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Around some results of G. Pisier and P. Saab on convolutions. I & II.

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1 Introduction

On the other hand, the eigenvalues result do not give a possibility to find out whether the above result on the factorization of T through an S_q-operator is the best one in the scale $S_{r,s}$. We need to proceed in another way.

A result of G. Pisier [17] gives us a possibility to get one more negative answer to the question considered in [18] on the product of two nuclear operators (see below..!!!). G. Pisier has shown that if a convolution operator

$$\star f:\ M(G)\to C(G),$$

where G is a compact Abelian group, $M(G)=C(G)^*$ and $f\in C(G)$, can be factored through a Hilbert space, then f has the absolutely summable set of Fourier coefficients. It is clear that the condition "the convolution operator ... can be factorized through a Hilbert space"

$$\star f: M(\mathbb{T})\to H\to C(\mathbb{T})$$

is the same as the condition "the operator $\star f$ can be factored through a bounded operator U in a Hilbert space":

$$\star f: M(\mathbb{T})\to H\stackrel{U}{\to} H\to C(\mathbb{T}).$$

We are going to generalize this result, so let us give some notes about it.

Let $S(H)$ be an ideal in the algebra $L(H)$ of all bounded operators in H (e.g., the ideal of compact operators). What is the condition on the set $\{\hat{f}(n)\}$ that gives a possibility to factorize the operator $\star f$ through an operator from $S(H)$?

$$\star f: M(\mathbb{T})\to H\stackrel{U\in S}{\longrightarrow} H\to C(\mathbb{T}).$$

We present some generalizations of the result of G. Pisier, giving answers to the question for the ideals $S_{p,q}(H)$ of operators from the Lorentz-Schatten classes (operators, whose singular numbers are in the Lorentz sequence space $l_{p,q}$) and for general compact Abelian groups G :

$$\star f: M(G)\to H\underset{S_p(H)}{\longrightarrow} H\to C(G),\ \ f\in C(G).$$

Moreover, we will consider even convolution operators in vector-valued function spaces, generalizing the result of G. Pisier and a result of P. Saab [20], Theorem 4.2, where it was shown that the Pisier's techniques in the scalar case can be extended to the vector-valued case (factorizations of a vector valued convolutions through Hilbert spaces). We will get also two theorems which are very close to to some generalizations of main theorem from [20]. generalizing also main results of P. Saab from [20].

2 Preliminaries

All the spaces $X, Y, Z, W, \dots$ are Banach, $x, x_n, y, y_k, \dots$ are elements of spaces $X, Y, \dots$ respectively. All linear mappings (operators) are continuous; as usual, $X^*, X^{**}, \dots$ are Banach duals (to X), and $x', x'', \dots$ (or $y', \dots$) are the functionals on $X, X^*, \dots$ (or on $Y, \dots$). By π_Y we denote the natural isometric injection of Y into its second dual. If $x \in X, x' \in X^*$ then $\langle x, x' \rangle = \langle x', x \rangle = x'(x)$. $L(X, Y)$ stands for the Banach space of all linear bounded operators from X to Y. We always consider the space X as the subspace $\pi_X(X)$ of its second dual X^{**} (denoting, if needed, by π_X the canonical injection).

2.1 Analysis on Groups

We refer to [19] on general topics of this subsection and to [4] for the information on vector-valued function spaces and vector measures.

Let G be a compact Abelian group, m be a Haar measure on G (i.e. the unique translation invariant normalized regular Borel measure, or, what is the same, Radon probability), Γ be the dual group of G, i.e., the group of all characters on G (multiplicative continuous complex functions γ so that $|\gamma(t)| = 1$ for all $t \in G$). Note that Γ is discrete. $C(G)$ is the Banach space of all continuous (complex-valued) functions on G with the natural uniform norm: if $\varphi \in C(G,)$ then

$$||\varphi||_\infty := \sup_{t \in G} |\varphi(t)|.$$

$L_p(G)$, $1 \leq p < \infty$, — the Banach space of all (m-equivalent classes of) absolutely p-summable Borel functions on G,

$$||\varphi||_p := (\int_G |\varphi|^p \, dm)^{1/p} < \infty \ \text{ for } \ \varphi \in L_p(G).$$

$M(G)$ is the Banach space of all (complex-valued) finite regular Borel measures on G with the variation norm $|\mu|(G)$ (or, what is the same, with the norm induced from $C^*(G)$ by the Riesz Representation Theorem).

If $f \in L_p(G)$, $1 \leq p < \infty$, and $\mu \in M(G)$, then

$$f \star \mu(g) := \int_G f(g - h) \, d\mu(h) \ \text{ for } \ g \in G,$$

$$||f \star \mu||_p \leq ||f||_p \, ||\mu||.$$

If $f \in C(G)$, then

$$f \star \mu(g) \in C(G) \text{ and } ||f \star \mu||_\infty \leq ||f||_\infty \, ||\mu||.$$

If $f \in L_1(G)$, and $\mu \in M(G)$, then the Fourier transform of f and μ are defined by

$$\hat{f}(\gamma) := \int_G \overline{\gamma(h)} f(h) \, dm(h) \text{ for } \gamma \in \Gamma;$$

(maps $L_1(G) \to C_0(\Gamma)$, $||\hat{f}||_\infty \leq ||f||_1$) and

$$\hat{\mu}(\gamma) := \int_G \overline{\gamma(h)} \, d\mu(h) \text{ for } \gamma \in \Gamma.$$

Here $C_0(\Gamma)$ is the subspace of $C(\Gamma)$, consisting of all functions which vanish at infinity.

2.2 Tensor products and summing operators

We refer to [4, 5, 6, 21] on tensor products of Banach spaces and to [16, 3] for the information on p-absolutely summing operators.

2.2.1 Tensor product and integral operators

For Banach spaces X, Y, denote by $F(X, Y)$ the linear subspace of the space $L(X, Y,)$ consisting of all finite rank operators. Algebraic tensor product $X^* \otimes Y$ will be identify with the linear space $F(X, Y)$: every tensor element $z := \sum_{n=1}^N x'_n \otimes y_n$ can be considered as an operator $\widetilde{z}(\cdot) := \sum_{n=1}^N \langle x'_n, \cdot \rangle y_n$. Also, $X \otimes Y$ can be considered as a subspace of the vector space $F(X^*, Y)$ (namely, as vector space of all linear weak*-to-weak continuous finite rank operators). We can identify also the tensor product (in a natural way) with a corresponding subspace of $F(Y^*, X)$. If $X = W^*$, then $W^* \otimes Y^{**}$ is identified with $F(X, Y^{**})$ (or with $F(Y^*, X^*)$.

The projective norm of an element $z \in X \otimes Y$ is defined as

$$||z||_\wedge := \inf\{\sum_{n=1}^N ||x_n|| \, ||y_n|| : \; z = \sum_{n=1}^N x_n \otimes y_n, \; (x_n) \subset X, (y_n) \subset Y\}.$$

The completion of the normed space $(X \otimes Y, || \cdot ||_\wedge)$ is called the projective tensor product of Banach spaces X and Y and denoted by $X \widehat{\otimes} Y$. Every element can be written in the form

$$z = \sum_{n=1}^\infty x_n \otimes y_n \text{ with } \sum_{n=1}^\infty ||x_n|| \, ||y_n|| < \infty. \tag{2.1}$$

Note, that $X\widehat{\otimes}Y = Y\widehat{\otimes}X$ in a sense. Every element z of $X\widehat{\otimes}Y$ generates an operator $\widetilde{z} : X^* \to Y$: If z has a representation 2.1, then $\widetilde{z}(x') := \sum_{n=1}^{\infty}\langle x_n, x\rangle y_n$.

