)}\big[T_{i,k}(a), T_{k,i}(b)\big] \pmod{\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))},$$

as claimed. □

Corollary 1. *Under the conditions of Lemma* 13, *further assume that* $b = b' - \lambda^{\varepsilon(i)}\overline{b'}$ *for some* $b' \in J$, *then*

$$\big[T_{i,-i}(a - \lambda^{\varepsilon(-i)}\overline{a}), T_{-i,i}(b - \lambda^{\varepsilon(i)}\overline{b})\big] \equiv [T_{i,k}(a), T_{k,i}(b')] \cdot [T_{i,k}(a), T_{k,i}(-\lambda^{\varepsilon(i)}\overline{b'})]$$

modulo $\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))$.

Proof. We keep the notation from the proof of Lemma 13. Under this additional assumption one has

$$z \equiv {}^{T_{k,-i}(-1)}\big[T_{i,k}(a), T_{k,i}(b')T_{k,i}(-\lambda^{\varepsilon(i)}\overline{b'})\big] \pmod{\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))}.$$

Expanding the commutator with respect to the second argument again, we see that

$$\begin{aligned}
&{}^{T_{k,-i}(-1)}\big[T_{i,k}(a), T_{k,i}(b')T_{k,i}(-\lambda^{\varepsilon(i)}\overline{b'})\big] \\
&\qquad = {}^{T_{k,-i}(-1)}\big([T_{i,k}(a), T_{k,i}(b')] \cdot {}^{T_{k,i}(b')}[T_{i,k}(a), T_{k,i}(-\lambda^{\varepsilon(i)}\overline{b'})]\big).
\end{aligned}$$

Applying Lemma 8, we get

$$z \equiv [T_{i,k}(a), T_{k,i}(b')] \cdot [T_{i,k}(a), T_{k,i}(-\lambda^{\varepsilon(i)}\overline{b'})] \pmod{\mathrm{EU}(2n, (I,\Gamma)\circ(J,\Delta))},$$

as claimed. □

Corollary 2. *If* $I = \Gamma$ *or* $J = \Delta$, *then for the second type of generators in Theorem* 1 *it suffices to take one pair* $(h, -h)$.

Corollary 3. *If* $\Gamma = I \cap \Lambda_{\min}$ *or* $\Delta = J \cap \Lambda_{\min}$, *then for the second type of generators in Theorem* 1 *it suffices to take one pair* (h, k), $h \neq \pm k$.

§9. Triple and quadruple commutators

Actually Theorem 7 easily follows by induction on m from the following two special cases, triple commutators, and quadruple commutators.

Lemma 14. *Let (A,Λ) be any associative form ring with 1, let $n \geqslant 3$, and let (I,Γ), (J,Δ), (K,Ω), be form ideals of (A,Λ). Then*

$$\begin{aligned}&[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],\mathrm{EU}(2n,K,\Omega)]\\&=[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,K,\Omega)].\end{aligned}$$

Proof. Let $i,j,k \in \{-n,\dots,-1,1,\dots,n\}$ with $i \neq \pm j \neq \pm k$. For any $a \in (I,\Gamma)$, $b \in (J,\Delta)$ and $c \in (K,\Omega)$, we have

$$\begin{aligned}&[Y_{k,-k}(a,b),T_{i,j}(c)] = e\\&[Y_{k,j}(a,b),T_{i,j}(c)] = T_{ij}(cba+cbaba)T_{ik}(-cbab).\end{aligned}$$

Both above commutators land in $[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,K,\Omega)]$. By Theorem 1, we deduce that

$$\begin{aligned}&[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],T_{i,j}(c)]\\&\subseteq[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,K,\Omega)].\end{aligned}$$

Similarly, we have

$$\begin{aligned}&[Y_{k,-k}(a,b),T_{i,-i}(c)] = e,\\&[Y_{k,j}(a,b),T_{i,-i}(c)] = e.\end{aligned}$$

which implies that

$$\begin{aligned}&[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],T_{i,-i}(c)]\\&\subseteq[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,K,\Omega)].\end{aligned}$$

We finish the proof by combining all above results and applying Theorem 1. □

Now, for $n \geqslant 4$ the only new case of quadruple commutators is considered in the following lemma, which immediately follows from Lemma 14 and Theorem 5. Of course, for the outstanding case $n = 3$ it requires a separate proof. All our assaults on this remaining case were crippled by forbidding calculations.

