Some recent results on probabilities of large and moderate deviations for L-statistics

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1 Introduction

The class of L-statistics is one of the most commonly used classes in statistical inferences. We refer to monographs [5], [11], [12], [14] for the introduction to the theory of L-statistics. A survey on some modern applications of them in the economy and theory of actuarial risks can be found in [7]. There is an extensive literature on asymptotic properties of L-statistics, but its part concerning the large deviations is not so vast. We can mention a few of highly sharp results on this topic for L-statistics with smooth weight functions established in [13], [2], [1]. As to the trimmed L-statistics, the first – and up to the recent time the single – result on probabilities of large deviations was obtained in [4], but under some strict and unnatural conditions. Recently, the latter result was strengthened in [10], where a different approach than in [4] was proposed and implemented.

In this note we present some of our recent results established in [8]-[9].

To conclude this short introduction, we want to mention a paper[3], and an interesting article [6], in which a general delta method in the theory of Chernoff's type large deviations was proposed and illustrated by many examples including M-estimators and L-statistics.

2 Moderate deviations for intermediate trimmed means

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed (i.i.d.) real-valued random variables (r.v.'s) with common distribution function (*df*) F, and for each integer $n \geq 1$ let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics based on the sample $X_1, \dots, X_n$. Introduce the left-continuous inverse function F^{-1} defined as $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, $F^{-1}(0) = F^{-1}(0^+)$, and let F_n and F_n^{-1} denote the empirical *df* and its inverse respectively.

Consider the intermediate trimmed mean

$$T_n = \frac{1}{n} \sum_{i=k_n+1}^{n-m_n} X_{i:n} = \int_{\alpha_n}^{1-\beta_n} F_n^{-1}(u)\, d\,u, \tag{2.1}$$

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where k_n, m_n are two sequences of integers such that $0 \le k_n < n - m_n \le n$, $\alpha_n = k_n/n,\ \ \beta_n = m_n/n$, where we assume that

$$\min(k_n, m_n) \to \infty, \quad \max(\alpha_n, \beta_n) \to 0, \quad \text{as } n \to \infty. \tag{2.2}$$

Define the population trimmed mean

$$\mu(u, 1-v) = \int_u^{1-v} F^{-1}(s)\, d\, s, \quad \text{where} \quad 0 \le u < 1 - v \le 1. \tag{2.3}$$

Let $\xi_\nu = F^{-1}(\nu)$ denote the ν-th quantile of F and let $W_i^{(n)}$ be the X_i Winsorized outside of $(\xi_{\alpha_n}, \xi_{1-\beta_n}]$, i.e. $W_i^{(n)} = \max(\xi_{\alpha_n}, \min(X_i, \xi_{1-\beta_n}))$, $i = 1, \ldots, n$. In order to normalize T_n, we define two sequences

$$\mu_n = \mu(\alpha_n, 1-\beta_n), \qquad \sigma^2_{W,n} = \mathbf{Var}(W_i^{(n)}), \tag{2.4}$$

and assume that $\liminf_{n\to\infty} \sigma_{W,n} > 0$.

Let $\mathbf{\Phi}$ denote the standard normal distribution function. Here is our main result on moderate deviations for intermediate trimmed means.

Theorem 2.1. ([9]) *Suppose that* $\mathbf{E}|X_1|^p < \infty$ *for some* $p > c^2+2$ $(c > 0)$. *In addition, assume that* $\frac{\log n}{\min(k_n, m_n)} \to 0$ *as* $n \to \infty$, *and that* $\max(\alpha_n, \beta_n) = O\big((\log n)^{-\gamma}\big)$, *for some* $\gamma > 2p/(p-2)$, *as* $n \to \infty$. *Then*

$$\mathbf{P}\Big(\frac{\sqrt{n}(T_n - \mu_n)}{\sigma_{W,n}} > x\Big) = [1 - \mathbf{\Phi}(x)](1 + o(1)), \tag{2.5}$$

as $n \to \infty$, *uniformly in the range* $-A \le x \le c\sqrt{\log n}$ $(A > 0)$.

It is known that the intermediate trimmed mean T_n can serve as a consistent and robust estimator for $\mathbf{E}X_1$ (whenever it exists), and that the large and moderate deviations results for T_n can be helpful to construct more attractive confidence intervals for the expectation of X_1 than those that arise from the CLT.

Our next result concerns the asymptotic behavior of the first two moments of T_n and the possibility of replacing the normalizing sequences in (2.5) (in particular, replacing of μ_n by $\mathbf{E}X_1$).