A linear functional "trace" is defined on each tensor product $z \in X^* \otimes X$: If $z = \sum_{n=1}^{N} x'_n \otimes x_n$, then trace $z = \sum_{n=1}^{N}\langle x'_n, x_n\rangle$ and the last sum does not depend on a representation of z. This functional has a unique extension to the completion $X^*\widehat{\otimes}X$ and its value at an element $z \in X^*\widehat{\otimes}X$ is denoted again by trace z. If $z = \sum_{n=1}^{\infty} x_n \otimes x_n$, then trace $z = z = \sum_{n=1}^{\infty}\langle x'_n, x_n\rangle$.

A dual space of the tensor product $X\widehat{\otimes}Y$ is $L(Y, X^*)$ with duality defined by

$$\langle T, z\rangle := \text{trace } T \circ z = \sum_{n=1}^{\infty}\langle x_n, Ty_n\rangle,\ z \in X\widehat{\otimes}Y, T \in L(Y, X^*).$$

Here, $T \circ z$ is an element $\sum_{n=1}^{\infty} x_n \otimes Ty_n \in X\widehat{\otimes}X^*$. In particular, $(X^*\widehat{\otimes}Y)^* = L(Y, X^{**}) = L(X^*, Y^*)$.

If $A \in L(X, W)$, $B \in L(Y, G)$ and $x \otimes y \in X \otimes Y$, then a linear map $A \otimes B : X \otimes Y \to W \otimes G$ is defined by $A \otimes B((x \otimes y) := Ax \otimes By$ (and then extended by linearity). Since $\widetilde{A \otimes B}(z) = B\widetilde{z}A^*$ for $z \in X \otimes Y$, we can use notation $B \circ z \circ A^* \in W \otimes G$ for $A \otimes B(z)$.

There is another natural norm on the tensor product $X \otimes Y$, namely, the norm induced from $L(X^*, Y)$, that is the uniform norm. The completion of $X \otimes Y$ with respect to this norm coincides with the closure of $X \otimes Y$ in $L(X^*, Y)$, is denoted by $X\widetilde{\otimes}Y$ and is called the injective tensor product of X and Y. In particular, the injective tensor product $X^*\widetilde{\otimes}Y$ is exactly the closure of all finite rank operators in $L(X, Y)$ and contained in the Banach space $K(X, Y)$ of all compact operators from X to Y.

The dual space to $X\widetilde{\otimes}Y$ can be identify with so-called integral operators from Y to X^*. We will use the following definition of an integral operator in Banach spaces: An operator $T : Z \to W$ is said to be integral (they say also "integral in the sense of Pietsch") if there exist a compact space K, a probability measure $\mu \in C^*(K)$ and two bounded operators $A : Z \to C(K)$ and $B : L_1(K, \mu) \to W$ so that T admits the following factorization:

$$T = BjA : Z \xrightarrow{A} C(K) \overset{j}{\hookrightarrow} L_1(K, \mu) \xrightarrow{B} W$$

where j is a natural inclusion. With the norm $i(T) := \inf ||A||\,||B||$ the space $I(Z, W)$ of all integral operators is Banach. For any $z = \sum_{n=1}^{N} x_n \otimes y_n \in X \otimes Y$ and $V \in I(Y, X^*)$ the composition $V \circ z$ lies in $X \otimes X^*$ and $||V \circ z||_{\wedge} \le ||\widetilde{z}||i(V)$. Thus V generates a linear continuous map from $X\widetilde{\otimes}Y$ into $X\widehat{\otimes}X^*$, the trace of $V \circ A$ is well defined for every $A \in X\widetilde{\otimes}Y$ and

$|\,\text{trace } V \circ A| \leq ||A|| i(V)$. The linear continuous functional trace $V \circ \cdot$ defines a duality between the spaces $X \widetilde{\otimes} Y$ and $I(Y, X^*)$ and the last space is the dual to the injective tensor product $X \widetilde{\otimes} Y$ with respect to this duality.

Let us mention that the above norms in $\widehat{\otimes}$ and in $\widetilde{\otimes}$ are the greatest and least crossnorms respectively (see, e. g., [4], p. 221). Projective and injective tensor products of several Banach spaces can be defined by induction.

Two important notion in connection with the just introduced notions: They say that a Banach space X has the approximation property if for every Banach space Y the natural mapping $Y^* \widehat{\otimes} X \to L(Y, X)$ is injective; X has the metric approximation property if for every Banach space Y the natural mapping $Y^* \widehat{\otimes} X \to I(Y, X^{**})$ is an isometric embedding. Such spaces as $L_p(\mu), C(K)$, $M(G) = C^*(G)$ and all their duals have the metric approximation property [5].

2.2.2 Absolutely summing operators

A series $\sum_{n=1} \infty x_n$ in a Banach space X is unconditionally convergent if for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the natural numbers the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is convergent too. It is the same as the convergence of the series $\sum_{n=1}^{\infty} b_n x_n$ for every bonded sequence (b_n) (see [3], 1.9). An operator $T : X \to Y$ is said to be absolutely p-summing if it takes any unconditionally convergent series in X to a absolutely convergent series in Y. A famous Dvoretzky-Rogers theorem says that the identity map in X is absolutely summing iff the space X is finite dimensional (see, e. g.,[3], 1.2).

Example: An inclusion $j : C(K) \to L_1(K, \mu)$, where μ is a probability measure on a compact set K.

An operator $T : X \to Y$ is said to be 2-absolutely summing if there is a constant $C > 0$ such that for every finite sequence $(x_n)_1^N \subset X$ one has

$$\sum_{n=1}^{N} ||Tx_n||^2 \leq C^2 \sup_{||x'|| \leq 1} \left| \sum_{n=1}^{N} |\langle x_n, x' \rangle|^2 \right| .$$

The set $\Pi_2(X, Y)$ of all such operators is Banach with a norm $\pi_2(T) = \inf C$.

Examples: 1) An inclusion $j : C(K) \to L_2(K, \mu)$, where μ is a probability measure on a compact set K. 2) $\Pi_2(H, H) = S_2(H, H)$. 3) Any operator from $C(K)$ to $M(K) = C^*(K)$ is 2-absolutely summing.

Generally, let $0 < r < \infty$. An operator $T : X \to Y$ is said to be r-absolutely summing if there is a constant $C > 0$ such that for every finite sequence $(x_n)_1^N \subset X$ one has

$$\sum_{n=1}^{N} ||Tx_n||^r \leq C^r \sup_{||x'|| \leq 1} \left| \sum_{n=1}^{N} |\langle x_n, x' \rangle|^r \right| .$$

The set $\Pi_r(X,Y)$ of all such operators is (quasi)Banach with a (quasi)norm $\pi_r(T) = \inf C$.

2.3 Lorentz-Schatten classes of operators in Hilbert spaces

The Lorentz-Schatten class $S_{p,q}$, $0 < p,q < \infty$, considered for the first time by H. Triebel in [22], can be defined in the following way. Let U be a compact operator in a Hilbert space H and (s_n) is the sequence of its singular numbers (see, e. g., [15], 2.1.13). An operator U belongs to the space $S_{p,q}(H)$, if $(s_n) \in l_{p,q}$. (see, e. g., [15], 2.11.15). The space $S_{p,q}(H)$ has a natural quasi-norm

$$\sigma_{p,q}(U) = ||(\mu_n)||_{p,q} = \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\mu_n^q\right)^{1/q}.$$

If $p = q$, then $S_{p,p}$ coincides with the class S_p (with a quasi-norm σ_p). Let us mention that, for $p,q \in (0,1]$, we have the equality $N_{p,q}(H) = S_{p,q}(H)$ (see, e. g., [7]) and the inclusions $S_{p,q} \subset S_{p,q'}$, if $0 < p < \infty$ and $0 < q \le q' < \infty$ or $S_{p,q} \subset S_{p',q'}$ if $0 < p < p' < \infty, 0 < q, q' < \infty$ (see [22], Lemma 2) and

$$S_{p,q} \circ S_{p',q'} \subset S_{s,r},\ 1/p + 1/p' = 1/s,\ 1/q + 1/q' = 1/r.$$

Moreover, if $V \in S_{p,q}$ and $U \in S_{p',q'}$, then $\sigma_{s,r}(UV) \le 2^{1/s}\sigma_{p',q'}(U)\,\sigma_{p,q}(V)$ (see [13], p. 155). In the case where $p = q, p' = q'$, one has the constant 1 instead of $2^{1/s}$ in the last inequality [8], [15], p. 128, [1], p.262.

