Article

The skies are falling: Mathematics for non-mathematicians

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Abstract: Mathematical education, both mass education, and university education of non-mathematicians, are in an abominable state, and rapidly degrading. We argue that the instruction of non-mathematicians should be dramatically reformed both as substance and style. With traditional approach, such a transformation would take decades, with unclear results. But we do not have this time. The advent of Computer Algebra Systems gives the mathematics community a chance to reverse the trend. We should make a serious attempt to seize this opportunity. In the present paper we describe one such project of reform implemented at the St Petersburg State University.

Keywords: Mathematical education, mathematics for non-mathematicians, mathematics and computers, computer algebra systems

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1. Introduction

We believe that the current situation with mathematical education, and the growing abyss between mathematicians and layman, even the educated ones, constitute an immediate desperate danger for our profession, and, eventually, for the whole human civilisation.

The problem has been aggravated by the advent of computers, which can address vast majority of the traditional tasks, where Mathematics is applied, and whose mathematical software has no user serviceable parts. This has created a wide-spread illusion that now for the end-users there is no need to study any Mathematics whatsoever.

Our own assessment of the situation is EXACTLY THE OPPOSITE. To successfully function within their subject fields most professionals would now need to grasp much more Mathematics, and at that much deeper and more advanced Mathematics. Teaching non-mathematicians the necessary Mathematics in the same style we did before is simply not feasible.

We believe though, that, being part of the problem, computers can be also a decisive part of its solution. We describe a current project "MATHEMATICS AND COMPUTERS" implemented at the St Petersburg State University for the last 15 years. The concept is to focus exclusively on understanding and big ideas, while replacing most of the proofs and actual computational skills — apart from the most basic and the most enlightening ones — by computer calculations, experiments, and visualisation. The hard part was, of course, to develop a set of a few hundred test problems that would require *both* mathematical and algorithmic thinking. A fraction of our experience in this direction is reflected in the recent textbook [32].

Although we mostly discuss our teaching experience in St Petersburg, the problem itself seems to be of a very general nature, apparent in all technologically developed societies for several decades now. Compare, for instance, the 1981 lecture by Vladimir

Rokhlin [27] or the 1990 article by William Thurston [29], which starts with the constatation: MATHEMATICS EDUCATION IS IN AN UNACCEPTABLE STATE. The interest of non-mathematicians in taking mathematics courses was constantly fading even then, see [12,16]. However, it seems to us that the situation has *dramatically* exacerbated over the last 10–15 years, after computers have turned the tables.

2. Mathematics in human culture

Let us start with some self-evident truths:

- Spiritually and noetically, mathematics is, together with other higher creative arts, such as music or visual arts, the *paramount* manifestation of human culture.
- On the other hand, pragmatically we live in the world created by mathematics and science, in the first place by the *mathematische Naturwissenschaft*.
- Overall, it would not be a great exaggeration to assert, as Oswald Spengler did, that the level of a civilisation is largely determined by the level of its mathematics.

Unfortunately, these simple facts are rarely — if at all! — fully recognised not only by the general public, such as taxpayers, entrepreneurs, and polititians, but even by philosophers, journalists, educationalists and other discoursemongers.

In fact, most of the things around us, inlcuding ourselves, could not have existed in the present form without science. It starts simply with the sheer numerical strength of the human race (and other synanthropic animal species, such as cattle, pig, or sheep), which BY SEVERAL ORDERS OF MAGNITUDE exceeds the population of any other animal species of comparable body mass, and which would had been IMPOSSIBLE TO MAINTAIN WITHOUT SCIENCE.

Similarly, it is IMPOSSIBLE TO MAINTAIN — let alone to develop! — many of the PRESENT-DAY TECHNOLOGIES without a large number of individuals deeply congnisant in mathematics and science.

At various periods of history, Mathematics has been *extremely* successful in fostering the development of natural sciences, initially Astronomy and Physics, later on also other sciences and engineering.

We strongly believe that nowadays Mathematics *could* play a similar role in the development of life sciences, such as Biology and Medicine, as also in Linguistics, Psychology, Economics, etc.

Today, we even have most of the requisite tools and computational resources. What is lacking, however, is the **awareness** on the part of those who have to apply Mathematics in the respective subject fields. They do not know any Mathematics, they do not understand it, and they do not even understand why it is relevant — that Mathematics is the only feasible mediator between spirit and reality.

3. Mathematical education

The above explains an absolutely exceptional role played by mathematical education in the functioning of a society. As Jean Pierre Kahane stated it: IN NO OTHER SCIENCE HAS TEACHING AND LEARNING SUCH SOCIAL IMPORTANCE (cited in [4]).

Here, one should clearly distinguish

- Pre-university level — spectators;
- Mathematics for non-mathematicians — gentlemen;
- Mathematics for mathematicians — players.

Of these three, training professional mathematicians is the least problematic. We fully agree with Rokhlin that TEACHING MATHEMATICS TO THE WOULD-BE MATHEMATICIANS IS INFINITELY EASIER THAN TEACHING MATHEMATICS TO NON-MATHEMATICIANS, see [27]. If you know, understand and love your subject, and if you are honest with your students, it does not matter, whether you are an accomplished teacher, and what you do exactly, and how you do it. If they are *already* interested in Mathematics, you can relax, since you are bound to get through, regardless.

However, when working with the general public, or with other professionals, you should be at all time aware that you are working at three completely different levels:

- Mathematics as part of general culture;
- Mathematics *per se*;
- Mathematics for specific applications.

The fundamental flaw of the traditional mathematical education is that it is focusing on, and advertising the third aspect alone, which is invariably the least important one of all, mostly the least interesting one of all, and usually fictitious.

In our view, the single most important aspect of teaching mathematics at the elementary level is the cultivation of **intellectual honesty.** In other words, the ability to distinguish what you understand from what you don't, what was defined and has a precise meaning from what doesn't, what is said from what is intended, plausible from improbable, true from false, proven from conjectured, etc.