Theorem 2.2. ([9]) *Suppose that the conditions of Theorem* 2.1 *are satisfied. Then*

$$n^{1/2}(\mathbf{E}T_n - \mu_n) = o\Big((\log n)^{-1}\Big), \quad \frac{\sigma_{W,n}}{\sigma} = 1 + o\big((\log n)^{-2}\big), \tag{2.6}$$

$$\frac{\sqrt{\mathbf{Var}(T_n)}}{\sigma_{W,n}/\sqrt{n}} = 1 + o\big((\log n)^{-1}\big), \quad \text{as } n \to \infty. \tag{2.7}$$

Moreover, μ_n *and* $\sigma_{W,n}$ *in relations* (2.5) *can be replaced respectively by* $\mathbf{E}T_n$ *and* σ *or* $\sqrt{n\mathbf{Var}(T_n)}$, *without affecting the result.*

Furthermore, if in addition

$$\max(\alpha_n, \beta_n) = o\big[(n \log n)^{-\frac{p}{2(p-1)}}\big], \tag{2.8}$$

then

$$n^{1/2}(\mathbf{E}X_1 - \mu_n) = o\Big((\log n)^{-1/2}\Big), \tag{2.9}$$

and μ_n *in* (2.5) *can be also replaced by* $\mathbf{E}X_1$.

3 Large and moderate deviations for trimmed L-statistics

In this section we consider the trimmed L-statistic given by

$$L_n = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i,n} X_{i:n}, \quad \text{where} \quad c_{i,n} \in \mathbb{R}. \tag{3.1}$$

Let $\alpha_n,\ \beta_n$ denote the same sequences as before, and suppose now that

$$\alpha_n \to \alpha, \quad \beta_n \to \beta, \quad \text{as} \quad n \to \infty, \quad 0 < \alpha < 1 - \beta < 1, \tag{3.2}$$

i.e. we focus on the case of heavy trimmed L-statistic. Let J be a function defined in an open set I such that $[\alpha, 1-\beta] \subset I \subseteq (0,1)$. Define the trimmed L-statistic

$$L_n^0 = n^{-1} \sum_{i=k_n+1}^{n-m_n} c_{i,n}^0 X_{i:n} = \int_{\alpha_n}^{1-\beta_n} J(u) F_n^{-1}(u)\, du \tag{3.3}$$

with the weights $c_{i,n}^0 = n \int_{(i-1)/n}^{i/n} J(u)\, du$ generated by the function J.

To state our results, we need the following set of assumptions.

(i) *J is Lipschitz in I.*

(ii) *F^{-1} satisfies a Hölder condition of order $0 < \varepsilon \le 1$ in some neighborhoods U_α and $U_{1-\beta}$ of α and $1-\beta$.*

(iii) $\max(|\alpha_n - \alpha|, |\beta_n - \beta|) = O\big(n^{-\frac{1}{2+\varepsilon}}\big)$, *where ε is the Hölder index from condition* **(ii)**.

(iv) $\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c_{i,n}^0| = O(n^{\frac{1}{2+\varepsilon}})$, *where ε is as in conditions* **(ii)**-**(iii)**.

Define the distribution function of the normalized L_n:

$$F_{L_n}(x) = \mathbf{P}\{\sqrt{n}(L_n - \mu_n)/\sigma \le x\}, \tag{3.4}$$

where $\mu_n = \int_{\alpha_n}^{1-\beta_n} J(u) F^{-1}(u)\, du$, and the asymptotic variance

$$\sigma^2 = \int_\alpha^{1-\beta} \int_\alpha^{1-\beta} J(u)J(v)(\min(u,v) - uv)\, dF^{-1}(u)\, dF^{-1}(v).$$

Here is our main result on Cramér type large deviations for L_n.

Theorem 3.1. ([10]) *Suppose that F^{-1} satisfies condition* **(ii)** *for some $0 < \varepsilon \le 1$ and the sequences α_n and β_n satisfy* **(iii)**. *In addition, assume that the weights $c_{i,n}$ satisfy* **(iv)** *for some function J satisfying condition* **(i)**.

Then for every sequence $a_n \to 0$ and each $A > 0$

$$1 - F_{L_n}(x) = [1 - \mathbf{\Phi}(x)](1 + o(1)), \tag{3.5}$$

as $n \to \infty$, uniformly in the range $-A \le x \le a_n n^{\varepsilon/(2(2+\varepsilon))}$.

Remark 3.1. Note that under somewhat stronger conditions **(iii')-(iv')** (cf. [10]) than **(iii)-(iv)**, the asymptotic variance σ in Theorem 3.1 can be replaced by $\sqrt{n\mathbf{Var}L_n}$, without affecting the result (see Theorem 1.2 [10]).

Corollary 3.1. *Suppose that the conditions of Theorem 2.1 are satisfied with $\varepsilon = 1$, i.e. F^{-1} is Lipschitz in some neighborhoods U_α and $U_{1-\beta}$ of α and $1-\beta$. Then for every sequence $a_n \to 0$ and each $A > 0$ relation* (3.5) *holds true, uniformly in the range $-A \le x \le a_n n^{1/6}$.*

Finally, we state our main results on probabilities of moderate deviations for L_n, i.e. the deviations in logarithmic ranges. We will need the following versions of conditions **(ii)-(iv)**.