Lemma 15. *Let (A,Λ) be any associative form ring with 1 and let (I,Γ), (J,Δ), (K,Ω), (L,Θ) be form ideals of (A,Λ). If either $n \geqslant 4$ or there exists an ideal that equals its corresponding relative form parameter and $n \geqslant 3$, then*

$$\begin{aligned}&\big[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],[\,\mathrm{EU}(2n,K,\Omega),\mathrm{EU}(2n,L,\Theta)]\big]\\&=[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))].\end{aligned}$$

Proof. From the previous lemma we already know that

$$\begin{aligned}&\big[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\big[\,\mathrm{EU}(2n,K,\Omega),\mathrm{EU}(2n,L,\Theta)\big]\big]\\&=\big[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))\big]\end{aligned}$$

and that

$$\begin{aligned}&\big[[\,\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))\big]\\&=\big[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))\big].\end{aligned}$$

Thus, it only remains to prove that

$$[Y_{ij}(a,b),Y_{hk}(c,d)]\in\big[\,\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta)),\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))\big],$$

where $a\in(I,\Gamma)$, $b\in(J,\Delta)$, $c\in(K,\Omega)$, and $d\in(L,\Theta)$. Conjugations by elements $x\in\mathrm{EU}(2n,A,\Lambda)$ do not matter, because they amount to extra factors from the above triple commutators, which are already accounted for.

Now, for $n\geqslant 4$ this already finishes the proof, because in this case we can move $Y_{hk}(c,d)$ modulo $\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))$ to a position where it commutes with $Y_{ij}(a,b)]$, either by Lemma 10 when $i\neq\pm j$ and $h\neq\pm k$ or by Lemma 12 when $i=-j$ or $h=-k$.

Suppose that there exists an ideal that equals its corresponding relative form paramerter, say $I=\Gamma$. If $i\neq\pm j$, then by Lemma 13 we have

$$Y_{i,j}(a,b)\equiv Y_{i,-i}(a,b-\lambda^{\varepsilon(i)}\overline{b}).$$

For $n\geqslant 3$, we can move $Y_{i,-i}(a,b-\lambda^{\varepsilon(i)}\overline{b})$ modulo $\mathrm{EU}(2n,(K,\Omega)\circ(L,\Theta))$ to a position where it commutes with $Y_{hk}(c,d)$ by Lemma 10. Otherwise, if $i=-j$, then we can also move $Y_{i,-i}(a,b)$ to a position where it commutes with $Y_{hk}(c,d)$ by Lemma 12. This finishes the whole proof. □

§10. Elementary multiple commutator formulas

In the current section, we show that multiple commutators of elementary subgroups can be reduced to double commutators of these kind.

To state our main results, we have to recall some further pieces of notation from [22, 23, 27, 31, 33, 64]. Namely, let $H_1,\dots,H_m\leqslant G$ be subgroups of G. There are many ways to form a higher commutator of these groups, depending on where we put the brackets. Thus, for three subgroups $F,H,K\leqslant G$ one can form two triple commutators $[[F,H],K]$ and $[F,[H,K]]$. Usually, we write $[H_1,H_2,\dots,H_m]$ for the *left-normed* commutator, defined inductively by

$$[H_1,\dots,H_{m-1},H_m]=[[H_1,\dots,H_{m-1}],H_m].$$

To stress that here we consider *any* commutator of these subgroups, with an arbitrary placement of brackets, we write $[\![H_1, H_2, \dots, H_m]\!]$. Thus, for instance, $[\![F, H, K]\!]$ refers to any of the two arrangements above.

Actually, a specific arrangement of brackets usually does not play a major role in our results — apart from one important attribute[3]. Namely, what will matter a lot is the position of the outermost pairs of inner brackets. Namely, every higher commutator subgroup $[\![H_1, H_2, \dots, H_m]\!]$ can be written uniquely as

$$[\![H_1, H_2, \dots, H_m]\!] = [[\![H_1, \dots, H_s]\!], [\![H_{s+1}, \dots, H_m]\!]],$$

for some $s = 1, \dots, m-1$. This s will be called the cut point of our multiple commutator. Now we are all set to finish the proof of Theorem 7. The proof is an easy adaptation of the proof of [78], Theorem 1, but we reproduce it here for the sake of completeness.

Proof. Denote the commutator on the left-hand side by H,

$$H = [\![\, \mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2), \dots, \mathrm{EU}(2n, I_m, \Gamma_m)]\!].$$

We argue by induction on m, with the cases of $m \leqslant 4$ as the base of induction — for the case of $m = 2$ there is nothing to prove, the case of $m = 3$ is accounted for by Lemma 14, and the case of $m = 4$ — by Lemma 14 if the cut point $s \neq 2$, and by Lemma 15 when s is not 2.