Another equally important aspect is the **callisthenics of mind**, as preparation to solve any kind of **difficult problems**. From this prospective, mathematics is a work-out[1] that allows to develop, train and maintain inner vision, aesthetic taste, memory, tenue, concentration, the abilities to observe, compare, generalise and specialise, draw conclusions, follow and construct chains of arguments, etc.

What becomes progressively more important at further stages, especially when you train professionals in other fields, is the mathematical **mode of thinking** itself. The ability to start with the first principles, to take the simplest possible case and build up from there, to express things in a different language, to use analogies, to argue symbolically, to compress huge bulks of arguments, etc.

If we are trying to sell specific applications, we lose! That's exactly what is happening now, with devastating effects.

4. Utilitarian prospective

It is our deep conviction that UTILITARIAN PRINCIPLE DESTROYS EDUCATION. The best possible education is the useless one. The same applies to the mathematics education, of course.

In Europe the controversy between the supporters of a comprehensive approach to education, and the proponents of the practically-oriented one never subsided for the last 5 centuries, it seems. It suffices to recall the discord over the study of Latin and Greek in schools. Of course, this is indeed a huge social and economical issue, as we allude below.

But the debate itself is terribly much older than that. The polemic between Mo Di and Chuang-tze is still as relevant today, after 24 centuries, as it was in their life-time. But we are on the side of Chuang-tze, anyway: EVERYBODY KNOWS THE USEFULLNESS OF USEFUL THINGS. NOBODY KNOWS THE USEFULLNESS OF USELESS THINGS.

As we all know, Mathematics is an art form working with *ideas*, see [18], and, as Oscar Wilde observed, ALL ART IS QUITE USELESS. It is amazing, how often the word "useful" is repeated in Hardy's "Apology", dozens of times. Here is the most famous such instance, and the one most applicable to education:

> One rather curious conclusion emerges, that pure mathematics is on the whole DISTINCTLY more useful than applied. A pure mathematician seems to have the advantage on the practical as well as on the aesthetic side. For what is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics.

Let us illustrate Hardy's thought in a typical example. Oftentimes, the time lapse between the initial idea and the subsequent discovery, and then between the discovery and its technical application, takes decades, or centuries. It would had been impossible to *discover* lasers in nature, they had to be *invented* on the basis of Quantum Mechanics. In

[1] When asked "What kind of exercise do you prefer?", our colleague Timothy O'Meara responded: "Well, I'M EXERCISING MY BRAIN".

turn, Quantum Mechanics could not have emerged without the preceding development of physics and mathematics, including, in particular, complex numbers, differential equations, or matrices.

However, the Italian XVI century algebraists, who introduced complex numbers, have done it for fun and for sport, rather than any practical applications. They have not been considering the possible role of complex numbers in Quantum Mechanics or lasers — or, for that matter, even in the alternating current or radio.

If you can summarise the XX century social and educational ideas with one word, that word would be "OVERSIMPLIFICATION". Yuri Manin [25] makes an incisive comment to this effect:

> The core intrinsic contradiction of the market metaphor (including the outrageous "free market of ideas") is this: we are projecting the multidimensional world of incomparable and incompatible degrees of freedom to the one-dimensional world of money prices. As a matter of principle, one cannot make it compatible with even basic order relations on these axes, much less compatible with non-existent or incomparable values of different kinds.
>
> In this respect, the most oxymoronic use of the market metaphor is furnished by the expression "free market of ideas".
>
> Only one idea is on sale at this market: that of "free market".

Similarly, "useful education" is trying to sell you only one idea: that of "usefulness".

5. Mathematics for the general public: sociology

Around 1905–1915 there were elite schools in St Peterburg, *Gymnasia*, whose students were studying Algebra from textbooks by Dmitry Grave, which *started* with the notion of field, complex numbers, and the like, and stopped short of Galois theory — that was his next textbook, for the University. Unfortunately, the mathematical awareness of the less privileged population strata was much lower than that.

Here is how Alexandre Borovik describes the corresponding choice nowdays, see [6], reiterated in [7]:

> Democratic nations, if they are sufficiently wealthy, have three options:
>
> (A) Avoid limiting children's future choices of profession, teach rich mathematics to every child—and invest serious money into thorough professional education and development of teachers.
>
> (B) Teach proper mathematics, and from an early age, but only to a selected minority of children. This is a much cheaper option, and it still meets the requirements of industry, defence and security sectors, etc.
>
> (C) Do not teach proper mathematics at all and depend on other countries for the supply of technology and military protection.
>
> Which of these options are realistic in a particular country at a given time, and what the choice should be, is for others to decide.
>
> I am only calling a spade a spade.

We do not immediately see, what it has to do with democracy — or wealth, for that matter — option (B) is not that much cheaper, after all. But the choice is obviously there, anyway.

In the 1990-ies one of us was teaching *Matematica generale* to a class of 200 economics and management students at the *Università commerciale Luigi Bocconi*. Then, he was shocked by the fact that in the same class there were students from *ragioneria*, who have never seen logarithms before, and other students from *liceo scientifico*, who were quite proficient with multiple integrals. In the last decades, Russia has rapidly evolved in the same direction, from option (A) to option (B), so that a similar lack of uniform preparation is now routine at some departments of our university. But again that was a social choice as much as an economic one.

What moderates the situation in Russia, and what makes recruiting excellent **Mathematics**[2] students relatively easy, is the system of specialised PHYSICS AND MATHEMATICS SCHOOLS, operating in all major Russian cities, starting with Moscow, St Petersburg, Novosibirsk, etc. The first such schools were created by Andrei Kolmogorov, Dmitry Faddeev, Mikhail Lavrentiev and others some 60 years ago and they are still *by far* the best, the most functional, and the most efficient component of the whole Russian educational system. The *Presidential Liceum* 239 is for St Peterburg what *Lyceé Louis-le-Grand* is for Paris, with all social implications. See the recent paper by Nikolai Konstantinov and Alexei Semenov [23] for a detailed description of the principles, the history, and the current state of the Physics and Mathematics Schools.

