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Notes on Characterizing Hilbert Space by Smoothness and Smooth Orlicz Spaces*

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1. GENERALITIES

In [9] it was shown that if a Banach (or B) space $\mathscr{X}$, and its adjoint $\mathscr{X}^*$, have twice Fréchet (or F) differentiable norms, then $\mathscr{X}$ *can be* given an equivalent norm under which it becomes a Hilbert space. From the point of view of applications, it is desirable that $\mathscr{X}$, in its original norm, is a Hilbert space. For instance, if $\mathscr{X}$ is finite-dimensional, then the above conclusion adds little. In reviewing [9] in Mathematical Reviews [Vol. 35 (1968), p. 402], R. Bonic states: "This, however, is a more difficult problem and remains open." However, the characterization given above, in terms of equivalence of norms, is the best possible result in the sense that there exist B spaces satisfying the differentiability hypotheses but which are not Hilbert spaces themselves. The following example illustrates this statement.

Recall that the norm $\|\cdot\|$ of $\mathscr{X}$ is twice F-differentiable iff (= if and only if)

$$\| x_0 + \alpha y \| = \| x_0 \| + \alpha G(x_0, y) + \frac{\alpha^2}{2} T_{x_0}(y, y) + o(\alpha^2), \qquad (*)$$

for all x_0, y in S, the unit sphere of $\mathscr{X}$. Here $G(x_0, y)$ is the first and $T_{x_0}(y, y)$ the second F-derivation at x_0 when as $\alpha \to 0$, the limits existing uniformly in $y \in S$. The proof of the result [9] consists in showing that $T_{x_0}(\cdot, \cdot)$ is a bounded bilinear form which is positive definite on $\mathscr{X}_1 \times \mathscr{X}_1$, where $\mathscr{X}_1 = \{y : G(x_0, y) = 0\}$. (It is here that the twice differentiability of the norm of $\mathscr{X}^*$ is needed.) Clearly, for any $x_i \in \mathscr{X}$, $x_i = y_i + \lambda_i x_0$, $y_i \in \mathscr{X}_1$, $i = 1, 2$ and λ_i real, the representation being unique. The inner product

$$(x_1, x_2) = T_{x_0}(y_1, y_2) + \lambda_1 \lambda_2 \| x_0 \|^2,$$

gives the equivalent norm $|||x||| = (x, x)$:

$$c_1 \| x \|^2 \leqslant ||| x ||| \leqslant c_2 \| x \|, \qquad c_i > 0, \quad i = 1, 2. \qquad (**)$$

With this set up, the counterexample is given as follows. Let

$$\Phi(x) = a\left(\frac{x^3}{3} + \frac{x^2}{2}\right), \qquad \Psi(x) = -\frac{x}{2} + \frac{a}{12}\left(\left[1 + \frac{4}{a}x\right]^{3/2} - 1\right),$$

where $0 < a \leqslant 1$, $x \geqslant 0$, is chosen such that $\Phi(1) + \Psi(1) = 1$. The latter is a convenient normalization (cf. [8]). The pair (Φ, Ψ) becomes what is called the complementary Young's functions (cf. Section 3 below). Let L^Φ be the space of (equivalence classes of) real measurable functions f on a general space (Ω, Σ, μ) such that $N_\Phi(f) < \infty$, where

$$N_\Phi(f) = \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leqslant \Phi(1)\right\}.$$

It is known [8, p. 683] that this particular space L^Φ is a uniformly rotund and smooth (= norm is F-differentiable) B space. By [12], rotundity and smoothness are dual properties in reflexive B spaces, and need not be present for the same space. However, the present L^Φ has both, (cf. [8]). If

$$k(t) = N_\Phi(f_0 + tf), \qquad N_\Phi(f_0) = 1, \qquad f_0 \geqslant 0, \quad f \in L^\Phi,$$

then an elementary computation shows:

$$k'(0) = G(f_0, f) = \int_\Omega f\Phi'(f_0)\, d\mu,$$

$$k''(0) = T_{f_0}(f, f) = 2k'(0)\,(1 - k'(0)) + \int_\Omega \Phi''(f_0)\,(f - f_0 k'(0))^2\, d\mu,$$

and similarly for the adjoint space L^Ψ of L^Φ. Here Φ', Ψ', Φ'', Ψ'' are the first and second derivatives of Φ and Ψ. Thus for the B space $\mathscr{X} = L^\Phi$, the hypothesis of the result stated above is satisfied so that $\mathscr{X}$ can be given an equivalent norm, under which it becomes a Hilbert space. However $(L^\Phi, N_\Phi(\cdot))$ is manifestly *not* a Hilbert space since $\Phi(x) \neq x^2/2$. If μ concentrates on a finite number of points, one gets even a simpler (finite dimensional) example from this one.