Now, let $m \geqslant 5$ and assume that our theorem is already proved for all shorter commutators. Consider an arbitrary arrangement of brackets $[\![\dots]\!]$ with the cut point s and let

$$[\![\, \mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2), \dots, \mathrm{EU}(2n, I_s, \Gamma_s)]\!],$$
$$[\![\, \mathrm{EU}(2n, I_{s+1}, \Gamma_{s+1}), \mathrm{EU}(2n, I_{s+2}, \Gamma_{s+2}), \dots, \mathrm{EU}(2n, I_m, \Gamma_m)]\!],$$

be the partial commutators, the first one containing the factors afore the cut point, and the second one containing those after the cut point.

- When the cut point occurs at $s = 1$ or at $s = m-1$, one of these commutators is a single elementary subgroup, $\mathrm{EU}(2n, I_1)$ in the first case or $\mathrm{EU}(2n, I_{m-1})$ in the second one. Then we can apply the inductive hypothesis to another factor. For $s = 1$, denote by $t = 2, \dots, m-1$ the cut point of the

[3]Actually, for noncommutative rings symmetric product of ideals is not associative, so that the initial bracketing of higher commutators will be reflected also in the bracketing of such higher symmetric products.

second factor. Then by inductive hypothesis

$$H = \Big[\operatorname{EU}(2n, I_1, \Gamma_1), [\![\operatorname{EU}(2n, I_2, \Gamma_2), \operatorname{EU}(2n, I_3, \Gamma_3), \ldots, \operatorname{EU}(2n, I_m, \Gamma_m)]\!] \Big]$$
$$= \Big[\operatorname{EU}(2n, I_1, \Gamma_1), \big[\operatorname{EU}(2n, (I_2, \Gamma_2) \circ \ldots \circ (I_t, \Gamma_t)),$$
$$\operatorname{EU}(2n, (I_{t+1}, \Gamma_{t+1}) \circ \ldots \circ (I_m, \Gamma_m))\big]\Big],$$

and we are done by Lemma 14. Similarly, for $s = m-1$ denote by $r = 1, \ldots, m-1$ the cut point of the first factor. Then by inductive hypothesis

$$H = \Big[[\![\operatorname{EU}(2n, I_1, \Gamma_1), \operatorname{EU}(2n, I_2, \Gamma_2), \ldots, \operatorname{EU}(2n, I_{m-1}, \Gamma_{m-1})]\!],$$
$$EU(2n, I_m, \Gamma_m)\Big]$$
$$= \Big[\big[\operatorname{EU}(2n, (I_1, \Gamma_1) \circ \ldots \circ (I_r, \Gamma_r)),$$
$$\operatorname{EU}(2n, (I_{r+1}, \Gamma_{r+1}) \circ \ldots \circ (I_{m-1}, \Gamma_{m-1}))\big],$$
$$\operatorname{EU}(2n, I_m, \Gamma_m)\Big],$$

and we are again done by Lemma 14.

• Otherwise, when $s \neq 1, m-1$, we can apply the inductive hypothesis to both factors. Let as above $r = 1, \ldots, s-1$ be the cut point of the first factor and let $t = s+1, \ldots, m-1$ be the cut point of the second factor. Then we can apply inductive hypothesis to both factors of

$$H = \Big[[\![\operatorname{EU}(2n, I_1), \operatorname{EU}(2n, I_2), \ldots, \operatorname{EU}(2n, I_s)]\!],$$
$$[\![\operatorname{EU}(2n, I_{s+1}), \operatorname{EU}(2n, I_{s+2}), \ldots, \operatorname{EU}(2n, I_m)]\!] \Big]$$

to conclude that

$$H = \Big[\big[\operatorname{EU}(2n, I_1 \circ \ldots \circ I_r), \operatorname{EU}(2n, I_{r+1} \circ \ldots \circ I_s)\big],$$
$$\big[\operatorname{EU}(2n, I_{s+1} \circ \ldots \circ I_t), \operatorname{EU}(2n, I_{t+1} \circ \ldots \circ I_m)\big]\Big],$$

and we are again done, this time by Lemma 15. □

§11. Further applications

Now, we are in a position to finish the proof of Theorem 8.

Proof. Since (I, Γ) and (J, Δ) are comaximal, there exist $a' \in I$ and $b' \in J$ such that $a' + b' = 1 \in R$. But then by Lemmas 10 and 13, for $i \neq \pm j$ one has

$$Y_{ij}(a, b) = Y_{ij}(a(a' + b'), b) \equiv Y_{ij}(aa', b) \cdot Y_{ij}(ab', b) \equiv e$$

modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$.