However, all of our gut instincts suggest that the sharpest possible form of option (A) is the only correct answer. We do believe, that comprehensive and profound *universal* education in mathematics and exact sciences would be an excellent idea. It was never attempted before in the history of mankind, and we agree with Rokhlin [27] that:

> Somehow we feel intuitively that it would be good if our children and grandchildren were familiar with the logical culture, with the mathematical culture, if they could understand the exact sciences better.

6. Mathematics for the general public: instruction

The present day elementary mathematics instruction is encumbered by an overly rigid tradition, and is not up to the requirements of the XVI century. It may sound too dramatic, but we strongly believe this is the case. The existing curricula are mostly oriented towards the development of [obsolete] computational skills and mechanical use of a small number of [outdated] standard algorithms.

In the past, such similar needlework was of undeniable value, but today the need for mass training in ancient craftsmanship looks suspicious. It is akin to extracting fire by friction: you may have to use it once in your lifetime — probably not! — but it would be stupid to practice it each and every day.

Of course, it's up to you, how far you are prepared to go. Do we have to memorise the multiplication table 100×100? What about 10×10? Our viewpoint is as follows. It is useful to understand the idea of long multiplication — to get a clear understanding of the relative size of numbers, that the decimal notation is logarithmic [— or to multiply two 8-digit numbers that nobody has multiplied before, to get some feel of probability]. But it is pointless to systematically practice this skill — none of today's schoolchildren will have to perform such operations manually, simply because any computing device makes it faster, in a more efficient and more reliable way.

6.1. Curricula

With respect to the actual inner architecture of mathematics, or its current applications, the choice of the subject matter in school curricula seems to be rather arbitrary and bizarre. Of course, in many cases such oddities have a historical explanation, sometimes more than one.

Thus, for instance, the prevalence of **trigonometry** is easily explained by the needs of ballistics, and navigation. Here is what Alexandre Borovik [7] writes in merit:

> It is worth to remember that in the first half of the 20th century, school mathematics curricula in many nations were dictated by the Armed Forces' General Staffs – this is why trigonometry was the focal point and apex of school mathematics: in the era of mass conscription armies, it was all about preparation for training, in case of war, of a sufficient number of artillery and Navy officers and aircraft pilots. With this legacy, we still cannot make transition to a more human mathematics.

[2] Well, actually, MATHEMATICS *and* COMPUTER SCIENCE, see `https://math-cs.spbu.ru/en/`

That's obvious, and obviously true. However, it does not explain why trigonometry is being taught in such an antediluvian manner, WITHOUT COMPLEX NUMBERS.

Of course, all of school trigonometry necomes immediately obvious once you explain that addition formulas for cosine and sine are *precisely* multiplication formulas for complex numbers, in various national traditions this is called **Euler formula**, or **de Moivre formula**, whatever. The father of one of us (who was an electrical engineer) explained this to him at the age of 10–11 years within half an hour.

This is not how it is done at schools, however. Instead, a child is forced to learn by rote dozens of seemingly unrelated special cases, and nobody explains the true meaning of signs, etc., one just has to memorise all of it.

The venial explanation due to Henri Lebesgue [24], is that this is done OUT OF PURE SADISM, just to torment and humiliate the child. A much more sinister interpretation is articulated by Yuri Neretin [26], who believes this was done on purpose, as part of a market strategy to create a separate field of knowledge, *elementary mathematics*.

The business plan behind is roughly as follows:

- to use mathematics as a barrier and filter — the so called *entrance mathematics*, or *exam mathematics*.
- to create a market for private or semi-private educational services – preparatory courses, private tutors, and the like + the corresponding literature, sites, etc.

Further, Neretin also observes that since this new field of knowledge does not have any relation whatsoever to any other branch of mathematics, pure or applied, the person who has perfectly mastered *entrance mathematics* does not thereby acquire any knowledge or skill remotely useful in mathematics or science.

Imagine the kind feelings the poor children and their parents must share towards *that* sort of mathematics! What is much worse, many of them are induced to think, this crossbreed of military training, bookkeeping and penmanship is authentic mathematics!

6.2. False rigour and misguided proofs

In many cases educators insist on obsolete ways of teaching certain things. It is obvious to all mathematicians for more than half a century now that one aspect of the school curriculum that should be *completely* revised, is geometry. Such a reform will not eliminate geometry, but, to the contrary, enhance and invigorate it! In fact, most of the geometric proofs along Euclid's line, which schoolchildren have to memorise for the sake of PRESUMED RIGOUR, are either incomplete, or incorrect, or incomprehensible.

At the same time, we all know that the XVII century approach by de Fermat and Descartes removes all such difficulties, and makes the whole subject transparent, open-ended and useful. It was clear to every competent mathematician for 40–60 years that this is how geometry should be taught at a mass school. Let us quote Jean Dieudonné [14], who was an exuberant advocate of this approach:

> For the trained mathematician of today, it is a triviality that the fundamental theorems of Euclidean geometry (in any number of dimensions, by the way) are very easily derived from the concept of a vector space equipped with a positive definite quadratic form. Why shouldn't this method be made available (in two or three dimensions) to high school students instead of the incredible, apparently irrelevant dissections of triangles, where every step is made to appear to be a conjurer's trick?

Nothing has changed since.