In view of the above discussion, it is necessary to strengthen the hypothesis in order to assert that $\mathscr{X}$ is a Hilbert space. A slight strengthening makes the problem very easy, however. Two simple results giving such characterizations are included in the next section. The final section is devoted to some inter-

CHARACTERIZING HILBERT SPACE BY SMOOTHNESS *)

BY

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1. *Introduction.*

While discussing the smoothenss properties of certain Banach (or B-) spaces, in [1], BONIC and FRAMPTON state: "It is likely that if a B-space and its dual are C^2-smooth, then it is a Hilbert space". The purpose of this note is to prove this statement in the following form.

Theorem 1. *If $\mathscr{X}$ is a B-space and $\mathscr{X}^*$ its dual and if the norms in $\mathscr{X}$ and $\mathscr{X}^*$ are twice Fréchet (or F-) differentiable at every point except possibly the origin, then $\mathscr{X}$ is isomorphic to a Hilbert space in the sense that $\mathscr{X}$ may be provided with an inner-product in such a way that the resulting (inner-product) norm is equivalent to the given norm of $\mathscr{X}$.*

Even though there are several characterizations of the Hilbert space (e.g., [3], [6], [8], [5]), they are mostly based on geometric considerations. The corresponding characterizations based on smoothness properties are of interest in certain analytical work (cf. [1]). Such a method is considered here. The terminology employed here essentially follows [2], and some results of the latter will be used freely.

2. Proof. The proof obtains from the following seven steps starting with the real case. Then the complex case will be obtained using ([3], p. 335).

(1) If $\mathscr{X}$ is a B-space and the norm of $\mathscr{X}^*$ is F-differentiable then $\mathscr{X}$ is reflexive. If also the norm of $\mathscr{X}$ is F-differentiable then $\mathscr{X}$ and $\mathscr{X}^*$ are homeomorphic (under a 'spherical image map', [2], p. 306).

For, by ([2], Cor. 3.18) the norm in $\mathscr{X}^*$ is F-differentiable iff the unit sphere of $\mathscr{X}$ is weakly uniformly rotund (and hence it is k-rotund, [2], p. 309). This implies in turn (cf. [2], Thm. 5.4 (i)) that the unit ball of $\mathscr{X}$ is weakly compact since $\mathscr{X}$ is complete. It follows that $\mathscr{X}$ is reflexive ([4], p. 38). The last statement is a consequence of ([2], Cor. 3.18 and Thm. 4.18). Moreover, by a well-known result of ŠMULIAN (cf. also [2], p. 289), $\mathscr{X}$ and $\mathscr{X}^*$ are simultaneously rotund and smooth.

(2) If $\mathscr{X}$ is a B-space whose norm is twice F-differentiable and if T_{x_0}

Smooth Banach Spaces*

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1. Introduction and basic definitions

This paper is the outcome of a study of the twice differentiability of the norm of a real Banach space. The basic definitions of the various second order differentials of the norm are formulated following the familiar pattern of the first differential, DAY [4]. The rest of the paper is divided into three sections. In Section 2 some properties of the second derivative according to one or the other of the definitions are established. With the help of these properties a polar characterisation of the twice differentiability of the norm and some isomorphism theorems are obtained. In particular it is proved that if the norms of a real Banach space E and its adjoint E^* are twice differentiable then E is isomorphic to a Hilbert space. In Section 3 the equivalence of the several definitions of the second order differentials of the norm when the space is finite dimensional is studied. In Section 4 the order of differentiability of the norms of the familiar $l_p(L_p)$ spaces is obtained.

We state the notations and definitions required in what follows.

E denotes a real Banach space and E^*, E^{**} are its first and second duals. $B(E)$, $\mathscr{L}(E, E^*)$ denote respectively the Banach space of bounded bilinear functionals on E and the Banach space of bounded linear operators on E into E^* respectively with the usual supremum norms. σ denotes the linear isomorphism on $\mathscr{L}(E, E^*)$ into $B(E)$ defined by $\sigma(T)(x)(y) = T(x, y)$. Let $U(U^*)$ and $S(S^*)$ denote respectively the unit ball and the unit sphere of $E(E^*)$. Once for all it is assumed that the norm of E is once Fréchet differentiable so that the spherical image map G on S into S^* is a function and indeed is continuous with respect to the norm topologies relativised to S and S^*, CUDIA [3].

