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MECHANICS

On Mathematical Modeling of a Hypersonic Flow Past a Thin Wing with Variable Shape

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Abstract—This work is devoted to the further study of a spatial flow past a thin wing of variable shape by a hypersonic flow of a nonviscous gas. The head shock wave is considered to be attached to the leading edge of the wing. The use of the thin shock-layer method for solving the system of gas-dynamics equations allows constructing a mathematical model of the flow in question. Also note that the analysis of boundary conditions makes it possible to determine the structure of the expansion of sought values in a series and to construct approximate analytical solutions. In this case, in determining the first-approximation corrections, two equations are integrated independently of other equations. The application of the Euler–Ampere transform allows constructing a solution depending on two arbitrary functions and an unknown shape of the front of the head shock wave. To determine these functions, the integrodifferential system of equations was obtained previously. This paper proposes a variant of the semi-inverse method for constructing a solution (of this system) such that the formula for one arbitrary function is given. This approach allows additional assignment of the equation for the leading edge of the wing, as well as (in the case in which the head wave is attached along the entire leading edge) the inclination of the wing surface on it. The variant of the semi-inverse method presented in this paper for the nonstationary spatial problem of flow makes it possible to obtain a particular solution, which is a model solution for various regimes of a flow past a wing. We obtain formulas to determine the shape of the front of the shock wave, the shape of the surface of the streamlined body, the distance between the shock wave and the surface of the body, and the flow parameters on the wing surface.

Keywords: mathematical modeling, hypersonic flowing of bodies, unsteady flows, partial differential equations, small parameter

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1. INTRODUCTION

When bodies flow around a hypersonic stream of a gas, a strong head shock wave is generated. The problem consists in determining flow parameters in the area between the head shock wave and the streamlined body. The shape of the head shock wave is also to be determined. The flow of a nonviscous gas in the shock layer is described by a system of nonlinear partial differential equations. To construct an approximate analytical solution, the method of a thin shock layer [1] is used, which rests on the smallness of the ratio of densities of a gas on the front of the shock wave. This small parameter is physically justified, and a good coincidence of the results of calculations obtained by this method with the results of numerical calculations and experimental data (see, e.g., [1–3]) allows us to conclude that it can be used in order to construct gas-flow models in various problems of gas dynamics with intense shock waves. Here, various environments in which the aircraft moves, as well as various forms of aircraft, can be considered. The variants of using the thin-shock-layer method to solve the problems of flowing around a thin wing in the stationary case are considered in [4–7]. We propose several particular solutions, which make it possible (for some stationary problems) to obtain formulas for determining parameters of a flow on the surface of a streamlined wing [5, 6]. The spatial nonstationary problem of a flow around a hypersonic flow of a thin wing of variable shape in the context of this method is investigated in [8]. Assuming that the head shock wave is attached to the leading edge of the wing at least at one point, in the first approximation, a solution of the simplified system of equations is constructed; this solution depends on two arbitrary functions and unknown shape of the front of the head shock wave. To determine these functions, an integrodifferential system of equations is obtained; for its solution, the semi-inverse method is proposed. In this paper, a variant of the semi-inverse method of constructing the solution of this system is proposed; in this variant, the formula for one of the arbitrary functions is given. When considering this variant of flowing around a thin wing of variable shape by a hypersonic flow of a perfect gas, we rely on the results of [8].

2. FORMULATION OF THE PROBLEM

We consider the flow past a thin wing of variable shape by a hypersonic flow of a perfect gas. The thickness, span, and chord of the wing have order $c = O(\varepsilon)$, $b = O(\sqrt{\varepsilon})$, and $L = O(1)$, respectively. Here, ε is a small parameter, which characterizes the ratio of gas densities in front of the head shock wave and directly behind it. We choose the coordinate system (x, y, z) in such a way that the surface of the wing is not so different from the plane (x, z) and velocity vector $\overline{V}$ is parallel to the plane (x, y). The origin of the coordinates is placed in the nose of the wing. The head shock wave is considered to be attached to the leading edge of the wing.