What is worse, many of the alleged proofs in the school geometry textbooks — including most of the proofs on lengths, areas and volumes — are overtly fake or fallacious. There are passionate narratives to this effect in the books by Lebesgue and Grothendieck [17,24]. In 1981 Rokhlin [27] mentions it casually, as a common knowledge:

> I went to high school (perhaps, it's still the same now), I was told what the area of a circle is. I was told that this is some sort of limit, and then something was written or was stated, and we got a formula for the area of the circle. What was said was difficult to understand then, but when I became a mathematician, it became totally clear to me, why it was so difficult to understand. IT WAS ALL SHEER NONSENSE.

Again, nothing has changed since.

6.3. Elementary mathematics

What annoys us most about hierophants of the so called "elementary mathematics", however, is their chicanery and hairsplitting. For us, trained as professional mathematicians, all of their discussions seem to be completely devoid of meaning, and extremely artificial.

Russian educational networks burst with messages of the following type. When you count, how many beer bottles are there in 3 boxes of 6 bottles each, should you muliply 3×6, or 6×3? It turned out, there is a sacral order, approved by a certain Areopagus some centuries ago, and they actually **lower grades** to the poor children who do it otherwise, even getting the right answer. Only that we could never memorise, which order of operations they consider correct.

Wu Hung-Hsi [33] describes this outrageous situation as follows:

> One of the flaws of the school mathematics curriculum is that it wastes time in fruitless exercises in notation, definitions, and conventions, when it should be spending the time on mathematics of substance. Such flaws manifest themselves in assessment items which assess, not whether students know real mathematics, but whether they could memorize arcane rules or senseless conventions whose raison-d'être they know nothing about.

At a later stage there comes all that fuss about staying real, all that harassment conserning "the domain of allowable values", and suchlike. As Felix Klein observes [22], the elementary mathematics of this sort is a late invention, not earlier than the last quarter of the XIX century. Before that the XVIII and XIX century classics were always working in the complex domain.

Yuri Neretin [26] concludes: THE ABOVE-MENTIONED SCIENCE CAUSES IN A NORMAL YOUNG MAN ONLY TEDIUM AND DISGUST, OR, WHAT IS INCOMPARABLY WORSE, TORPIDITY.

7. Mathematics for non-mathematicians: what it is

The situation with training other professionals at the university level is similarly disgraceful. Obviously, in many executive respects it is much less odious than the mass mathematical education. But in terms of teaching content it is dominated by an obsolete tradition, which oftentimes makes it even less meaningful.

Historically, these "higher mathematics" courses are just diluted (or, as Rokhlin designates it, "watered-down") early XX century courses for mathematicians. These courses start with the same sequences, series and limits, and then pass on to the same derivatives, integrals, differential equations, etc., dealed with in a sterile and perfunctory manner.

Calculus textbooks, when they attempt at proving anything, are full of direct mathematical mistakes anyway, see [30]. Only that "higher mathematics" textbooks are usually worse than that, since they remove all deeper theorems and mathematically interesting examples, making the leftovers unsavoury, boring and impossible to digest[3].

Traditional mathematics courses for non-mathematicians — not just the absolutely stale and futile calculus courses, but most of the archaic service mathematics courses

[3] What Peter Taylor [28] says of the school curriculum is even more applicable at the university level: "The secondary-school mathematics curriculum is narrow in scope and technical in character; this is quite different from the nature of the discipline itself".

in linear algebra, differential equations, probability theory and discrete mathematics – are also focused almost exclusively on the mechanical exercise of rudimentary computational skills, without any deeper understanding of the true structure of the subject, its applications, its current state, or wider context.

Let us give an illustration of how slavishly the textbooks of higher mathematics follow traditional courses for mathematicians. We were shocked by seeing in a mathematics textbook for *philosophers* trigonometric substitutions, derivation of the function $x \mapsto x^x$, and the like. We recognise that the IDEA OF FUNCTORIALITY and the chain rule themselves could be extremely useful for philosopers. But we do not see any use in teaching them specific *technical* tricks for calculation of derivatives and integrals, whose gist they won't be able to grasp anyway.

As in school, there is a lot of insistence on "foundations" and the false "rigour". One of such completely artifical roadblocks is the "theory of limits". The emphasis on limits creates conceptual difficulties for many students, and it is absolutely irrelevant both for exposition of analysis itself, and for applications[4]. Here is what Rokhlin [27] says:

> ...the limits are part of the course that is most difficult to understand, and, what is interesting, absolutely unnecessary. Differential calculus, integral calculus, and, in general, all the classical mathematics, to say nothing of the finite mathematics, can be easily explained without the limits. More than that, they are not needed there. They are an absolutely extraneous phenomenon, extraneous subject that has been introduced into this area by the people who wanted to build a proper foundation for analysis.

8. Proofs and other evidence

We believe that the teaching of mathematics to non-mathematicians should be completely reformed. We do not see, why it should stay a downgraded version of training of mathematicians, either as far as the subject matter, or as style.

8.1. Proofs in education

Traditionally, it is claimed that most results stated in the elementary courses must be accompanied by complete proofs. Such a viewpoint seems to us HOPELESSLY OUTDATED, UNREALISTIC AND HYPOCRITICAL.

As it happens, in most cases, the presence or absence of proofs does not influence the confidence of students in the results themselves. We believe that the primary role of proofs in lectures and textbooks for non-mathematicians amounts to the following:

- To convince the students that they correctly understand the statement.
- To clarify the purport of a statement and its connection with other statements.

In the training of professional mathematicians proofs may have also other functions:

- To drill general patterns of mathematical reasoning (induction, reduction, partition into cases, general position, specialization, ...) and standard techniques in a specific area.
- To develop a habit and taste for precise arguments as such, and to exercise the ability to distinguish assumptions, evidence and plausible guesses from well-established facts.
- As they say in Cambridge, to illustrate some of the tedium.

All of these goals *might* be pursued also when teaching non-mathematicians — with some moderation, though, especially the last one!