For $i=-j$, one has

$$Y_{i,-i}(a,b)=Y_{i,-i}((a'+b')a\overline{(a'+b')},b)=Y_{i,-i}(a'a\overline{a'}+b'a\overline{a'}+a'a\overline{b'}+b'a\overline{b'},b).$$

Applying multiplicativity of commutators to the first argument of the above commutator and then using Lemma 9, we deduce that

$$z\equiv Y_{i,-i}(a'a\overline{a'},b)Y_{i,-i}(b'a\overline{a'},b)Y_{i,-i}(a'a\overline{b'},b)Y_{i,-i}(b'a\overline{b'},b) \pmod{\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))}.$$

By Theorem 10, each of the above factors is trivial modulo $\mathrm{EU}(2n,(I,\Gamma)\circ(J,\Delta))$. This finishes the proof. □

Let us state another amusing corollary of Theorem 10. For the form ideals themselves, one has an obvious inclusion

$$\begin{aligned}&\big((I,\Gamma)+(J,\Delta)\big)\circ\big((I,\Gamma)\cap(J,\Delta)\big)\\&=\big((I+J)\circ(I\cap J),\Gamma_{\min}((I+J)\circ(I\cap J))+{}^{(\Gamma\cap\Delta)}(\Gamma+\Delta)+{}^{(\Gamma+\Delta)}(\Gamma\cap\Delta)\big)\\&\leqslant\big(I\circ J,\Gamma_{\min}(I\circ J)+{}^{J}\Gamma+{}^{I}\Delta\big)=(I,\Gamma)\circ(J,\Delta).\end{aligned}$$

Only very rarely this inclusion is always an identity.

Theorem 11. *For any two form ideals (I,Γ) and (J,Δ) of (A,Λ), $n\geqslant 3$, one has*

$$\big[\mathrm{EU}\,(2n,(I,\Gamma){+}(J,\Delta)),\mathrm{EU}\,(n,(I,\Gamma){\cap}(J,\Delta))\big]\leqslant[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)].$$

Proof. The observation immediately preceding the theorem shows that the level of the left-hand side is contained in the level of the right-hand side,

$$\mathrm{EU}\,\big(2n,R,((I,\Gamma)+(J,\Delta))\circ((I,\Gamma)\cap(J,\Delta))\big)\leqslant\mathrm{EU}\,(2n,R,(I,\Gamma)\circ(J,\Delta)).$$

Thus, it only remains to prove that the elementary commutators $Y_{ij}(a{+}b,c)$, with $a\in(I,\Gamma)$, $b\in(J,\Delta)$, $c\in(I,\Gamma)\cap(J,\Delta)$, on the left-hand side belong to the right-hand side.

By Theorem 10, one has

$$Y_{ij}(a+b,c)\equiv Y_{ij}(a,c)\cdot Y_{ij}(b,c) \pmod{\mathrm{EU}\,(2n,R,((I,\Gamma)+(J,\Delta))\circ((I,\Gamma)\cap(J,\Delta)))}.$$

Thus, this congruence holds true also modulo the larger subgroup

$$\mathrm{EU}(2n,R,(I,\Gamma)\circ(J,\Delta)).$$

On the other hand, Theorem 6 implies that

$$Y_{ij}(b,c)\equiv Y_{ij}(c,-b) \pmod{\mathrm{EU}(2n,R,(I,\Gamma)\circ(J,\Delta))}.$$

Combining the above congruences, we see that

$$Y_{ij}(a+b,c) \equiv Y_{ij}(a,c)\cdot Y_{ij}(c,-b)\,(\mathrm{mod}\ \mathrm{EU}(2n,R,(I,\Gamma)\circ(J,\Delta)))\,,$$

where both commutators on the right-hand side belong to

$$[\mathrm{EU}(2n,I,\Gamma),\mathrm{EU}(2n,J,\Delta)],$$

which proves the desired inclusion. □

§12. Final remarks

Here we make some further observations concerning the context of this work and also state some unsolved problems and reiterate some further problems from [27, 31], which are still pending.

12.1. How we got here. The study of birelative standard commutator formulas goes back to the foundational work by Hyman Bass [10]. As early successes one should also mention important contributions by Alec Mason and Wilson Stothers [39–42] and by Hong You [84]. Our own research in this direction started in 2008–2010 in the joint works with Alexei Stepanov and Roozbeh Hazrat [32, 74, 75] and was then continued in 2011–2017 in a series of our joint works based on relative versions of localization methods, in particular[4] [27–31, 33]. Simultaneously, Stepanov developed his universal localization and applied it to multiple commutator formulas and commutator width, see [63, 64]. One can find systematic description of that stage of development in our surveys and conference papers [22–24, 30].

The present work is a natural extension of our more recent papers [71, 72, 76–80]. It owes its existence to the two following momentous observations we made in October 2018, and in September 2019, respectively.