Examples of the $S_{p,q}$-operators are the diagonal operators D in l_2 with the diagonals (d_n) from $l_{p,q}$; in such cases we write $D = (d_n)$.

Given two complex Hilbert spaces H_1 and H_2, we denote by $H_1 \otimes_2 H_2$ the completion of the tensor product $H_1 \otimes H_2$ with respect to the natural scalar product.

3 On a Pisier's result

In this section we are going to prove some generalizations of the Pisier's theorem mentioned in Introduction to the cases of S_p-factorizations of operators for scalar cases. Some applications are given

3.1 Definition. *An operator $T \in L(X,Y)$ is said to be r-nuclear, where $0 < r \le 1$, if it admits a representation*

$$Tx = \sum_{n=1}^{\infty} \mu_n \langle x'_n, x\rangle y_n, \quad \text{for} \ \ x \in X, \tag{3.2}$$

where $(x_n) \in X, (y_n) \in Y$ $||x_n|| \leq 1, ||y_n|| \leq 1$ *and* $(\mu_n) \in l_r$. *We put* $\nu_r(T) :=$ $\inf ||(\mu_n)||_{l_r}$, *where the infimum is taken over all possible factorizations of* T *in the form (3.2).*

It is clear that we can assume that μ_n are real and non-negative. With the quasi-norm ν_r, the space $N_r(X,Y)$ of all r-nuclear operator from X to Y is a complete quasi-normed space. We need the following well known fact. The proof is given for completeness.

3.3 Proposition. *If* $T \in N_r(X,Y)$ $(0 < r \leq 1)$, *then* T *can be factored through an operator from* $S_v(H)$, *where* $1/v = 1/r - 1$. *Moreover,* $\gamma_{S_v}(T) \leq \nu_r(T)$

Proof. $T : X \to Y$ admits the following factorization:

$$T:\, X \overset{W}{\to} l_\infty \overset{\Delta}{\to} l_1 \overset{V}{\to} Y, \tag{3.4}$$

where $||V|| = ||W|| = 1$ and Δ is a diagonal operator with a diagonal $(d_n) \in l_r$. Indeed, it is enough to put $Wx := (\langle x'_k, x\rangle)$, $V(\alpha_n) := \sum \alpha_n y_n$ and $\Delta(\beta_n) := (d_n \beta_n)$ (where $d_n := \mu_n$). rewrite the factorization (3.4) as follows:

$$T:\, X \overset{W}{\to} l_\infty \overset{\Delta_1}{\to} l_2 \overset{\Delta_0}{\to} l_2 \overset{\Delta_2}{\to} l_1 \overset{V}{\to} Y, \tag{3.5}$$

where $\Delta_1 := (\sqrt{d_n^r})$, $\Delta_2 := (\sqrt{d_n^r})$ and $\Delta_0 := (d_n^{1-r})$.

Suppose that $\varepsilon > 0$ and in the factorization (3.4) $||V|| = ||W|| = 1$ and $||(d_n)||_{l_r} \leq (1+\varepsilon)\nu_r(T)$. Then

$$\begin{aligned} ||\Delta_2|| &= ||\Delta_1|| \leq \pi_2(\Delta_1) \leq ||(\sqrt{d_n^r})||_{l_2} \\ &= ||(d_n^r)||_{l_1}^{1/2} \leq [(1+\varepsilon)\nu_r(T)]^{r/2}. \end{aligned} \tag{3.6}$$

Also $\Delta_0 \in S_v(l_2)$, where $1/v = 1/r - 1$. Moreover, since $1 - r = r/v$, we have

$$\begin{aligned} \sigma_v(\Delta_0) &= \left(\sum d_n^{1-r}]^v\right)^{1/v} \\ &= \left(\sum [d_n]^r\right)^{1/v} \leq [(1+\varepsilon)\nu_r(T)]^{1-r}. \end{aligned} \tag{3.7}$$

□

3.8 Remark. *As a matter of fact we see that* $T = A\Delta_0 B$, *where* $A \in \Pi_2^{dual}(l_2, Y)$, $B \in \Pi_2(X, l_2)$ *and* $\Delta_0 \in S_v(l_2, l_2)$.

3.9 Example. *Let* $T \in N_r(X,Y)$ *and* $U \in N_1(Y,X)$, *where* $0 < r \leq 1$. *We have the diagram*

$$UT:\, X \overset{B_2}{\to} l_2 \overset{\Delta_0}{\to} l_2 \overset{A_2}{\to} Y \overset{B_1}{\to} l_2 \overset{A_1}{\to} X,$$

where $B_2, B_1 \in \Pi_2$ *and* $\Delta_0 \in S_v(l_2)$, $1/v = 1/r - 1$. *Eigenvalues of* UT *are the same as ones of the operator* $V := \Delta_0 A_2 B_1 A_1 B_2$:

$$l_2 \xrightarrow{\Delta_0} l_2 \xrightarrow{A_2} Y \xrightarrow{B_1} l_2 \xrightarrow{A_1} X \xrightarrow{B_2} l_2.$$

Since $B_2, B_1 \in \Pi_2$, *we have* $A_2 B_1 A_1 B_2 \in S_1(l_2)$. *Therefore,* $V \in S_r$ $(1+1/v = 1/r)$. *Thus, the sequence of all eigenvalues of* UT *lies in* l_r.

3.10 Remark. *It can be shown that if* $U \in N_p(Y, X)$ *in Example 3.9, then eigenvalues of* UT *belong to the Lorentz space* $l_{s,q}$ *with* $1/s = 1/r + 1/p - 1$ *and* $1/q = 1/r + 1/p$. *Example 3.9 shows the specificity of the particular case* $p = 1$.

We are going to show that the results from Proposition 3.3 and Example 3.9 are sharp. For this we need the following first generalization of the Pisier result, mentioned in Introduction.

3.11 Theorem. *Let* $f \in C(G)$, $0 < s \leq 1$ *and* $1/r = 1/s - 1$. *Consider a convolution operator* $\star f : M(G) \to C(G)$. *The set of Fourier coefficients* $\hat{f}$ *belongs to* l_s *if and only if the operator* $\star f$ *can be factored through a Schatten* S_r*-operator in a Hilbert space.*

Proof. 1) Let there exists $U \in S_r(H)$ such that

$$\star f = AUB : \ M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G).$$

If $j : C(G) \hookrightarrow M(G)$ is a natural injection, then the Fourier coefficients of f are the eigenvalues of the operator $AUBj : C(G) \to M(G) \to C(G)$. Consider a diagram

$$C(G) \overset{j}{\hookrightarrow} M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G) \overset{j}{\hookrightarrow} M(G) \xrightarrow{B} H.$$

The operators $AUBj$ and $BjAU$ have the same sequences of eigenvalues. Since $B \in \Pi_2(M(G), H)$ $j : C(G) \hookrightarrow L_2(G) \hookrightarrow M(G) \in \Pi_2(C(G), M(G))$ and $U \in S_r$, we get that

$$(*) \ \ BjAU \in S_r \circ S_1 \subset S_s,$$

where $1/s = 1+1/r$. Therefore, the eigenvalues of $AUBj$ lie in l_s. So $\{\hat{f}(\gamma)\} \in l_s$.