In many cases proofs in educational literature, especially long, badly structured and purely computational ones, merely *disorient* the student, hazing and alienating the

[4] This discussion is not new either. Already "Lüshi Chunqiu" compiled not later than III century B.C. mentions that A TRUE SCIENTIST DOES NOT KNOW LIMITS.

meaning. In research papers, bad proofs are better than none, but in teaching it's the other way round.

8.2. Other evidence

What many mathematicians seem to ignore, is that there is nothing sacrosanct about the current ("modern") forms of mathematical expression. The ways how we organise and record our arguments are provisional and historically determined. For the purposes of education, our present day "proofs" are no better, than the ancient Egyptian "proofs", or ancient Chinese "proofs", or ancient Indian "proofs" — or ancient Greek "proofs", for that matter, they are just *different*. And, more probably than not, our current standards of reasoning and exposition are as transitory as these older forms.

A traditional proof, even less so a formal proof, are not the only ways to understand a mathematical result, and even for a professional mathematician they are rarely *the* best ways. There are smart proofs that explain things, that make us wiser, and such proofs should be cherished.

But otherwise to understand a statement you should look at examples, special cases, corollaries, experiments, heuristic arguments, analogies, applications, visualisation, etc. — this will usually tell you more about the true nature of a mathematical result, than most proofs. Much more so for students!

Just 100–150 years ago many mathematicians would *claim* that they verify the proofs of all results they quoted[5]. Today, a similar claim would sound pathetic. We have to rely always more on the work of others, and that's a one-way road. It is inevitable that we distribute our trust, see [9]:

> In all these settings, modern computational tools dramatically change the nature and scale of available evidence. Given an interesting identity buried in a long and complicated paper on an unfamiliar subject, which would give you more confidence in its correctness: staring at the proof, or confirming computationally that it is correct to 10,000 decimal places?

It is ridiculous to pretend that the students can meet the standard we have long abandoned ourselves.

9. Mathematics for non-mathematicians: what should we teach, really?

Our short answer to the question in the section heading is: WE DO NOT KNOW — and nobody does! There are several possible answers, the following most immediate ones:

- The same as always — whatever, limits, eigenvalues, . . .
- What is used in the corresponding subject field today — well, between "the same as always", and "nothing".
- Nothing — no joke! This viewpoint has more and more supporters!
- The mathematics of mathematicians.

Our answer is that we should teach mathematics as we, mathematicians, understand it. What we think important — the language, some general concepts that would allow to assimilate further concepts, and, above all, the mathematical thinking itself: basic techniques, some most productive arguments and ways of reasoning, some classical constructions, etc.

As far as the subject matter, it is our belief that it does not matter much, what exactly we teach. Nobody knows what exactly will be used in a specific field — certainly we do not know, but, as we said, nobody does.

We believe that the only way for the science and technology to advance, is to expose the professionals in these fields to more mathematics, more advanced mathematics — and, above all, meaningful mathematics, both classical and modern. But to do it

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5 Whether they were actually doing that, is a completely different story. We bet, not [30].

differently, focusing on conceptual aspects, **understanding**, applications, rather than on technical details of the proofs or specific computational skills.

It's not that they should stop studying mathematics relegating all computations to computers instead — quite to the contrary, they should be exposed to richer and deeper mathematics.

10. Mathematics and computers

We have already quoted, on several occasions, the following observation by Doron Zeilberger [34]:

> The computer has already started doing to mathematics what the telescope and microscope did to astronomy and biology.

We cannot agree more! In fact, we are convinced that mathematicians today have better access to the mathematical reality, than most experimental sciences have to physical reality, see [20,31]. And we tend to agree with Borovik [8] that the current ineffectiveness of Mathematics in Biology and other applications *might* be explained by the fact that the requisite Mathematics is simply too large for an individual human mind.

10.1. Mathematics for end-users

Computers have *already* completely redefined the applications of mathematics, the way in which mathematics will be handled in any predictable future by most end-users. We should be completely honest with ourselves that *none* of our students will *ever again* solve a system of linear equations, invert a matrix, calculate an integral or graph a function *by hand*, outside of the mathematics class. Why should we insist that they do it without the use of *computing machines* in the mathematics class?

They say that the mother of Carl Friedrich Gauß could observe with the naked eye phases of Venus and some moons of Jupiter. Unfortunately, for the vast majority of the usual people this is not possible, they have to resort to the help of *magnifying machines*.

This is clear to the end-users in the corresponding fields, as this is absolutely clear to our students. But we still prefer to pretend that we are doing something useful by feeding them badly chewed cardboard, which they do not need, and cannot digest anyway. As a result, many end-users start to complain, louder and louder.

In the last years, we've heard from more than one engineer, and not some imposters, but rather serious professionals, that there is no need to teach mathematics to [all of] engineering students anymore, just computers. We know they are wrong and that even the present day imperfect and retarded mathematical instruction is better then none. And a real sensible course of conceptual mathematics — **mathematics of mathematicians** would start a Golden Age in some subject fields. But at the end of the day, they will decide!

10.2. Mathematics for players

There is another closely related aspect, which we do not touch here, and which will eventually change the scene completely.

Most mathematicians tend to dramatically underestimate to which extent the development of mathematics is determined by the external circumstances, in the first place by the available computational resources. But whether we appreciate it or not, mathematics itself is in the process of an immense metamorphosis, one of the greatest in its history.

Already today the progress of computers and computer algebra systems strongly influences research in many areas of pure mathematics itself — such as group theory, combinatorics, number theory, commutative algebra, algebraic geometry, etc. Predictably, in the nearest future this influence will expand to all of pure mathematics and will produce *Umwertung aller Werte*: radical revision of research directions and style.

11. Computer algebra systems

For us, it is obvious that teaching science and engineering students to calculate derivatives and integrals, to solve algebraic or differential equations, to multiply or invert matrices by hand, or the like, is a sheer waste of time. These skills are as osbolete as the use of a slide rule or a logarithm table.