In October 2018 the first author proved a special case of Theorems 2 and 3 for the general linear group $\mathrm{GL}(n,R)$, $n\geqslant 3$, over commutative rings, see [71]. The initial proof employed a version of decomposition of unipotents [65], which was already used for a similar purpose in his joint work with Alexei Stepanov [74]. The second author then immediately observed that Theorem 2 implies the first claim of Theorem 1 and that it should be possible to proceed conversely, first establish a version of Theorem 1 by elementary calculations, and then derive Theorems 2 and 3. This is exactly what was done for Chevalley groups in our paper [76], again over commutative rings.

In July–September 2019 the first author was discussing bounded generation of Chevalley groups in the function case with Boris Kunyavsky and Eugene

[4]At least three our scheduled works of that period, which were essentially completed by 2016, viz., the general multiple commutator formula for $\mathrm{GL}(n,R)$, unitary commutator width, and the analysis of the case $\mathrm{GU}(4,R,\Lambda)$, still remain unpublished.

Plotkin. One of the tricks used in many published papers consisted of splitting an elementary conjugate/elementary commutator and then reassembling it in a different position. We noticed that the same calculation of rolling elementary conjugates to a different position appeared over and over again in many different contexts.

• *Congruence subgroup problem.* In a preliminary mode it was already present in the precursory article by Jens Mennicke [43] and then already in full-fledged form in the epoch-making memoir by Hyman Bass, John Milnor, and Jean-Pierre Serre [12], behold the proof of Theorem 5.4.

• *Bounded generation.* Post factum, we discerned the same calculation in the classical papers by David Carter, Gordon Keller, and Oleg Tavgen [15, 68], but we only became aware of that perusing a recent article by Bogdan Nica [44].

• In fact, Wilberd van der Kallen and Alexei Stepanov [34, 62, 63] used a very similar calculation to reduce the generating sets of relative elementary subgroups.

Here we attached merely a handful of references. Retrospectively, we spotted the same or very similar calculations in oodles of further papers, but apparently it was hardly ever applied in the birelative context.

At the end of September the first author used essentially the same calculation[5] to prove that when R is commutative and $n \geqslant 3$, the mixed relative commutator subgroup $[E(n,A),E(n,B)]$ is contained in another birelative group

$$\mathrm{EE}(n,A,B)=\langle t_{ij}(c),\ \text{where } c\in A, i<j,\ \text{and } c\in B, i>j\rangle,$$

see [72], Theorem 3. Within a few days of vehement correspondence we observed that everything works over arbitrary associative rings and can be further enhanced to entail Theorems 1 and 5 for $\mathrm{GL}(n,R)$. This was done in [76], and soon thereafter in a more mature form, implying also Theorems 6, 7 and 8, in [78].

Morally, the present paper, and a parallel paper that addresses the case of Chevalley groups [80], are direct offsprings of this development. However, technically these cases turned out to be way more demanding, and we had to spend quite some time to supply detailed proofs of all auxiliary results.

12.2. Degree improvements. Of course, the first question that immediately occurs is whether Theorem 7 is true also for $n=3$. For *quasifinite* rings this is indeed the case [30], and we are pretty more inclined to believe in the positive answer.

Problem 1. *Prove that Lemma* 15 *and Theorem* 7 *hold also for* $n=3$.

[5]Simultaneously and independently exactly the same calculation was applied by Andrei Lavrenov and Sergei Sinchuk [38] at the level of K_2.

Getting a proof in the same style as that of Lemma 14 seems to be highly nontrivial from a technical viewpoint. However, the possibility to construct a counterexample appears even more remote.

In the main body of the present paper we always assumed that $n \geqslant 3$. Obviously, due to the exceptional behavior of the orthogonal group $\mathrm{SO}(4, A)$, these results do not fully generalize to the case of $n = 2$. It is natural to ask whether results of the present paper are true also for the group $\mathrm{GU}(4, A, \Lambda)$. However, this obviously fails in general without some strong additional assumptions on the form ring and/or form ideals.

Still, we believe they do generalize, provided $\Lambda A + A\Lambda = A$, or the like. Known results[6] clearly indicate both that this should be possible, and that the analysis of the case $n = 2$ will be considerably harder from a technical viewpoint than that of the case $n \geqslant 3$.

Problem 2. *Generalize results of the present paper to the group* $\mathrm{GU}(4, A, \Lambda)$, *provided that* $\Lambda A + A\Lambda = A$, $\Gamma J + J\Gamma = I$, $\Delta I + I\Delta = J$, *or the like.*

Actually, some 8 years ago we obtained various headways towards the relative standard commutator formula and all that for $\mathrm{GU}(4, A, \Lambda)$, but even these results are unpublished, due to their fiercely technical character.

12.3. Presentations and stability. As a counterpart to Theorem 9 we can ask, whether the stability map for this quotient is also injective. A natural approach to this would be to tackle the following much more ambitious project.