The problem of flowing around a thin wing with a hypersonic flow of a gas in the coordinate system chosen in such a way, consists in determining, first, the flow parameters (components of the velocity vector of gas particles u, v, and w; pressure p; and the value inverse to density τ) in the shock layer of the wing and, second, the shape of the front of the head shock wave. This problem is solved in dimensionless variables [8]: components of the velocity vector of gas particles u, v, and w are assigned to a certain characteristic velocity of motion V_0; density ρ is assigned to density ρ_0 before the front of the head shock wave; pressure p is assigned to the product $\rho_0 V_0^2$; the x coordinate is assigned to characteristic size L; the y coordinate is assigned to $\varepsilon L \tan\alpha$; the z coordinate is assigned to $\sqrt{\varepsilon} L \tan\alpha$; time t is assigned to the value $(V_0 \cos\alpha)/L$; and α is the incidence.

The solution of the system of nonlinear partial differential equations describing a flow of a gas in the shock layer, with boundary conditions on the front of the head shock wave and on the surface of the wing is constructed by the method of a thin shock layer [8]. The system of equations for determining corrections of the first approximation is split in such a way that two equations can be solved independently of other equations; here, one of them can be written in the divergence form. The introduction of a new function (by analogy with the flow function) reduces the solution of the problem to integration of a nonlinear partial differential equation of the second order. The use of the Euler–Ampere transform allows constructing a solution that depends on two arbitrary functions and an unknown shape of the head shock wave front; to find them, the following integrodifferential system of equations is written [8]:

$$\begin{cases} \left(\dfrac{\partial \nu}{\partial t} + \dfrac{\partial \nu}{\partial x} + \nu \dfrac{\partial \nu}{\partial z}\right) H(\nu, \mu, \lambda) = -1, \quad \lambda = x - t, \quad \mu = \nu x - z, \\ \nu = -\Phi_z, \\ \Phi(x, z, t) = F(x, z, t) - \displaystyle\int_{q_b(x,z,t)}^{\nu(x,z,t)} H(s, sx - z, \lambda) ds, \end{cases} \tag{1}$$

where $y = F(x, z, t)$ and $y = \Phi(x, z, t)$ are the equations of the wing surfaces and the shape of the front of the head shock wave, respectively; H is an arbitrary function introduced in [8]; $\nu(x, z, t)$ and $q_b(x, z, t)$ are values of parameter q on the surface of the head shock wave and the surface of the wing, respectively; and $q = w$ is the projection of the velocity of a gas particle on the z axis. Here, $q = \text{const}$ on flow lines.

In this case, q_b is determined using the equation

$$\frac{\partial q_b}{\partial t} + \frac{\partial q_b}{\partial x} + q_b \frac{\partial q_b}{\partial z} = 0. \tag{2}$$

In [8], the semi-inverse method for solving integrodifferential system of Eqs. (1) is proposed; in this method, instead of the wing-surface equation $y = F(x, z)$, we assign the formula for the function $H(\nu, \mu)$, where $\mu = \nu x - z$. By analogy with [8], in the present paper, we assign the formula for function ν. With this approach, the task of integrating system (1) is greatly simplified; in addition, sufficient arbitrariness remains and allows additional assigning the equation of the leading edge of the wing $x = x_0(z)$, as well as (for a flow regime with a head shock wave attached along the entire leading edge) inclination $F_{x_0(z)}$ of the wing surface on the leading edge.

3. CONSTRUCTING THE SOLUTION

We turn to the first equation of system (1). Introduce the new variables as follows: assume $x = \tilde{x}$, $z = \tilde{z}$, and $t = (x - \lambda)$; then, the equation takes the form

$$\left(\frac{\partial \nu}{\partial x} + \nu \frac{\partial \nu}{\partial z}\right) H(\nu, \mu, \lambda) = -1. \tag{3}$$

In (3) and thereafter, the “tilde” sign is omitted.