Definition 1. The norm $\|.\|$ of the Banach space E is said to be *twice directionally differentiable* at an element $x \neq 0$ if there exists a symmetric bounded bilinear function T_x such that if $y \in S$ then

$$\|x + ty\| = \|x\| + tG(x)\,y + t^2 T_x(y, y) + \theta(t)$$

where $\theta(t)$ depends on x and y and $\frac{\theta(t)}{t^2} \to 0$ as $t \to 0$. The Banach space E is said

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to be *twice directionally differentiable* if the norm is twice directionally differentiable at all $x \neq 0$.

Definition 2. The norm of E is said to be twice differentiable at $x \neq 0$ if the mapping G on $E \sim \{0\}$ into E^*, where $G(z)$ is the first order derivative at z, is differentiable at x; i.e., there exists a linear operator T_x on E into E^* such that

$$G(x+h) = G(x) + T_x(h) + \theta_x(h)$$

where $\frac{\theta_x(h)}{\|h\|} \to 0$ as $\|h\| \to 0$. If G has a derivative at all $x \neq 0$ then the norm is said to be twice differentiable.

Remark 1. It is known (DIEUDONNÉ [5]) that if there exists an operator T_x satisfying the condition in the preceding definition then T_x is bounded.

Definition 3. The norm of E is said to be *twice Fréchet differentiable* at an $x \neq 0$ if there exists a symmetric bounded bilinear functional T_x such that

$$\|x+h\| = \|x\| + G(x)\,h + T_x(h,h) + \theta_x(h)$$

where $\frac{\theta_x(h)}{\|h\|^2} \to 0$ as $\|h\| \to 0$. As in definition 2 if T_x exists for all $x \neq 0$ then E is said to be *twice Fréchet differentiable.*

Remark 2. If the mapping $x \to T_x$, where T_x is the second derivative according to definition 2 (3), is well defined in an open set A in E and is a continuous mapping into $\mathscr{L}(E, E^*)$ $(B(E))$ then the norm is twice continuously differentiable *(twice continuously Fréchet differentiable)* in A.

Remark 3. It is directly verified that the norm is *twice differentiable* at $x \neq 0$ if and only if it is *twice Fréchet differentiable* at x.

2. Twice differentiability of the norm of a Banach space

In theorem 1 some useful properties of the second derivative of the norm are established.

Lemma 1. *If the norm is once differentiable at* $x \neq 0$ *then*

$$\lim_{t \to 0} \frac{\|x+t\xi\| - \|x\|}{t} = G(x)\,\xi \quad \textit{for all} \quad \xi \in E.$$

Further, $G(\lambda x)$ *exists for all* $\lambda \neq 0$ *and*

$$G(\lambda x) = \operatorname{sign} \lambda\, G(x). \quad \textit{Also,} \quad \|G(x)\| = 1.$$

Theorem 1. *If the norm is twice differentiable at* $x \neq 0$ *and* T_x *is the corresponding derivative then*

(i) T_x *is symmetric.*

(ii) *Range* $T_x \subset \{x\}^{\perp}$.

(iii) *If* $\lambda \neq 0$ *then* $T_{\lambda x}$ *exists and* $T_{\lambda x} = \frac{1}{|\lambda|} T_x$.

(iv) $T_x(\xi, \xi) \geqq 0$ *for all* $\xi \in E$.

Proof. (i) The property (i) holds for any twice differentiable function, DIEUDONNÉ [5] theorem 8.12.2.

Now we proceed to characterize Hilbert spaces among twice differentiable Banach spaces. The Gateaux gradient of the norm in E^* is denoted by G_1.

Theorem 5. *If the Banach space E and its dual E^* are twice differentiable then E is isomorphic to a Hilbert space.*

Proof. Let $x \in S$ and let T_x and $T_{G(x)}$ denote the second derivatives of the norms in E and E^* at x and $G(x)$ respectively. It is verified that $G_1(G(x)) = Q\dfrac{x}{\|x\|}$ where Qx is the canonical image of x in E^{**}. Further we note that the mapping Q is a linear isometric mapping on E into E^{**}. Now consider any $y \in E_x$. Since the norm in E^* is twice differentiable at $G(x)$ and G is a continuous mapping on S into S^* the following equations are obtained.