Problem 3. *Give a presentation of*

$$[\,\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]/\,\mathrm{EU}(2n, A, (I, \Gamma) \circ (J, \Delta))$$

by generators and relations, does this presentation depend on $n \geqslant 3$?

In Theorems 6 and 10 and Lemma 13 we established some of the relations among the elementary commutators modulo $\mathrm{EU}(2n, A, (I, \Gamma) \circ (J, \Delta))$. However, easy arithmetic examples show this is not a defining set of relations, so that there must be some further relations. Compare [76, 78, 79] for discussion of a similar problem for $\mathrm{GL}(n, A)$.

12.4. Higher relations. In [79] we established some further congruences for the elementary commutators in $\mathrm{GL}(n, A)$, $n \geqslant 3$, where A is an arbitrary associative ring. The highlight of that paper is the following remarkable triple congruence, a version of the Hall–Witt identity.

[6] Compare the work by Bak and the first author [8], and references therein.

Let I, J, K be two-sided ideals of R. Then for any three distinct indices i, j, h such that $1 \leqslant i, j, h \leqslant n$, and all $a \in I$, $b \in J$, $c \in K$, one has

$$y_{ij}(ab, c) y_{jh}(ca, b) y_{hi}(bc, a) \equiv e \pmod{E(n, R, IJK + JKI + KIJ)},$$

see [79, Theorem 1]. This identity has lots of applications, including many new inclusions among double and multiple mixed relative elementary commutator subgroups.

Specifically, it allows us to solve an analog of Problem 3 for $\mathrm{GL}(n, A)$ in the particularly agreeable case of Dedekind rings. Thus, it would be most natural to seek out similar higher congruences also in the unitary case.

Problem 4. *Generalize the results of* [79] *to the unitary groups* $\mathrm{GU}(2n, A, \Lambda)$, $n \geqslant 3$.

One such congruence among *short* root type elementary commutators is immediately clear. But the congruences involving long root type elementary commutators will be fancier and longer.

12.5. Other birelative groups. We briefly discuss two further groups depending on two form ideals of a form ring. First of all, it is the partially relativized group $\mathrm{FU}(2n, I, \Gamma)^{\mathrm{FU}(2n, J, \Delta)}$. It seems that in view of the identity

$$\mathrm{FU}(2n, I, \Gamma)^{\mathrm{FU}(2n, J, \Delta)} = [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)] \cdot \mathrm{FU}(2n, I, \Gamma),$$

our Theorem 1 readily implies the following generalization of [9, Proposition 5.1], to $\mathrm{FU}(2n, I, \Gamma)^{\mathrm{FU}(2n, J, \Delta)}$. Namely, we assert that it is generated by the appropriate elementary conjugates.

Problem 5. *Prove that the partially relativized groups* $\mathrm{FU}(2n, I, \Gamma)^{\mathrm{FU}(2n, J, \Delta)}$ *are generated by* ${}^{T_{ji}(b)}T_{ij}(a)$, *where* $a \in (I, \Gamma)$, $b \in (J, \Delta)$.

Another birelative group $\mathrm{EEU}(2n, (I, \Gamma), (J, \Delta))$ is defined as follows

$$\mathrm{EEU}(2n, (I, \Gamma), (J, \Delta)) = \big\langle T_{ij}(a), \text{ where } c \in (I, \Gamma), i < j, \text{ and } c \in (J, \Delta), i > j \big\rangle.$$

The following problem proposes a unitary generalization of [72, Theorem 3], where a similar result was established for $\mathrm{GL}(n, A)$.

Problem 6. *Prove that*

$$[\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)] \leqslant \mathrm{EEU}(2n, (I, \Gamma), (J, \Delta)).$$

12.6. General multiple commutator formula. Now we recall another major unsolved problem as stated already in [27, 30] and [31, Problem 1]. We proffer to prove a general multiple commutator formula for unitary groups.

Problem 7. *Let* (I_i, Γ_i), $1 \leqslant i \leqslant m$, *be form ideals of the form ring* (A, Λ) *such that* A *is module-finite over a commutative ring* R *that has finite Bass–Serre dimension* $\delta(R) = d < \infty$. *Prove that for any* $m \geqslant d$ *one has*

$$\begin{aligned} &[\![\, \mathrm{GU}(2n, I_0, \Gamma_0), \mathrm{GU}(2n, I_1, \Gamma_1), \ldots, \mathrm{GU}(2n, I_m, \Gamma_m)]\!] \\ &= [\![\, \mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{EU}(2n, I_1, \Gamma_1), \ldots, \mathrm{EU}(2n, I_m, \Gamma_m)]\!]. \end{aligned}$$

Observe that the arrangement of brackets in the above formula should be the same on both sides because the mixed commutators are not associative. A similar problem for algebraic groups over *commutative* rings, in particular for Chevalley groups, was solved by Alexei Stepanov [64], by his remarkable *universal localization* method.