As of today, the *default* tools for all these things are the **general purpose CAS** = Computer Algebra Systems. There are many low-end products with limited functionality. There are also many **specialised CAS**, which are very good at some things, like polynomial calculations, or linear algebra, but do not cover the full range of symbolic mathematics.

Dropping the systems that are obsolete, not powerful enough, not supported anymore, too complicated or too expensive, do not have convenient Front End, or do not support graphics, you are left with an amazingly limited choice, essentially only four products: `Axiom`, `Maple`, `Mathematica`, and `SageMath`.

All of these four systems are very very good. All of them are, in the first place, very high-level programming languages, whose expressive power approaches fragments of a natural language. All of them can perform *all* usual computations, anything that a non-mathematician is likely to see in *any* possible present day application.

Nowadays, teaching top end computer scientists or mathematicians we would probably choose `Axiom` and `SageMath`. However, for a number of reasons, teaching non-mathematicians you have to choose between `Maple` and `Mathematica`, which is purely a matter of taste. In our courses we used *both*, but for a number of *extra-mathematical* reasons eventually opted for `Mathematica`.

12. Some tapas of computer algebra

We would usually start our class with a dozen or so demonstrations, of what is mathematics, really, and how computer can help. The actual examples would vary each year, below we reproduce some typical computations we were showing to our students at the first lecture, as a warm up for our course.

12.1. Elkies counter-example

Obviously, our students heard of Fermat problem. So we asked them whether they heard that Euler suggested a broad generalisation of that. Namely, he claimed that for $m \geq 4$ the equation

$$x^m + y^m + z^m = u^m$$

does not have solutions in natural numbers. That for $m \geq 5$ the equation

$$x^m + y^m + u^m + v^m = z^m$$

does not have solutions in natural numbers, etc.

However, in 1988 Noam Elkies [15] discovered that

$$2682440^4 + 15365639^4 + 18796760^4 = 180630077292169281088848499041 = 20615673^4.$$

Of course, finding such a solution with a home computer without knowing some rather advanced algebraic number theory and algebraic geometry is not feasible.

However, a similar counter-example for the fifth powers

$$27^5 + 84^5 + 110^5 + 133^5 = 61917364224 = 144^5$$

can be found by any student by brute force, within a few hours.

12.2. Ramanujan for low-brows

Polynomials can tell you many stories as well. Let us reproduce the famous 6-10-8-Ramanujan identity, see [5]. Set

$$f_n(x,y) = (1+x+y)^n + (x+y+xy)^n - \\ - (1+x+xy)^n - (1+y+xy)^n + (1-xy)^n - (x-y)^n \quad (1)$$

Then

$$64 f_6(x,y) f_{10}(x,y) = 45 f_8(x,y)^2$$

Of course, we would demostrate this by brute force, simply by opening all brackets and evaluating both sides to

$$46080x^2y^2 + 322560x^3y^2 + 887040x^4y^2 + 1128960x^5y^2 + 241920x^6y^2 - \dots$$

Ramanujan identities are in a sense most peculiar, since even for a mature mathematician it is not always easy to guess what goes on inside. But otherwise usually any of the Liouville identites, or even the corollaries of the Newton—Waring identities suffice to impress a student.

12.3. High precision fraud

We would usually show a couple of examples illustrating the difference between the mathematical and computational viewpoints, and the need for infinite precision calculations.

For instance, $e^{\pi\sqrt{163}}$ is so close to being an integer, that even the calculation with 12 positions after the decimal point still does not allow to tell, whether it's integer, or not

$$262537412640768743.999999999999$$

Of course, this only looks weird. Every competent mathematician knows that there is an obvious explanation, consisting in the fact that $\mathbb{Z}\big[\sqrt{-163}\big]$ is a principal ideal domain. The numbers $e^{\pi\sqrt{67}}$ and $e^{\pi\sqrt{43}}$ are also very close to integers, though not with such marvellous precision.

12.4. BBP-formulas

Another highlight of computer mathematics is the formula which allows to compute any *hexadecimal* digit of π separately, without computing the previous ones, see [2,3]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\Big(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\Big).$$

12.5. Inverting a 1000×1000 *matrix*

As another *tapas*, we would generate a random real 1000×1000 matrix, with values in the range, say $[-10, 10]$, machine precision. And then invert it, machine precision, which would normally take 3–4 seconds. Then we would comment that the amount of numerical computation involved in this individual evaluation far exceeds all numerical computation that all students in the class will perform, or could possibly perform, during their life-time.

Usually, the students were shocked, excited and amazed. We told them we could not teach them *discover* such things, but within a year or so we can certainly bring them closer to understanding and appreciating some of the mathematics behind such examples, and perform such similar calculations — and in fact *all* usual calculations! — with confidence. Thereafter, we usually had their attention.

We do not know, how to teach students who are not impressed by this kind of examples. It is our belief that in such extreme cases any medicine is powerless. As observed at the very beginning of the treatise [11] by Nicolas Bourbaki:

> Nous ne discuterons pas de la possibilité d'enseigner les principes de mathématique à des êtres dont le développement intellectuel n'irait pas jusqu'à savoir lire, écrire et compter.

13. Borwein's joke

Here is a similar (fancier!) example we were not showing to our students. But next time we certainly will! Consider the following sequence of integrals, see [10]:

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11}\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11}\frac{\sin(x/13)}{x/13}\, dx = \frac{\pi}{2}.$$

Guess the value of the next one.

Well, actually the pattern breaks at the next step:

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11}\frac{\sin(x/13)}{x/13}\frac{\sin(x/15)}{x/15}\, dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi.$$

The reason is of course that

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13} < 1, \quad \text{but} \quad \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15} > 1,$$

and it's a [highly non-trivial!] exercise in harmonic analysis and integral transforms to work out what goes on here! There are more such remarkable examples, see [1,13,21] and references there.