$$\begin{aligned} G_1[G(x+ty)] &= G_1[G(x)] + T_{G(x)}[G(x+ty)-G(x)] + \theta_x(t) \\ &= Qx + T_{G(x)}[tT_x(y)] + \varphi_x(t) + \theta_x(t) \\ &= Qx + tT_{G(x)}[T_x(y)] + T_{G(x)}(\varphi_x(t)) + \theta_x(t) \end{aligned}$$

whereas $t \to 0$ we have $\dfrac{\varphi_x(t)}{t} \to 0$ and $\dfrac{\theta_x(t)}{t} \to 0$ since

$$\left\| \frac{G(x+ty)-G(x)}{t} \right\| \to \|T_x(y)\| \quad \text{as} \quad t \to 0.$$

Hence

$$\begin{aligned} T_{G(x)}[T_x(y)] &= \lim_{t\to 0} \frac{Q\dfrac{x+ty}{\|x+ty\|} - Qx}{t} \\ &= \lim_{t\to 0} \frac{Q(x+ty) - \|x+ty\|\, Qx}{t\,\|x+ty\|} \\ &= \lim_{t\to 0} \frac{Qx + tQy - [1 + tG(x)\,y + t^2 T_x(Y, Y) + \varepsilon(t)]\, Q(x)}{t\,\|x+ty\|} \\ &= Qy \text{ since } G(x)\,y = 0 \quad \text{and} \quad \lim \frac{\varepsilon(t)}{t} = 0 \quad \text{as} \quad t \to 0. \end{aligned}$$

Thus $T_{G(x)}[T_x(y)] = Qy$ for all $y \in E_x$. Since Q is an isometry, $\|T_{G(x)}[T_x(y)]\| = \|y\|$. Clearly $T_{G(x)} \neq 0 \neq T_x$, and for $y \in E_x \cap S$, $\|T_x(y)\| \geq \dfrac{1}{\|T_{G(x)}\|}$.

Thus by theorem 4, E is isomorphic to a Hilbert space.

Remark 6. Since a Hilbert norm (and its dual norm) are twice differentiable theorem 5 is a characterization of Hilbert spaces up to an isomorphism. Further it is known that if the dual of a Banach space E is once differentiable then E is

SMOOTHNESS OF ORLICZ SPACES [1]). II

BY

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4. *Rotundity and smoothness*

A B-space $\mathscr{X}$ is said to be *rotund* if its unit ball $U=\{x: \|x\| \leq 1\}$ has the property that every open segment in U is disjoint from the boundary, $S=\{x: \|x\|=1\}$, (cf., [3], p. 112). Also $\mathscr{X}$ is *smooth* if at every point of S there is only one supporting hyperplane of U. This concept is known to be equivalent to the weak differentiability of the norm functional of $\mathscr{X}$ at every point of S. In [2], further classifications were introduced, but the above terminology, which follows [3], will be sufficient in this paper. A B-space $\mathscr{X}$ is *uniformly rotund* if $x_n \in U$, $y_n \in U$ and $\|x_n+y_n\| \to 2$ implies $\|x_n-y_n\| \to 0$. Also $\mathscr{X}$ is *uniformly smooth* if the norm is almost additive in a narrow cone, i.e., for $\varepsilon>0$, there is $\eta_\varepsilon>0$, such that $\|x-y\|<\eta_\varepsilon$ implies $\|x+y\|(1+\varepsilon) \geq \|x\|+\|y\|$. As may be expected from definitions, there is a duality between smoothness and rotundity and the results of the preceding section play a key role in what follows. It should be remarked that *rotundity* and *uniform rotundity* are also referred to (in the older literature) as *strict convexity* and *uniform convexity* respectively. The present nomenclature is due to DAY, [3].

The first conditions on rotundity and smoothness are given by

Theorem 4. *Let L^Φ, L^Ψ be (complementary) Orlicz spaces and $\Phi(\cdot)$ be strictly convex. Then L^Φ is rotund. In particular, if $M^\Phi=L^\Phi$, $M^\Psi=L^\Psi$, (and Φ', Ψ' are continuous by normalizations) then $L^\Phi[L^\Psi]$ is both rotund and smooth.*

Proof. The last part is easy. For, by Corollary 1.1, $L^\Phi[L^\Psi]$ is reflexive, and $(L^\Phi)^*[(L^\Psi)^*]$ is isometrically isomorphic to $L^\Psi[L^\Phi]$. But in a reflexive space, the unit ball is rotund (i.e., the space is rotund) if and only if the unit sphere of its conjugate space is smooth, by ([3], p. 114; [2], p. 289). Also by Corollary 2.1, since $M^\Psi=L^\Psi$, the norm in L^Ψ is strongly (hence weakly) differentiable (and the same is true for L^Φ). It follows that $L^\Phi[L^\Psi]$ is both rotund and smooth.