2) Suppose that $\{\hat{f}(\gamma)\} \in l_s$, where $1/s = 1 + 1/r$. Let $\{c_{\gamma_n} = |\hat{f}(\gamma_n)|\}$ Consider the operators $B : M(G) \to L_2(G)$, $U : L_2(G) \to L_2(G)$ and $A : L_2(G) \to C(G)$, defined by

$$B\mu := \sum_n \hat{\mu}(\gamma_n) c_{\gamma_n}^{s/2} \gamma_n, \ \ U\varphi := \sum_n \hat{\varphi}(\gamma_n) c_{\gamma_n}^{1-s} \gamma_n$$

and

$$A\psi := \sum_n \hat{\psi}(\gamma_n)\ \mathrm{sign}\hat{f}(\gamma_n) c_{\gamma_n}^{s/2}\gamma_n$$

The operators are well defined since the series

$$\sum \hat{\psi}(\gamma_n)\ \mathrm{sign}\hat{f}(\gamma_n) c_{\gamma_n}^{s/2}\gamma_n$$

is absolutely convergent.

We have

$$\star f = AUB : \ M(G) \xrightarrow{B} l_2 \xrightarrow{U} l_2 \xrightarrow{A} C(G),$$

where A, B are bounded and U is from $S_r(l_2)$. □

3.12 Corollary. *Let $f \in C(G)$, $0 < s \leq 1$ and $1/r = 1/s - 1$. For the convolution operator $\star f : M(G) \to C(G)$, The following are equivalent:*

1. *$\star f$ can be factored through a Lorentz-Schatten S_r-operator;*
2. *$\hat{f} \in l_s$;*
3. *$\star f \in N_s(M(G), C(G))$.*

Proof. 3) $\Longrightarrow$ 1) is valid for any s-nuclear operator (Proposition 3.3).

2) $\Longrightarrow$ 3). It is enough to consider the diagram

$$\star f = AUB : \ M(G) \xrightarrow{B} l_\infty \xrightarrow{\Delta} l_1 \xrightarrow{A} C(G),$$

where $B\mu := \{\hat{\mu}(\gamma)\}$, $\Delta\{a_\gamma\} := \{|\hat{f}(\gamma)|a_\gamma\}$, $A\{b_\gamma\} := \sum_\gamma\ \mathrm{sign}\hat{f}(\gamma) b_\gamma \gamma$. □

It follows from the above corollary that the result of Proposition 3.3 is sharp.

We now give an application to the products of two nuclear operators . A. Grothendieck [5] proved that the eigenvalue sequence of a product of two nuclear operators is absolutely summable. The following corollary shows that the result is sharp in the scale $l_{r,s}$.

3.13 Corollary. *There exist two nuclear operators t_1 and t_2 whose product has the eigenvalues in $l_1 \setminus \cup_{s<1} l_{1,s}$.*

Proof. Let $f \in C(G)$ with $\widehat{f} \in l_1 \setminus \cup_{s<1} l_{1,s}$, $T_1 = \star f : M(G) \to C(G)$ and $T_2 : C(G) \to M(G)$ be a natural inclusion. By Corollary 3.12, the operator T_1 is nuclear. By using a natural factorization of T_1 through a diagonal operator from l_∞ to l_1, we can represent T_1 as a product kt_1 of a nuclear operator $t_1 : M(G) \to l_1$ and a compact operator $k : l_1 \to C(G)$. Since T_2 is an integral operator, the product operator $T_2 k$ is nuclear. Put $t_2 := T_2 k$. Then, $t_1 t_1 = T_2 T_1$. The sequence of the eigenvalues of $t_2 t_1$ coincides with $\widehat{f}$. □

3.14 Remark. *Of course, the assertion of the corollary follows implicitly from the Pisier's theorem. Also, note that Corollary 3.12 gives us one more proof of the sharpness of the Grothendiek's theorem about eigenvalues of nuclear operators as well as his factorization result (take $p = q = 1$ and f with $\widehat{f} \in l_1 \setminus \cup_{s<1} l_{1,s}$).*

A slightly more general corollary (compare Example 3.9):

3.15 Corollary. *For every $r \in (0, 1)$ there exist two nuclear operators $t_1 \in N_r$ and $t_2 \in N_1$ whose product has the eigenvalues in $l_r \setminus \cup_{s<r} l_{r,s}$.*

Proof. Take $f \in C(G)$ with $\widehat{f} \in l_r \setminus \cup_{s<r} l_{r,s}$ and proceed as in the previous proof. □

3.16 Remark. *The same can be done for the products of type*

$$N_1 N_p N_1 N_q \cdots N_1 N_r.$$

4 Around Saab's theorem

We will get here two theorems which are in a sense generalize the main Saab's theorem from [20].

Following [20], for $\overline{f} \in C(G, X)$, define a convolution operator $\star\overline{f} : M(G) \to C(G, X)$ as

$$\star f(\mu) = \overline{f} * \mu(s) := \int_G \overline{f}(s - t)\, d\mu(t) \in X.$$

4.1 Definition. *$T \in L(X, Y)$ possesses the property $(\mathfrak{l})$ if for every $\overline{f} \in C(G, X)$ such that $\star\overline{f}$ can be factored through a Hilbert space*

$$\star\overline{f} : M(G) \xrightarrow{A} H \xrightarrow{B} C(G, X)$$

the family $T\widehat{\overline{f}}$ is absolutely summable.

4.2 Theorem. *For functions $\overline{f} \in C(G, X)$, consider the convolution operators $\star\overline{f} : M(G) \to C(G, X)$ and let $T \in L(X, Y)$. Consider the following assertions:*

1) $T \in L(X, Y)$ possesses the property $(\mathfrak{l})$.

2) $T \in \Pi_2(X, Y)$.

3) If $d \in l_1^{weak}(X) \cap l_2(X)$, then $Td \in l_1(Y)$.

We have: 1) $\implies$ 2) *and* 3) $\implies$ 1).

Proof. 1) $\implies$ 2). Fix any infinite sequence (γ_n) of distinct elements of Γ. Take a weakly 2-summable family $(x_n)_1^\infty \in X$. It is enough (by a theorem of E. Landau) to show that for every sequence $a = (a_k) \in l_2^\infty$

$$\sum_{k=1}^{\infty} ||Tx_k||\,|a_k| < \infty.$$

Fix $a = (a_k) \in l_2^\infty$ and consider the series

$$\sum_{k=1}^{\infty} a_k x_k \gamma_k. \tag{4.3}$$

Since for $n, m \in \mathbb{N}$

$$||\sum_{k=n}^{n+m} a_k x_k \gamma_k||_{C(G,X)} = \sup_{s\in G}\sup_{||x'||\le 1} |\langle \sum_{k=n}^{n+m} a_k x_k \gamma_k(s), x'\rangle| \le$$

$$\sqrt{\sum_{k=n}^{n+m} |a_k|^2}\sqrt{\sup_{||x'||\le 1}\sum_{k=1}^{\infty} |\langle x_k, x'\rangle|^2}$$

and the space $C(G,X)$ is complete, the series (4.3) converges in $C(G,X)$.