14. The course "Mathematics and Computers"

In 2005 we started to teach a two-semester course "MATHEMATICS AND COMPUTERS" at the Economics Department of St Petersburg State University, the Spring semester of the 1st undergraduate year + the Fall semester of the 2nd undergraduate year.

For administrative reasons[6] the second semester of this course was sometimes called "MATHEMATICAL SOFTWARE", but it was a direct sequel of the same course anyway, so that one should think of our course as "Mathematics and Computers, I" and "Mathematics and Computers. II".

The course was taught not to all economics students, just to those specialising in "MATHEMATICAL METHODS IN ECONOMICS"[7] and in "APPLIED INFORMATICS IN ECONOMICS"[8], about 25 students per year each, 50 students per year total.

Another person actively involved in the development of this project at the initial stage was Oleg Ivanov. Later he and Grigory Fridman have launched a similar project at the St Petersburg State University of Economics and Finance, see [19], for instance.

A normal class was mixed format. It usually started with introducing some new mathematical concepts and ideas, and a few key statements with occasional proofs. The proofs were only explained when they were especially short and transparent and contained powerful general ideas which work in many situations. After that we gave suggestions for further reading, for those who wanted to study these concepts deeper and passed to algorithms and computer demonstrations, computations, graphics, etc. After that we distributed small standard problems and larger semi-research projects, both individual and for small groups of 2–3 students. Both were subsequently discussed in the class, very selectively, though, sometimes only in case of difficulties, otherwise only answers, ideas, and/or parts of the code.

The course would concentrate on basic mathematical ideas, rather than specific applications. Below we list the topics which were covered sort of each year. Otherwise, we allowed a lot of flexibility and any given year could mention different examples and subject fields.

Usually, we started with warm up material on subjects which were [partly] familiar to many of the students — but not to all of them! Part of the idea was that the students begin facile coding with topics where mathematics is either familiar or amusing [or both!], and feel some initial confidence.

- **Arithmetics**. We started with integers, rational numbers, real and complex numbers, and modular arithmetics. Various formats, basic algortihms, elementary functions, calculation of powers, Euler formula and de Moivre theorem, roots of 1, congruences up to, say, Euclidean algorithm, finite fields and Chinese Remainder Theorem. Sometimes this part included some fancier topics, like continuos fractions, denesting of radicals, harmonic numbers, Bernoulli numbers, etc.
- **Basic number theory**. That would normally include primes, Eulcid's theorem and the Fundamental theorem of Arithmetics, some dainties like Fermat and Mersenne primes, the prime number theorem and Dirichlet theorem on primes in arithmetic progression[9], Fermat and Euler theorems, pseudoprimes, Legendre symbol, quadratic reciprocity. We would mention also some classical problems in additive number theory, but no part of that was required for the exam, it served only as a source of research projects in the style of recreational mathematics.

The part on discrete mathematics and combinatorics was the central part of the course, at least the focus of the 1st semester, in view of the fact that we were teaching prospective high-end computer users.

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6 The absurd bureaucratic requirement that courses in different semesters should have different names.

7 This major was created at St Petersburg State University in the 1930-ies, by Leonid Kantorovich.

8 This major was relatively new, and only created in the early 2000-ies. Presently it changed the name to "BUSINESS INFORMATICS".

9 Both without the faintest sketch of proof, just as experimental facts! The students had to verify them up to certain limits and in certain special cases as experimental facts.

- **Combinatorics I**. That would normally include factorials, raising and falling factorials, binomial and multinomial coefficients, Stirling and Bell numbers, Catalan numbers, generating functions, and the like. Here, we would present as many proofs as possible, to practice such ideas as induction, partition into cases, Dirichlet principle, recurrencies, etc.
- **Discrete Mathematics I**. Lists: generation of lists, parts of a list, basic structure manipulations, nested lists, trees and other data structures, various algorithms for sampling, search and sorting. Sets and multisets: subsets, chains and antichains, Boolean operations, Cartesian products, enumeration theory, inclusion-exclusion, partitions, Gray code.
- **Discrete Mathematics II**. Maps: functions, Dirichlet principle, surjective and injective maps, pure and anonymous functions, λ-calculus, compositions and iterations, orbits, trajectories, and fixed points. Relations: Binary relations, graphs, equivalence relations, order relations, Hasse diagrams, Möbius inversion, Ramsey theorem, Hall theorem (with proofs!)
- **Combinatorics II**. Permutations: algebra of permutations, symmetric group, generation of permutations, lexicographically and otherwise, transpositions, change ringing, sign of permutations via decrement and inversions (with proofs!), alternating group, involutions. Cycles: canonical decomposition, long cycles, multiplication of cycles, cycle type and conjugacy classes, statistics of cycles, maximal order, and the like.

That would normally take most of the first semester, after which most students would feel quite confortable in translating mathematical problems into fully functional code in `Mathematica`, and eager to apply this skill to other fields of mathematics which they studied.

The end of the first semester, and the beginning of the second semester were a medley of further basic mathematics and [mathematical] applications. Here, we would normally cover some further basic constructions, and various somewhat deeper topics.

Typically, this material would start with the two following classical constructions, with some proofs (but by far not all of them!)

- **Polynomials**. Structure manipulation with polynomials, rational functions, power series, and the like, coefficients, roots, effective evaluation, fast multiplication and division, convolution, various flavours of interpolation (Newton, Taylor, Lagrange, Hermite,. . .), fast Fourier transform, algebraic equations and factorisation of polynomials, Gauss theorem, Chebyshev polynomials, cyclotomic polynomials, classical orthogonal polynomials, etc. Polynomials in several variables, symmetric polynomials (Viète, Newton, Waring,. . .), etc.
- **Matrices**. Structure manipulations with rows, columns, matrices and other tensors, parts of a matrix, multipication of matrices and other operations, matrices and linear maps, eigenvalues and eigenvectors, various notions of rank, elementary transformations, systems of linear equations, inverse matrix, various classical types of matrices (symmetric, orthogonal, circulant, etc.), block matrices and efficient algroithms, Kronecker product and sum of matrices, determinants and other invariants, canonical forms.