For the more general case, it should be shown that L^Φ is rotund if Φ is strictly convex. Suppose that the statement is false. Then there exist f_1, f_2 in S^Φ, $f_1 \neq t f_2$, a.e., (t scalar) and $N_\Phi(f_1+f_2)=N_\Phi(f_1)+N_\Phi(f_2)$, A **uniformly convex space** is a normed vector space such that, for every $0 < \varepsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with $\|x\| = 1$ and $\|y\| = 1$, the condition

$$\|x - y\| \geq \varepsilon$$

implies that:

$$\left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

Intuitively, the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

1.3. *Young's functions*

In his studies on Fourier series, W. H. Young has analyzed certain convex functions $\Phi : I\!R \to I\!R^+$ which satisfy the conditions: $\Phi(-x) = \Phi(x)$, $\Phi(0) = 0$, and $\lim_{x \to \infty} \Phi(x) = +\infty$. With each such function Φ, one can associate another convex function $\Psi : I\!R \to I\!R^+$ having similar properties, which is defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \qquad y \in I\!R. \tag{1}$$

Then Φ is called a *Young function*, and Ψ the *complementary* function to Φ. It follows from the definition that $\Psi(0) = 0$, $\Psi(-y) = \Phi(y)$, and, what is important, $\Psi(\cdot)$ is a convex increasing function satisfying $\lim_{y \to +\infty} \Psi(y) = +\infty$. From (1) it is evident that the pair (Φ, Ψ) satisfies *Young's inequality*:

$$xy \leq \Phi(x) + \Psi(y), \qquad x, y \in I\!R. \tag{2}$$

Definition 1. A Young function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is said to satisfy the Δ_2-condition (globally), denoted $\Phi \in \Delta_2$ ($\Phi \in \Delta_2$ (globally)) if

$$\Phi(2x) \leq K\Phi(x), \qquad x \geq x_0 \geq 0 \; (x_0 = 0) \tag{1}$$

for some absolute constant $K > 0$.

Considering $\Phi(x) = |x|^p$, $p \geq 1$, we see that $K \geq 2$. Also in (1), 2 can be replaced by $\alpha > 1$, and one gets a condition equivalent to (1). Further note that all $\Phi : x \mapsto a|x|^p$, $p \geq 1, a > 0$, belong to Δ_2. On the other hand, if $\Phi_0 : x \mapsto e^{|x|} - 1$, then this $\Phi_0 \notin \Delta_2$.

In the opposite direction to (1), one can introduce the following condition studied by Andô (1960).

Definition 2. A Young function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is said to satisfy the ∇_2-condition (globally), denoted $\Phi \in \nabla_2$ ($\Phi \in \nabla_2$ (globally)) if

$$\Phi(x) \leq \frac{1}{2\ell}\Phi(\ell x), \qquad x \geq x_0 > 0 \; (x_0 = 0) \tag{2}$$

for some $\ell > 1$.

For instance, if $\Phi(x) = (1 + |x|)\log(1 + |x|) - |x|$, then its complementary function Ψ is given by $\Psi(y) = e^{|y|} - |y| - 1$. It is quickly verified that $\Phi \in \Delta_2$ (but not ∇_2) and $\Psi \in \nabla_2$ (but not Δ_2). We may present a characterization of these conditions:

Recall that if $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ its adjoint which is always a Banach space under the adjoint norm, let $(\mathcal{X}^*)^*$ be the second adjoint, simply denoted $\mathcal{X}^{**}$. If the image $\hat{\mathcal{X}}$ of $\mathcal{X}$ is all of $\mathcal{X}^{**}$, one says that $\mathcal{X}$ is a *reflexive* Banach space, and writes $\mathcal{X} = \mathcal{X}^{**}$ where equality is understood as isometric isomorphism (or topological equivalence if equivalent [but different] norms are used). We can characterize the reflexive Orlicz spaces using the preceding result.

Theorem 10. *Let (Ω, Σ, μ) be a measure space and (Φ, Ψ) be a pair of complementary Young functions. Then $L^\Phi(\mu)$ [$L^\Psi(\mu)$] is reflexive iff $L^\Phi(\mu) = M^\Phi$ and $L^\Psi(\mu) = M^\Psi$, or equivalently both $L^\Phi(\mu)$ and $L^\Psi(\mu)$ have absolutely continuous norms. In particular this holds if both Φ and Ψ are Δ_2-regular.*