Now fix $s \in G$ and take an $x' \in X^*$ with $||x'|| \le 1$. For the operator $u : l_2 \to C(G,X)$, defined by $ub := \sum_{k=1}^\infty b_k x_k \gamma_k$ for $b := (b_k) \in l_2$ we have

$$|\sum_{k=1}^{\infty} b_k \langle x_k, x'\rangle \gamma_k(s)|^2 \le ||b||^2 \sum_{k=1}^{\infty} |\langle x_k, x'\rangle|^2.$$

Hence,

$$||\sum_{k=1}^{\infty} b_k x_k \gamma_k(s)||_X^2 \le \sup_{||x'||\le 1} ||b||^2 \sum_{k=1}^{\infty} |\langle x_k, x'\rangle|^2$$

and

$$||ub||^2 = \sup_{s\in G} ||\sum_{k=1}^{\infty} b_k x_k \gamma_k(s)||_X^2 \le ||b||^2 \sup_{||x'||\le 1}\sum_{k=1}^{\infty} |\langle x_k, x'\rangle|^2.$$

Therefore,

$$\begin{aligned} ||u||^2 &\le \sup_{||x'||\le 1}\sum_{k=1}^{\infty} |\langle x_k, x'\rangle|^2 \\ &\le \sup_{\substack{F\in C(G,X)^* \\ ||F||\le 1}} \sum_{k=1}^{\infty} |\langle x_k\gamma_k, F\rangle|^2 = ||(x_k\gamma_k)||_2^{weak} = ||u||^2. \end{aligned} \tag{4.4}$$

Thus, we have:

$$||(x_k)||_2^{weak} = ||u|| \tag{4.5}$$

Now, for our fixed sequence $a = (a_k) \in l_2$ define an operator $A : M(G) \to l_2$ by

$$A\mu := (a_k \widehat{\mu}(\gamma_k)),\ \mu \in M(G).$$

For $s \in G$, put $\overline{f}(s) := \sum_{n=1}^{\infty} a_n x_n \gamma_n(s)$. Then $\overline{f} \in C(G, X)$ and

$$uA\mu = \sum_{k=1}^{\infty} a_k \widehat{\mu}(\gamma_k) x_k \gamma_k = \star\overline{f}(\mu).$$

By assumption, the family $(T\widehat{\overline{f}})$ is absolutely summable. It means that $(a_n T x_n) \in l_1(Y)$ and this is true for any sequence $a \in l_2$. Therefore, $(Tx_n) \in l_2(Y)$.

3) $\implies$ 1). We may (and do) assume that X is separable (since the subspace $\overline{f}(G) \subset X$ is separable).

Let $\star\overline{f} = BA$ be a continuous factorization of $\star\overline{f}$ through a Hilbert space:

$$\star\overline{f} : M(G) \xrightarrow{A} H \xrightarrow{B} C(G, X) = C(G)\widetilde{\otimes}X.$$

Note that, for $s \in G$, $\gamma \in \Gamma$,

$$\star\overline{f}(\gamma dt)(s) = \int \overline{f}(t)\gamma(s-t)\, dt = \int \overline{f}(t)\gamma(s)\overline{\gamma(t)}\, dt = \gamma(s)\widehat{\overline{f}}(\gamma).$$

Also, $T \circ \star\overline{f}$ maps γ to $\gamma T(\widehat{\overline{f}}(\gamma))$ (we identify γdt with γ). Let

$$x_\gamma := \widehat{\overline{f}}(\gamma),\ y_\gamma := Tx_\gamma \quad (\text{so} \ \ \gamma y_\gamma := \gamma \otimes y_\gamma \in C(G)\widetilde{\otimes}Y).$$

If $x' \in X^*$ and $i : C(G) \to C(G)$ is the identity map, then $\langle x', \overline{f}(\cdot)\rangle \in C(G)$ and we have a diagram:

$$x' \circ BA = x' \circ \star\overline{f} : M(G) \xrightarrow{A} H \xrightarrow{B} C(G, X) = C(G)\widetilde{\otimes}X \xrightarrow{i\otimes x'} C(G)$$

with $x' \circ BA(\gamma) = \star\overline{f}(\gamma) = \langle x', x_\gamma\rangle\gamma$. By the Pisier's result, for every $x' \in X$ the series $\sum_\gamma |\langle x', x_\gamma\rangle|$ is convergent. Hence, only countably many of x_γ's are not zero (recall that X is assumed to be separable). Since $A \in \Pi_1(M(G), H)$, the mapping $\star\widehat{\overline{f}} = BA$ is absolutely 2-summing. So, the family (x_γ) is stronly 2-summable (and countable). Thus, $(x_\gamma) \in l_1^{weak}(X) \cap l_2(X)$. By 3), $(y_\gamma) = T(x_\gamma) \in l_1(Y)$. □

It follows immediately:

4.6 Corollary. *If $T \in \Pi_1(X,Y)$, then $T \in L(X,Y)$ possesses the property $(\mathfrak{l})$.*

Consider a generalization of Definirion 4.1.

4.7 Definition. *Let $0 < s \leq 1$ and $1/r = 1/s - 1$. $T \in L(X,Y)$ possesses the property $(\mathfrak{l}_r)$ if for every $\overline{f} \in C(G,X)$ such that $\star\overline{f}$ can be factored through an S_r-operator in a Hilbert space*

$$\star\overline{f} : M(G) \stackrel{A}{\to} H \stackrel{U}{\to} H \stackrel{B}{\to} C(G,X)$$

the family $T\widehat{\overline{f}}$ is absolutely s-summable.

The proof of the following result is the same as the proof of the implication 3) $\implies$ 1) of the previous theorem (but instead of the Pisier's result one must use Theorem 3.11 for the general case). Cf. Corollary 4.6.

4.8 Theorem. *For functions $\overline{f} \in C(G,X)$, consider the convolution operators $\star\overline{f} : M(G) \to C(G,X)$ and let $T \in L(X,Y)$. Consider the following assertions:*

1) $T \in L(X,Y)$ possesses the property $(\mathfrak{l}_r)$.

2) $T \in \Pi_r(X,Y)$.

We have: 2) $\implies$ 1).

Now, a concrete corollary from the theorems.

4.9 Corollary. *For $1 \leq p \leq 2$ it follows from $T \in \Pi_2(L_p(\nu),Y)$ that T possesses the properties $(\mathfrak{l}_r)$ for all $r \in (0,1]$. On the other hand, $T \in (\mathfrak{l})$ implies $T \in \Pi_1(L_p(\nu),Y)$.*

Indeed, as is well known, for any $s \in (0,2]$ we have $\Pi_s(L_p(\nu),Y) = \Pi_2(L_p(\nu),Y)$ (see [16]).

5 On a Pisier's result: Vector-valued case

In this section we are going to give some generalizations of the Pisier's theorem mentioned in Introduction to the cases of $S_{p,q}$-factorizations of operators for vector-valued cases. We will generalize also a result of P. Saab [20], Theorem 4.2, where it was shown that the Pisier's techniques in the scalar case can be extended to the vector-valued case (factorizations of a vector valued convolutions through Hilbert spaces). In the end of the section, we consider the factorizations through $S_{p,q}$-operators linear mappings between tensor products of several Banach spaces.

Let $f \in C(G)$ and $T \in L(X,Y)$. All operators under considerations are supposed to be not identically zero.

Denote by $M(G,X)$ the Banach space of all regular Borel X-valued measures of bounded variation, C(G,X) the Banach space of all continuous X-valued functions defined on G equipped with the supremum norm.

Note that

$$M(G) \otimes X \subset M(G)\widehat{\otimes}X \subset M(G,X),$$

where $\widehat{\otimes}$ is the projective tensor product.

Define a map $T_f := T \circ \star f : M(G,X) \to C(G,Y)$ by

$$T_f(\overline{\mu})(s) = \int_G f(s-t)\, dT\overline{\mu}(t),\ \overline{\mu} \in M(G,X).$$

5.1 Theorem. *Let $f \in C(G)$, $0 < r, s < \infty$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operator $T : X \to Y$. If the operator*

$$T_f : M(G,X) \to C(G,Y)$$

can be factored through an $S_{r,s}$-operator then the operators $\star f$ and T possess the same property. The same is true for the case where $r = s = \infty$ (or $q_1 = q_2 = 1$).