Recall that the proof of a similar result for $\mathrm{GL}(n, R)$ over *noncommutative* rings is based on the following result of Mason–Stothers [42], Theorem 3.6 and Corollary 3.9, see [30, Theorem 13], for an easy modern proof. Of course, that we can unrelativize the right-hand side was only established in [76, Theorem 2], so formally this theorem was never stated in this form.

Theorem 3. *Let* A *be a ring, and let* I *and* J *be two two-sided ideals of* A. *Assume that* $n \geqslant \mathrm{sr}(R), 3$. *Then*

$$[\mathrm{GL}(n, A, I), \mathrm{GL}(n, A, J)] = [E(n, I), E(n, J)].$$

For unitary groups, even such basic facts at the stable level seem to be missing.

Problem 8. *Find appropriate stability conditions under which*

$$[\mathrm{GU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)] = [\mathrm{FU}(2n, I, \Gamma), \mathrm{FU}(2n, J, \Delta)].$$

After that, the proof in our unpublished paper proceeds by induction on d, which depends on Bak's results [3], the precise form of injective stability for K_1, such as the Bass–Vaserstein theorem, etc. It seems that to solve Problem 7 one has to rethink and expand many aspects of the structure theory of unitary groups, starting with stability theorems for KU_1.

The first complete[7] generally accepted proof of injective stability for KU_1 was obtained (but not published!) by Maria Saliani [56], and first published

[7] In late 1960s and mid 1970s Anthony Bak and Manfred Kolster obtained stability under stronger assumptions, with very sketchy proofs. Leonid Vaserstein worked in smaller generality as far as groups, and his proof of injective stability for unitary groups contained serious gaps and inaccuracies. In 1980 Mamed-Emin Oglu Namik Mustafa-Zadeh announced surjective stability for KU_2 — and thus also injective stability for KU_1 — in full generality.

by Max Knus in his book [35]. After that, generalizations and improvements were proposed by Anthony Bak, Guoping Tang, Victor Petrov, and Sergei Sinchuk [5, 7, 60], and then very recently by Weibo Yu, Rabeya Basu and Egor Voronetsky [14, 82, 87].

Problem 7 is also intimately related to the nilpotent structure of KU_1. In the absolute case the corresponding results for unitary groups were obtained by Roozbeh Hazrat in his Ph. D. Thesis [20, 21], and in the relative case in a joint paper by Bak, Hazrat and the first author [4]. To fully cope with Problem 7, we need more powerful results on the superspecial unitary groups than what was established in [4]. Part of what is demanded here was recently established by Weibo Yu, Guoping Tang and Rabeya Basu [13, 88], but there is still a lot of work to be done.

12.7. Subnormal subgroups. Initially, one of our main motivations to pursue the work on birelative commutator formulas were prospective applications to the study of subnormal subgroups of $\mathrm{GU}(2n, A, \Lambda)$. As was observed by John Wilson [83], technically this amounts to description of subgroups of $\mathrm{GU}(2n, A, \Lambda)$, normalized by a relative elementary subgroup $\mathrm{EU}(2n, J, \Delta)$, for *some* form ideal (J, Δ).

A major early contribution is due to Günter Habdank [17, 18], who additionally assumed that the form ring was subject to some stability conditions. Definitive results for quasifinite rings were then obtained by the second author and You Hong [85, 90–92]. However, we are convinced that the bounds in these papers can be further improved and hope to return to the following problem with our new tools.

Problem 9. *Obtain optimal bounds in the description of subgroups of*

$$\mathrm{GU}(2n, A, \Lambda),$$

normalized by the relative elementary subgroup $\mathrm{EU}(2n, J, \Delta)$, *for a form ideal* $(J, \Delta) \trianglelefteq (A, \Lambda)$.

Until recently, for the unitary groups the proofs of structure theorems were in bad shape even in the absolute case.[8] However, now the situation has changed. In 2013 Hong You and Xuemei Zhou [86] published a detailed proof for commutative form rings. Finally, in 2014 Raimund Preusser in his Ph. D. Thesis [49]

However, a complete proof was never published, and the exposition in his 1983 Ph. D. Thesis is blurred by serious mistakes.

[8]As indicated in [26], the proof in the work by Leonid Vaserstein and Hong You [69] contained a major omission, and only established the *weak* structure theorem. The details of the purported global proof by Bak and the first author, that was around since the early 1990s, and that was harbingered in [9], remained unpublished.

gave a first complete *localization proof* for quasifinite form rings, which was published in [50].