To construct a solution of the problem of flowing around a thin wing of variable shape by a hypersonic flow of a gas, instead of the surface of the wing, we assign the formula for function ν. Assume that

$$\nu = g_z(z, \lambda). \tag{4}$$

From (3), we obviously obtain

$$H = -\frac{1}{g_z(z,\lambda) g_{zz}(z,\lambda)}. \tag{5}$$

The formula for function ν is given, and now the second equation of system (1) is easily integrated. We have

$$\Phi = g(z, \lambda) + \varphi(x, \lambda, t). \tag{6}$$

If the leading-edge equation of the wing at $t = 0$ has the form $x = x^\circ(z)$, then, for the attached head shock wave, we obtain from (6)

$$0 = \Phi^\circ = g(z, \lambda) + \varphi(x^\circ(z), \lambda, 0).$$

It follows that

$$g(z, \lambda) = -\varphi(x^\circ(z), \lambda, 0).$$

Substituting this expression in (6), for the shape of the front of the head shock wave, we obtain, finally, in the original variables

$$\Phi = \varphi(x, \lambda, t) + \varphi(x^\circ(z), \lambda^\circ, 0), \tag{7}$$

where $\lambda^\circ = x^\circ(z)$. Function φ is given arbitrarily.

The solution of Eq. (2) can be presented parametrically:

$$\begin{cases} q_b = \nu(\zeta) = g_\zeta(\zeta, \lambda^\circ(\zeta)), \\ z - \zeta = g_\zeta(\zeta, \lambda^\circ(\zeta))(x - x^\circ(\zeta)), \\ x - t = x^\circ(\zeta), \end{cases} \tag{8}$$

where ζ is the applicate of point $x = x_0(z)$ of intersection of the considered flow line with the leading edge of the wing.

We can now find the distance between the shock wave and the wing surface:

$$\delta(x, z, t) = -\int_{q_b(x,z,t)}^{\nu(x,z,t)} H(s) ds = \int_{\zeta}^{z} \frac{d\sigma}{g'(\sigma, \lambda)}, \tag{9}$$

where $s = g'(\sigma, \lambda)$, $H(s) = -[g'(\sigma, \lambda) \cdot g''(\sigma, \lambda)]^{-1}$, $\nu(x, z, t) = g'(z, \lambda)$, and $q_b(x, z, t) = g'(\zeta, x^\circ(\zeta))$.

The shape of a streamlined surface is determined using the formula

$$F(x, z, t) = \Phi(x, z, t) - \delta(x, z, t). \tag{10}$$

4. GAS PARAMETERS BEHIND THE SHOCK WAVE SURFACE

The constructed solution of the formulated problem allows determining parameters of a flow in the shock layer of a thin variable-shaped wing with a shock wave attached along the entire leading edge of the wing.

By [8], we obtain

$$y = \Phi(x,z,t) + \int_q^{\nu} H(s)ds. \tag{11}$$

Hence, for component of the vector of the velocity, v we have

$$v = Dy = y_t + y_x + qy_z = \varphi_t + \varphi_x - q\left[g'(z,\lambda) + \frac{1}{g'(z,\lambda)}\right]. \tag{12}$$

On the surface of the wing, for $q = q_b$, we obtain

$$v_b = \varphi_t + \varphi_x - q_b\varphi_z(x^\circ(z),\lambda^\circ,0) + D\left(\int_{q_b}^{\nu} H(s)ds\right). \tag{13}$$

Corrections to pressure and the longitudinal component of the velocity vector are determined from the following equations in the variables (q, x, z, t):

$$Du = 0, \quad y_q Dv = -p_q.$$

5. CONCLUSIONS

The solution proposed in this paper can be considered as a model for various regimes of a wing flow. Such solutions can have an independent meaning, as well as being applicable for calculating specific problems in combination with numerical methods to speed up a computational procedure. In addition, these solutions can be used to check the results of numerical calculations. We can quite well take into account the influence of real properties of a gas on flow parameters by introducing the effective adiabatic index [3, 9, 10]. If the gas behind the shock wave is in the state of thermodynamic equilibrium, then its flow can be simulated by the movement of a certain perfect gas, whose adiabatic index is determined depending on the Mach number and the thermodynamic state of the gas in the shock layer. In this case, the equation of state can be taken in the quasi-perfect form. The proposed mathematical model can supplement numerical modeling in the study of nonequilibrium flows as well (see, e.g., [11]).

ADDITIONAL INFORMATION

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