As applications we would usually mention some further topics, discussing them very briefly in the class, and offering all more complicated themes as projects for homework (at this stage it was assumed that the students spend *at least* 3 homework hours for each class hour).

- **Calculus**. Derivatives, integration, differential equations, whatever.
- **Linear Algebra**. Aplications to geometric and/or applied problems of linear algebra.

In the second semester, we would also discuss the topics required to produce a document containing complex mathematical formulas and computations, and, maybe something else, text, graphics, and other elements.

- **Algorithms with strings**. Transformation of text, formulas and tables: search, sorting, formatting, etc., rudimentary typesetting issues.
- **Basic Graphics**. Graphs of functions of one and two variables, geometric transformations of objects in 2 and 3 dimensions: translations, rotations, symmetries. Usually up to, say, regular and semi-regular solids, tilings and wallpaper groups.

This was a rather intensive course, and we do not believe we could do much more than that within a year at such an early stage, given the preparation of the students, and the share of their time they could devote to our course.

15. Reservations

Overall, we judge this project as a complete and overwhelming success. It was certainly a refreshing and gratifying experience for ourselves. Much more fun than teaching the usual service courses anyway!

With active participation and interest on part of students we succeeded in covering much more Mathematics, more varied Mathematics, more interesting and useful Mathematics, with much better results, than would ever be possible with more traditional approaches.

It was, as we know, quite an experience for our students, many of whom later indicated that as a result of our course they understood what mathematics was about, stopped being scared by mathematics, started to love formulas, numbers, graphs, and as a result routinely use specialised mathematics tools for other courses.

Whether a similar project is portable and would be equally successful at a different university and/or within a different subject field, is not quite clear to us. We fully realise that we were in a privileged position in more than one respect.

1. St Petersburg State Univ. is one of the two universities in Russia (the other one is the Moscow State Univ.) that enjoy full academic autonomy. We can introduce new courses without any authorisation or approval of the Ministry of Science and Higher Education, or any other administrative body.
2. The project had full support of the Dean's office, both administrative, and financial. We had to present the course at the Teaching committee and the Departmental council, but essentially we had free hand as far as its outline and contents.
3. We had *two* fully equipped computer classes, with blackboards and 25+1 computers joined to a local network, with licenced copies of `Mathematica`, `Maple` and other necessary software installed + friendly technical support.
4. The programs "Mathematical Methods in Economics" and "Applied Informatics in Economics" are fairly competitive and select [mostly] good students, who were prepared to work with computers anyway. Many of them had preceding experience of programming in low level languages.
5. Many of these students were coming from good St Petersburg schools and had previous exposure to some calculus, vector analysis and the like at school, others were taking traditional courses of calculus and/or linear algebra in parallel.
6. Virtually all of the students had home computers with *some* mathematical software, and full access to the departmental computers with licensed copies of `Mathematica`, `Maple`, etc., outside of the class hours.
7. Most of the students had good working command of English, so that we did not have to translate for them help files, problems, instructions, jokes, etc.

Obviously, any of these points could break even at an equally excellent university, and all of them will break if you consider passage to lower level education.

In fact, it is not feasible that every school class could be equipped with comparable hardware, to install licenced commercial CAS such as `Mathematica`, `Maple` or `Axiom`.

One of the points to start should be creation of a simpler and less demanding CAS with front end in national languages.

16. Conclusions

Below, we outline our general convictions about teaching mathematics to non-mathematics students, summarising a few decades on teaching experience.

1. Teaching of Mathematics for non-mathematicians must be fascinating, vivid, inspiring. It is much more important to demonstrate the beauty and power of Mathematics, than to teach any specific topic. Mathematics is fun, any teaching that ignores this basic fact is harmful in times of peace, and dangerous in wartime.
2. The choice of specific content is mostly immaterial, since we do not know what kind of mathematics they will use during their careers anyway. The mathematical culture, the mathematical way of thinking themselves, positive attitude and willingness to study new topics and to use Mathematics are way more important.
3. The value of most of specific computational skills is negligible. Most of the students will never use these skills during their careers. Most of the specific calculations will be relegated to a computer, and difficult cases require professional advice anyway. Conceptual understanding and awareness are by far more valuable.
4. Most of the proofs have subordinate value. The student can understand a mathematical concept or result and sensibly use it without knowing the proofs. In most cases examples, special cases, corollaries, applications, analogues, experimental data, visualisations can do as much or more to explain a result, than a formal proof.
5. Computers have dramatically changed applications of mathematics. But computers have not made Mathematics obsolete. They have made obsolete only the current teaching of mathematics that was obsolete anyway, even before the advent of computers. Quite to the contrary, today we have to teach most professionals more Mathematics, more profound Mathematics, more advanced Mathematics, but we have to do it differently.
6. If you cannot beat them, join them. We have to welcome symbolic calculations and computer algebra systems in mathematics class, and widely use them as a medium of instruction. Of course, the corresponding conversion of all mathematical courses, curricula, tests, exams, etc. will require a lot of work. But if done right it entails no dangers for mathematical education, just possibilities.

To finish on a slightly more cheerful note, let us quote Asterix:

> Gauls! We have nothing to fear; except perhaps that the sky may fall on our heads tomorrow. But as we all know, tomorrow never comes!!

Tomorrow does come. It is *almost* there. Our only hope is that its arrival is leisurly enough to give us, the mathematical community, time to adapt and reform the teaching of mathematics before it is too late.

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