Proof. We may (and do) assume that $T \neq 0$ and $f \neq 0$.

We use partially (in the first part of the proof) an idea from [20]. Let $T_f = BUA$, where $A \in L(M(G,X),H)$, $U \in S_{r,s}(H)$ and $B \in L(H, C(G,Y)$. Fix a point $s_0 \in G$ for which $f(s_0) \neq 0$. Define the operators $i : X \to M(G,X)$ and $j : C(G,Y) \to Y$ by $ix = \delta_e \otimes x$ (e is a neutral element of G) and $jh = h(s_0)/f(s_0)$. For $s \in G$

$$\begin{aligned}(T_f ix)(s) &= \int f(s-t)\, dT(\delta_e \otimes x)(t) \\ &= \int_G f(s-t) Tx\, d\delta_e(t) = f(s)Tx \in C(G,Y).\end{aligned} \tag{5.2}$$

Hence, $jT_f ix = f(s_0)Tx/f(s_0) = Tx$ or $T = jBUAi$.

Now, let $k : M(G) \to M(G,X)$ be defined by $k\mu = \mu \otimes x_0$, where x_0 is such that $||Tx_0|| = 1$. Then $BUAk\mu(M(G)) =: C_1 \subset C(G,Y)$ and $UAk(M(G)) = H_1 \subset H$. Denote by P an orthogonal projector from H ontoH_1 and by R the composition Bl, where $l : H_1 \to H$ is the identity injection. We have a diagram:

$$M(G) \overset{k}{\to} M(G,X) \overset{A}{\to} H \overset{U}{\to} H \overset{P}{\to} H_1 \overset{R}{\to} C_1 \subset C(G,Y).$$

If $\mu \in M(G)$, then

$$RPUAk\mu = T_f k\mu = T_f(\mu \otimes x_0) = Tx_0 \int_G f(\cdot - t)\, d\mu(t) = Tx_0\, f * \mu.$$

Therefore, $C_1 = \{Tx_0\, f * \mu : \mu \in M(G)\} \subset C(G) \otimes \operatorname{span}\{Tx_0\} \subset C(G,Y)$. Take a functional $y' \in Y^*$ with $\langle y', Tx_o\rangle = ||y'|| = 1$ and define an operator $V : C(G) \otimes \operatorname{span}\{Tx_0\} \to C(G)$ putting $V(h \otimes y) = h\langle y', y\rangle$ for $y \in \operatorname{span}\{Tx_0\}$. Then, for $\mu \in M(G)$,

$$VRPUAk\mu = V(f * \mu \otimes Tx_0) = f * \mu.$$

Thus, the convolution operator $\star f$ is factorized through an operator from $S_{r,s}$. $\square$

5.3 Remark. *It is clear from the proof that the condition "the operator $T_f : M(G,X) \to C(G,Y)$" can be changed by the condition "the restricted operator $T_f : M(G)\widehat{\otimes}X) \to C(G,Y)$."*

5.4 Corollary. *Let $f \in C(G)$, $0 < p \le \infty$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operator $T : X \to Y$. If the operator*

$$T_f : M(G,X) \to C(G,Y)$$

can be factored through an S_p-operator then the operators $\star f$ and T possess the same property.

In a partial case where $X = Y$, $T = \mathrm{id}_X$ and $p := p_1 = p_2 = \infty$ we get a result of E. Saab [20]:

5.5 Corollary. *Let $f \in C(G)$, Consider a Banach space X and a convolution operator $\star f : M(G) \to C(G)$. If the operator*

$$T_f : M(G,X) \to C(G,X)$$

can be factored through a Hilbert space then $\hat{f} \in l_1$ and $X \cong H$.

We proof now a general theorem:

5.6 Theorem. *Let $0 < s \le r < \infty$, $T_i \in L(X_i, Y_i)$ for $i = 1, 2, \ldots, m$. If the operators T_i can be factored through the $S_{r,s}$-operators then the tensor product*

$$T := T_1 \otimes T_2 \otimes \cdots \otimes T_m : X_1 \widehat{\otimes} X_2 \widehat{\otimes} \ldots \widehat{\otimes} X_m \to Y_1 \widetilde{\otimes} Y_2 \widetilde{\otimes} \ldots \widetilde{\otimes} Y_m$$

possesses the same property.

Proof. It is enough to consider the case of the product of two operators. Let

$$T_i : X_i \xrightarrow{A_i} H_i \xrightarrow{U_i} H_i \xrightarrow{B_i} Y_i$$

be the factorizations of operators $T_i, i = 1, 2$, Here $A_i \in L(X_i, H_i), B_i \in L(H_i, Y_i)$ and $U_i \in S_{s,r}(H_i)$ for $i = 1.2$. Let $U_i := \sum_{n=1}^{\infty} s_n^i e_n^i \otimes f_n^i$, where $(e_n^i), (f_n^i)$ are orthonormal systems in corresponding Hilbert spaces and $(s_n^i) \in l_{s,r}$ are the sequences of singular numbers of the operators U_i $(i = 1, 2.)$ Tensor product $U_1 \otimes U_2$ has the following representation:

$$\begin{aligned} U_1 \otimes U_2(h_1 \otimes h_2) &= \sum_{n=1}^{\infty} s_n^1 (h_1, e_n^1) f_n^1 \otimes \sum_{k=1}^{\infty} s_k^2 (h_2, e_k^2) f_k^2 \\ &= \sum_{k,n} s_n^1 s_k^2 (h_1 \otimes h_2, e_n^1 \otimes e_k^2) f_n^1 \otimes f_k^2. \end{aligned} \tag{5.7}$$

The last series is convergent in $H_1 \otimes_2 H_2$ since it follows from the conditions on r, s that, e. g., $U_1 \otimes U_2 \in S_{2r}$ (we have $\sum_{k,n} (s_n^1)^{2r} (s_k^2)^{2r} < \infty$). Therefore, the sequence $(s_n^1 s_k^2)_{k,n}$ is the sequence of all singular numbers of $U_1 \otimes U_2$. Thus, $U_1 \otimes U_2 = \sum_{k,n} s_n^1 s_k^2 (e_n^1 \otimes e_k^2) \otimes (f_n^1 \otimes f_k^2)$. Since the sequences (s_n^1) and (s_k^2) belong to $l_{s,r}$ and $r \leq s$, their product $(s_n^1 s_k^2)$ is also in $l_{s,r}$ by the O'Neil's theorem (see [?], Theorem 7.7).

Now, consider the mappings $A_1 \otimes A_2$ and $B_1 \otimes B_2$. The first one acts from the projective tensor product $X_1 \widehat{\otimes} X_2$ to the projective tensor product $H_1 \widehat{\otimes} H_2$. The second one maps $H_1 \widetilde{\otimes} H_2$ to $Y_1 \widetilde{\otimes} Y_2$. Denoting by φ and ψ the canonical injections $H_1 \widehat{\otimes} H_2 \to H_1 \otimes_2 H_2$ and $H_1 \otimes_2 H_2 \to H_1 \widetilde{\otimes} H_2$ respectively, we obtain a factorization of $T_1 \otimes T_2$ through an $S_{r,s} - operator$: $T = B_1 \otimes B_2 \psi U_1 \otimes U_2 \varphi A_1 \otimes A_2$:

$$T : X_1 \widehat{\otimes} X_2 \xrightarrow{A_1 \otimes A_2} H_1 \widehat{\otimes} H_2 \xrightarrow{\varphi} H_1 \otimes_2 H_2 \xrightarrow{U_1 \otimes U_2} H_1 \otimes_2 H_2 \xrightarrow{B_1 \otimes B_2} Y_1 \widetilde{\otimes} Y_2.$$

□

5.8 Remark. *The converse of the theorem is also true (cf. the proof of Theorem 5.1).*

For the case where one of the space is $M(G)$ we can get a more general result. Recall that if the space X has the RN property, then $M(G, X) = M(G) \widehat{\otimes} X$. This is rather simple: Let $\overline{\mu} \in M(G, X.)$ If $X \in RN$, then there is a function $\overline{f} \in L^1(G, |\overline{\mu}|; X)$ such that $\overline{\mu}(E) = \int_E \overline{f} \, d|\overline{\mu}|$ for every Borel set E. Identifying the space $L^1(G, |\overline{\mu}|; X)$ with a subspace of $M(G, X)$ in a natural way, we see that $\overline{\mu} \in L^1(G, |\overline{\mu}|; X) = L^1(G, |\overline{\mu}|) \widehat{\otimes} X$ (see [4]).