In 2017 Raimund Preusser [51, 52] also finally succeeded in completing a *global proof* as envisaged in [9]. These papers constitute a major breakthrough because, at least for commutative rings, they give explicit polynomial expressions of nontrivial transvections as products of elementary conjugates of a given matrix and its inverse. (See also [53, 55] for further results in this spirit for $\mathrm{GL}(n, A)$ over various classes of noncommutative rings.) The first author immediately recognized that the results by Preusser procure an effectivization for the description of normal subgroups in much the same sense as the *decomposition of unipotents* [65] does for the normality of the elementary subgroup. This prompted him to call this method *reverse decomposition of unipotents* [70]. Moreover, he noticed that in the case of $\mathrm{GL}(n, A)$ these results can be generalized (with only marginally worse bounds) to the description of subgroups normalized by a *relative* elementary subgroups [73].

We are confident that, combining the methods developed by Preusser in the above papers with our methods, we could easily improve bounds in all published results for unitary groups. Of course, to prove that the bounds thus obtained are themselves the best possible would be quite a challenge.

12.8. Commutator width. Another related problem that initially motivated our work was the study of commutator width. Alexander Sivatsky and Alexei Stepanov [61] discovered that over rings of finite Jacobson dimension $\text{j-dim}(A) = d < \infty$ any commutator $[x, y]$, where $x \in \mathrm{GL}(n, A)$, $y \in E(n, A)$, is a product of at most L elementary generators, where $L = L(n, d)$ only depends on n and d. This result was then generalized to all Chevalley groups $G(\Phi, A)$ by Stepanov and the first author [66], with the bound depending on the type Φ and on d.

Ultimately, Stepanov discovered that for *reductive groups* similar results hold for *arbitrary* commutative rings and that the bound L therein depends on the type of the group alone and not on the ring A. Also, he discovered that similar results hold at the relative and birelative level, with elementary conjugates and our generators (like those in Theorem B) as the generating sets of $[E(\Phi, A, I), E(\Phi, A, J)]$, again with bounds that depend on the type alone, and not on A, I, or J. See [24] for statements and a detailed discussion of these results.

However, Bak's unitary groups are not always algebraic and similar results on commutator width are not yet published even in the absolute case and even over finite-dimensional rings.

Problem 10. *Let* (A, Λ) *be a commutative form ring such that* $\text{j-dim}(A) < \infty$*. Prove that the length of commutators in* $[\mathrm{GU}(\Phi, A, I), E(\Phi, A, J)]$ *in terms of the generators listed in Theorem* 1 *is bounded, and estimate this length.*

Alexei Stepanov maintained that the above length is bounded in the absolute case, without actually producing any specific bound. To obtain an exponential bound depending on d by relative localization methods [27, 30, 31] would be simply a matter of patience. Actually, this was essentially done by ourselves and Roozbeh Hazrat, but even in the absolute case all of this still remains unpublished.

On the other hand, to achieve a *uniform* polynomial bound, similar to the one established in [61] for $\mathrm{GL}(n, A)$ but not depending on d, one would need to combine a full-scale generalization of Stepanov's universal localization to unitary groups, with full-scale unitary versions of decomposition of unipotents, including explicit polynomial formulas for the conjugates of root unipotents. This seems to be a rather ambitious project.

12.9. Unitary Steinberg groups. It is natural to ask to which extent our methods and results carry over to the level of KU_2.

Problem 11. *Prove analogs of the main results of the present paper for the unitary Steinberg groups* $\mathrm{StU}(2n, A, \Lambda)$.

For the definition of unitary Steinberg groups, see [2,36] and references there (or [37] for odd unitary Steinberg groups). Here, we do not discuss subtleties related to the definition of relative unitary Steinberg groups, as also the relationship with excision in the unitary algebraic K-theory, etc.

12.10. Description of subgroups. The methods of the present paper may have applications also in description of various classes of subgroups of unitary groups. Not in the position to discuss this at any depth here, we only cite the works by Victor Petrov, Alexander Shchegolev, and Egor Voronetsky [46, 57–59, 81] where one can find many further references. Observe that the result by Voronetsky [81] is especially powerful, because it simultaneously generalizes also the description of EU-normalized subgroups (in the context of odd unitary groups!)

12.11. Odd unitary groups. Finally, we are positive that all results of the present paper generalize also to odd unitary groups introduced by Victor Petrov [47, 48].

Problem 12. *Generalize the results of* [27,29,30] *and the present paper to odd unitary groups, under suitable isotropy assumptions.*

Of course, this is not an individual clear-cut problem, but rather a huge research project. Clearly, in most cases the proofs in this setting will require much more onerous calculations. Let us cite some important recent papers by Yu Weibo, Tang Guoping, Li Yaya, Liu Hang, Anthony Bak, Raimund Preusser, and Egor Voronetsky [6, 54, 81, 82, 88, 89] that address normal structure and stability for odd unitary groups.

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