(ii") *There exists a positive ε such that for each $t \in \mathbb{R}$* when $n \to \infty$

$$\begin{aligned} F^{-1}\big(\alpha + t\sqrt{\log n/n}\big) - F^{-1}\big(\alpha\big) &= O\big((\log n)^{-(1+\varepsilon)}\big), \\ F^{-1}\big(1-\beta + t\sqrt{\log n/n}\big) - F^{-1}\big(1-\beta\big) &= O\big((\log n)^{-(1+\varepsilon)}\big). \end{aligned} \tag{3.6}$$

(iii") $\max(|\alpha_n - \alpha|, |\beta_n - \beta|) = O\big(\sqrt{\frac{\log n}{n}}\big)$, $n \to \infty$.

(iv") *For some $\tilde{\varepsilon} > 0$* $\sum_{i=k_n+1}^{n-m_n} |c_{i,n} - c^0_{i,n}| = O\big(\frac{1}{\log^{\tilde{\varepsilon}} n}\sqrt{\frac{n}{\log n}}\big)$, $n \to \infty$. .

Theorem 3.2. ([8]) *Suppose that F^{-1} satisfies condition* **(ii")** *and that condition* **(iii")** *holds for the sequences α_n and β_n. In addition, assume that there exists a function J satisfying condition* **(i)** *such that* **(iv")** *holds for the weights $c_{i,n}$. Then relation* (3.5) *holds true, uniformly in the range $-A \le x \le c\sqrt{\log n}$, for each $c > 0$ and $A > 0$.*

Theorem 3.3. ([8]) *Suppose that the conditions of Theorem 3.2 hold true. In addition, assume that $\boldsymbol{E}|X_1|^\gamma < \infty$ for some $\gamma > 0$. Then*

$$\sqrt{n\mathbf{Var}(L_n)}/\sigma = 1 + O\big((\log n)^{-(1+2\nu)}\big), \tag{3.7}$$

where $\nu = \min(\varepsilon, \tilde{\varepsilon})$, ε, $\tilde{\varepsilon}$ are as in conditions **(ii")** *and* **(ii")** *respectively.*

Moreover, relation (3.5) *remains valid for each $c > 0$ and $A > 0$, uniformly in the range $-A \le x \le c\sqrt{\log n}$, if we replace σ in definition of $F_{L_n}(x)$* (cf. (3.4)) *by $\sqrt{n\mathbf{Var}(L_n)}$.*

References

[1] *Aleskeviciene, A.* Large and moderate deviations for L-statistics. Lithuanian Math. J. Vol. 31, 1991, P. 145–156.

[2] *Bentkus, V., Zitikis, R.* Probabilities of large deviations for L-statistics, Lithuanian Math. J., Vol. 30, 1990, P. 215–222.

[3] *H. Boistard*, *Large deviations for L-statistics*, Statistics & Decisions, 25 (2007), pp. 89–125.

[4] *Callaert, H., Vandemaele, M. and Veraverbeke, N.* A Cramér type large deviations theorem for trimmed linear combinations of order statistics, Comm. Statist. Th. Meth., Vol 11, 1982, P. 2689–2698.

[5] *David, H., Nagaraja, H. N.* Order Statistics, 3rd. ed., Wiley, New York, 2003.

[6] *Gao,F. and Zhao, X.* Delta method in large deviations and moderate deviations for estimators, Ann. Statist., Vol. 39, 2011, P. 1211-1240.

[7] *Greselin, F.,Madan, L., Puri, M. L. and Zitikis, R.* L-functions, processes, and statistics in measuring economic inequality and actuarial risks, Stat. Interface, Vol. 2, 2009, P. 227-245.

[8] *Gribkova, N. V.* Cramér type moderate deviations for trimmed L-statistics. Math. Methods Statist., Vol. 25, no. 4, 2016, P. 313-322.

[9] *Gribkova, N. V.* Cramér type moderate deviations for intermediate trimmed means. Commun. Statist. Th. Meth./N.Y., 2017, Vol. 46, no 23, P. l9l8-11932.

[10] *Gribkova, N. V.* Cramér type large deviations for trimmed L-statistics. Probab. Math. Statist., Vol. 37, no. 1, 2017, P. 101-118.

[11] *Serfling, R. J.* Approximation theorems of mathematical statistics, Wiley, New York, 1980.

[12] *Shorack, G.R., Wellner, J.A.* Empirical processes with application in statistics, Wiley, New York, 1986.

[13] *Vandemaele, M., Veraverbeke, N.* Cramér type large deviations for linear combinations of order statistics, Ann. Probab., Vol. 10, 1982, P. 423-434.

[14] *van der Vaart, A. W.* Asymptotic statistics, Cambridge Series in Statistical and Probabilistic Mathematics, 3, Cambridge Univ. Press, Cambridge, 1998.