Thus, it follows from the theorem above that if $0 < s \leq r < \infty$ and $X \in RN$, then the possibility of factorization through $S_{s,r}$-operators of the operators $\star f : M(G) \to C(G)$ and $T : X \to Y$ implies the possibility of such a factorization for the operator $T_f : M(G,X) \to C(G,Y)$. However, we can prove such a theorem without any assumption on the Banach space X.

Below we will use the following simple fact: If an operator $S : Z \to W$ in Banach spaces can be factored as

$$S : Z \xrightarrow{L} H \xrightarrow{V} H \xrightarrow{M} W$$

and $W_0 := \overline{S(Z)} \subset W$, then there is an operator $M_0 : H \to W_0$ such that S has the factorization

$$S : Z \xrightarrow{L} H \xrightarrow{V} H \xrightarrow{M_0} W_0 \overset{j}{\hookrightarrow} W,$$

where j is an inclusion. Indeed, consider the subspace $H_0 := \overline{VL(Z)} \subset H$, take an orthonormal projector $P : H \to H_0$. Put $M_0 := M|_{H_0} PVL$.

5.9 Theorem. *Let $f \in C(G)$, $0 < s \le r < \infty$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operator $T : X \to Y$. If the operators $\star f$ and T can be factored through the $S_{r,s}$-operators then the operators*

$$T_f : M(G,X) \to C(G,Y)$$

possesses the same property.

Proof. Denote the restriction of the operator T_f onto $M(G)\widehat{\otimes}X$ by $\widetilde{T}_f$ We have:

$$M(G,X) = I(C(G),X) \;\text{ and }\; (X^*\widetilde{\otimes}C(G))^* = I(C(G),X^{**}) \supset M(G,X)$$

(for the first equality, see **??** (Diestel-Uhl, Vector Measures, p. 162, Th. 3)).

By Theorem 5.6, the restricted operator $\widetilde{T}_f : M(G)\widehat{\otimes}X \to C(G,Y)$ can be factored through a $S_{r,s}$-operator. Then the dual operator

$$\widetilde{T}_f^* : I(Y,M(G)) = (C(G)\widetilde{\otimes}Y)^* \to L(X,C(G)^{**})$$

can be factored through a $S_{r,s}$-operator (we use the equality $C(G,Y) = C(G)\widetilde{\otimes}Y$). But

$$Y^*\widehat{\otimes}M(G) \subset I(Y,M(G))$$

since the space $C(G)$ and all of its duals have the metric approximation property. Therefore, $\widetilde{T}_f^*$ maps $Y^*\widehat{\otimes}M(G)$ into $X^*\widetilde{\otimes}C(G)$ (apply definition of T_f) and its restriction to the first tensor product can be factored through a $S_{r,s}$-operator.

Let τ be a restriction of $\widetilde{T}_f^*$ to the subspace $Y^*\widehat{\otimes} M(G)$. Consider the dual operator τ^* :

$$\tau^* : I(C(G), X^{**}) \to L(M(G), Y^{**}).$$

Since $I(C(G), Z) = \Pi_1(C(G), Z)$ for any Banach space Z and the ideal Π_1 of 1-absolutely summing operators is injective, the space $M(G, X)$ can be naturally identify with a subspace of $I(C(G), X^{**})$ and the restriction of τ^* to this subspace is nothing that T_f. It follows that T_f can be factored through a $S_{r,s}$-operator. □

5.10 Corollary. *Let $f \in C(G)$, $0 < s \leq r < \infty$. Consider a convolution operator $\star f : M(G) \to C(G)$ and an operators $T_k : X_k \to Y_k$, $k = 1, 2, \ldots n$, If the operators $\star f$ and T_k can be factored through the $S_{r,s}$-operators then the corresponding operator*

$$T_f : M(G, \widehat{\bigotimes}_{k=1}^n X_k) \to C(G, \widetilde{\bigotimes}_{k=1}^n Y_k)$$

possesses the same property.

Proof. Apply Theorem 5.6 to $X = \widehat{\otimes}_{k=1}^n X_k$ and to the tensor product of the operators T_k. Then apply Theorem 5.9. □

5.11 Corollary. *Let $f_k \in C(G)$, $k = 1, 2, \ldots n$, $0 < s \leq r < \infty$. Consider the convolution operators $\star f_k : M(G) \to C(G)$ and an operators $T_k : X_k \to Y_k$, $k = 1, 2, \ldots n$, If the operators $\star f_k$ and T_k can be factored through the $S_{r,s}$-operators then the corresponding operator*

$$T_f : \widehat{\bigotimes}_{k=1}^n M(G, X_k) \to \widetilde{\bigotimes}_{k=1}^n C(G, Y_k)$$

possesses the same property.

Proof. By Theorem 5.9, for every k the operator $T_{f_k} : M(G, X_k) \to C(G, X_k)$ can be factored through an $S_{r,s}$-operator. By Theorem 5.6, the operator T_f possesses the same property. □

6 Appendix

6.1 On Lorentz sequence spaces

6.1 Theorem. *Let $p, q \in (0, \infty)$ and $q > p$. There exist $x, y \in l_{p,q}$ such that $x \otimes y \notin l_{p,q}$.*

This will be prooved below,

We need some more information on Lorentz spaces (see [15] for details).

The n-th *approximation number* of $x \in l_\infty(I)$:

$$a_n(x) := \inf\{||x - u||_{l+\infty(I)} : u \in l_\infty(I), \text{card}(u) < n\}.$$

$$a_n(x) := \inf\{c \geq 0 : \text{card}(i \in I : |x_i| \geq c) < n\}.$$

When $x = (x_n)_{n=1}^\infty$ and $|x_1| \geq |x_2| \geq \cdots \geq 0$ we have $a_n(x) = |x_n|$. In general case, $(a_n(x))$ is the non-increasing rearrangement of x. An usual notation: (x_n^*).

The Lorentz space $l_{r,w}(I)$ consists of all complex-valued families $x = (x_i)$ such that

$$(n^{1/r-1/w} a_n(x)) \in l_w.$$

$l_{r,w}(I)$ is a quasi-normed space:

$$||x||_{l_{r,w}} := \Big(\sum_{n=1}^\infty n^{w/r-1} a_n(x)^w\Big)^{1/w} \quad \text{if} \quad 0 < w < \infty$$

and

$$||x||_{l_{r,\infty}} := \sup\{n^{1/r} a_n(x) : n \in \mathbb{N}\}.$$

If $I = \mathbb{N}$, then we use the notation $l_{r,w}$.

From [15](2.1.10):

6.2 Lemma.

$$x \in l_{r,w} \quad \textit{if and only if} \quad (2^{k/r} a_{2^k}(x)) \in l_w.$$

$||(2^{k/r} a_{2^k}(x))||_{l_w}$ *is an equivalent quasi-norm.*

Moreover, we have the following from [14] (getting a more general result).

A sequence (n_k) of natural numbers beginning with $n_0 = 1$ is called quasi-geometric if there are constants a and b such that

$$1 < a \leq \frac{n_{k+1}}{n_k} \leq b > \infty \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

For example,

$$n_k := (k+1)2^k.$$

6.3 Lemma. *Let (n_k) be a quasi-geometric sequence and $x \in l_\infty(I)$.*

$$x \in l_{r,w}(I) \quad \textit{if and only if} \quad (n_k^{1/r} a_{n_k}(x)) \in l_w.$$

Notation:

$$l_{r,u} \otimes l_{r,u} := \{h \in l_\infty(\mathbb{N} \times \mathbb{N}) : \exists\, x, y \in l_{r,u},\ h(n,m) = (x_n y_m)\}.$$

Also, $h = x \otimes y$.

6.4 Remark. *If*

$$l_{r,u} \otimes l_{r,v} \subset l_{r,u},$$

then there is a constant $c = c(r,u,v)$"0 so that for $x \in l_{r,u}$ and $y \in l_{r,v}$

$$||x \otimes y||_{l(r,u)} \leq c||x||_{r,u}||y||_{r,v}.$$

6.5 Proposition. *Let $r, u \in (0, \infty)$. If*

$$l_{r,u}(\mathbb{N}) \otimes l_{r,u}(\mathbb{N}) \subset l_{r,u}(\mathbb{N} \times \mathbb{N}),$$

then $u \leq r$.

Proof. Sketch: Consider a partial case where $r = 1, u = 2$ (trying get a contradiction).

Take the finite sequences $X_m := (x_{mn})$ defined by

$$x_{mn} := 2^{-i} \text{ if } 2^i \leq n < 2^{i+1} \text{ and } i \leq m$$

and $x_{mn} = 0$ otherwise.

It turns out that

$$||x_m||_{l_{1,2}} \asymp m^{1/2}.$$

Recall that the natural numbers $n_k := (k+1)2^k$ constitute a quasi-geometric sequence. Since the double sequences $x_m \otimes x_m$ contain n_k-times the coordinate 2^{-k} whenever $k \leq m$, we obtain

$$||x_m \otimes x_m||_{l_{r,u}(N \times N)} \asymp \Big(\sum_{k=0}^{\infty} [n_k a_{n_k}(x_m \otimes x_m)]^2\Big)^{1/2}$$

$$\asymp \Big(\sum_{k=0}^{\infty} (k+1)^2\Big)^{1/2} \asymp m^{3/2}.$$

On the other hand, we must have

$$||x \otimes y||_{l(1,2)} \leq c||x||_{1,2}||y||_{1,2}.$$

This yields

$$m^{3/2} \prec m^{1/2+1/2}.$$

□

6.2 Returning to convolutions

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Almost the same proof as the proof of Theorem 3.11 gives us

6.6 Theorem. *Let* $f \in C(G)$, $0 < q, s \leq 1$ *and* $1/r = 1/s - 1, 1/p = 1/q - 1$. *Consider a convolution operator* $\star f : M(G) \to C(G)$. *If the operator* $\star f$ *can be factored through a Lorentz-Schatten* $S_{r,p}$*-operator in a Hilbert space, then the set of Fourier coefficients* $\hat{f}$ *belongs to* $l_{s,q}$.

Also, for the tensor products we have (as a consequence of Theorem 5.6):

6.7 Theorem. *Let* $0 < s \leq r < \infty$, $f_1, f_2 \in C(G)$. *Consider the convolution operators* $\star f_1, \star f_2 : M(G) \to C(G)$. *If these operators can be factored through the* $S_{r,s}$*-operators then the tensor product*

$$T := \star f_1 \otimes \star f_2 : M(G) \widehat{\otimes} M(G) \to C(G) \widetilde{\otimes} C(G)$$

possesses the same property. The converse is also true.

Take now $0 < q, s < 1$ and $r, u \in (0, \infty), u > r$ with $1/r = 1/s - 1, 1/u = 1/q - 1$. Using Proposition 6.5, take two function $f_1, f_2 \in C(G)$ with $\widehat{f_1}, \widehat{f_2} \in l_{s,q}$ but $\widehat{f_1} \otimes \widehat{f_2} \notin l_{s,q}$.

Then the tensor product $T := \star f_1 \otimes \star f_2$ can not be factored through $S_{r,u}$-operator since else $\widehat{f_1} \otimes \widehat{f_2} \in l_{s,q}$ (the proof is similar to the proof of 6.6).

Finally, note that all the fact where we have the restrictions of type $s \leq r$ (cf. Theorem 6.6) are sharp (i.e., the conditions of type $s \leq r$ are essential).

References

[1] Michael Sh. Birman, M.Z. Solomjak. *Spectral Theory of Self-Adjoint Operators in Hilbert Space.* D. Reidel Pub. Co., 1987.

[2] T. Carleman. Uber die Fourierkoeffizienten einer stetigen Funktion. Aus einem Brief an Herrn A. Wiman. *Acta Math.*, 41(3) : 77–384, 1916. [Ca16]

[3] J. Diestel, H. Jarchow, A. Tonge. *Absolutely Summing Operators.* Cambridge University Press, 1995.

[4] J. Diestel & JJ. Uhl, Jr. *Vector Measures.* Amer. Math. Soc, 1977.

[5] A. Grothendieck. *Produits tensoriels topologiques et espases nucléaires. Mem. Amer. Math. Soc.*, Volume 16, 1955.

[6] Alexander Grothendieck. La theorie de Fredholm *Bulletin de la S. M. E*, 84 : 19–384, 1956).

[7] A. Hinrichs, A. Pietsch. p-nuclear operators in the sense of Grothendieck. *Math. Nachr.* 283(2), 232–261, 2010.

[8] A. Horn. On the singular values of a product of completely continuous operators. *Proc. Nat. Acad. Sci. USA*, 36 : 374–375, 1950.

[9] J. P. Kahane, Y. Katznelson, and K. de Leeuw. Sur les coefficients de Fourier des fonctions continues. *C.R. Acad. Sci. Paris*, 285 : 1001–1004, 1977.

[10] R. J. Kaiser and J.R. Retherford. Preassigning eigenvalues and zeros of nuclear operators. *Studia Mathematica*, 81(2) : 127–133, 1985.

[11] H. König. Weyl-type inequalities for operators in Banach spaces. *Proc. Conf. Funct. Anal. Paderborn 1979.* North Holland, 297–317, 1980.

[12] H. König. Hermann König. *Eigenvalue distribution of compact operators*, 1986.

[13] A. Pietsch. Weyl numbers and eigenvalues of operators in Banach spaces. *Math. Ann.*, 247 : 149–168, 1980.

[14] Albrecht Pietsch (1982). Tensor products of sequences, functions, and operators. Archiv der Mathematik, 38(1), 335Ũ344.

[15] A. Pietsch. *Eigenvalues and s-numbers.* Cambridge University Press, Cambridge, 1987.

[16] A. Pietsch. *History of Banach Spaces and Linear Operators.* Birkhauser, Boston, 2007.

[17] Gilles Pisier. *Factorization of Linear Operators and Geometry of Banach Spaces*, volume 60 of *CBMS*. Amer. Math. Soc., Providence, Rhode Island, 1985.

[18] O.I. Reinov. On products of nuclear operators. *Funct. Anal. Its Appl.*, 51 : 316–317, 2017.

[19] W. Rudin. *Fourier analysis on groups.* Interscience, New York, 1962.

[20] Paulette Saab. Convolution Operators that Factor Through a Hilbert Space, *Quaestiones Mathematicae*, 31(1) : 79-87, 2008.

[21] R. Schatten. *A theory of cross-spaces*. Volume 26 of *Annals of Math. Studies*, Princeton Univ. Press, 1950.

[22] H. Triebel. Uber die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev- Besov-Raumen. *Invent. Math.*, 4 : 275–293, 1967.