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# Study of the hexapod dynamics using equations in redundant coordinates

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**Abstract.** The dynamics of hexapods (Stewart platforms) has been extensively studied for several decades. In this problem, the equations of motion are usually constructed using the basic theorems of mechanics. Lagrange equations of the second kind are often constructed for the same purpose. In the present paper, a new form of dynamic equations is considered. These equations are a special form of equations of motion of a system of rigid bodies (equations of dynamics in redundant coordinates). This approach is used to obtain the differential equations of motion of a hexapod in redundant coordinates. In this case, the loaded Stewart platform is considered as a rigid body, whose position is determined by setting the radius vector of the center of mass and the unit vectors of the body-axes system. From the vector form of the Lagrange equations of the first kind scalar differential equations of motion of the mechanical system under consideration are obtained. The obtained equations for some standard motions of the hexapod are numerically integrated. It is noted that the stable motion of the mechanical system under consideration can be obtained only with the introduction of feedbacks.

## 1. Differential equations of motion of a rigid body in redundant coordinates

Consider the fixed coordinate system  $O\xi\eta\zeta$  and the movable coordinate system  $Cxyz$ , whose axes are directed along the main central axes of inertia of the body; the point  $C$  is the center of mass of the rigid body. Let  $\boldsymbol{\rho} = \overrightarrow{OC}$  (see Figure. 1). Following [1], the position of a rigid body in the space will be determined from the radius vector of its center of mass and the unit basis vectors of the system  $Cxyz$ :

$$\boldsymbol{\rho}, \quad \mathbf{i}, \quad \mathbf{j}, \quad \mathbf{k}. \quad (1.1)$$

In the process of motion of the body, its vector coordinates (1.1) are subject to the holonomic constraints

$$\begin{aligned} f^1 = \mathbf{i}^2 - 1 = 0, \quad f^2 = \mathbf{j}^2 - 1 = 0, \quad f^3 = \mathbf{k}^2 - 1 = 0, \\ f^4 = \mathbf{i} \cdot \mathbf{j} = 0, \quad f^5 = \mathbf{j} \cdot \mathbf{k} = 0, \quad f^6 = \mathbf{k} \cdot \mathbf{i} = 0. \end{aligned} \quad (1.2)$$

The elementary work of the forces  $\mathbf{F}_\nu$  applied to the body at the points  $(x_\nu, y_\nu, z_\nu)$  can be written as

$$\begin{aligned} \delta A &= \sum_{\nu} \mathbf{F}_\nu \cdot (\delta \boldsymbol{\rho} + x_\nu \delta \mathbf{i} + y_\nu \delta \mathbf{j} + z_\nu \delta \mathbf{k}) = \\ &= \mathbf{Q}_\rho \cdot \delta \boldsymbol{\rho} + \mathbf{Q}_i \cdot \delta \mathbf{i} + \mathbf{Q}_j \cdot \delta \mathbf{j} + \mathbf{Q}_k \cdot \delta \mathbf{k}, \end{aligned}$$



where

$$\begin{aligned} \mathbf{Q}_\rho &= \sum_\nu \mathbf{F}_\nu, & \mathbf{Q}_i &= \sum_\nu x_\nu \mathbf{F}_\nu, \\ \mathbf{Q}_j &= \sum_\nu y_\nu \mathbf{F}_\nu, & \mathbf{Q}_k &= \sum_\nu z_\nu \mathbf{F}_\nu. \end{aligned}$$

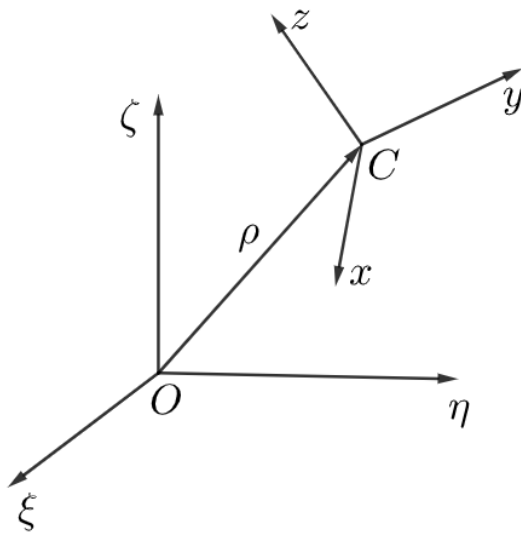


Figure 1. Coordinate systems.

The kinetic energy is as follows:

$$T = \frac{M\dot{\rho}^2}{2} + \frac{I_x \dot{\mathbf{i}}^2}{2} + \frac{I_y \dot{\mathbf{j}}^2}{2} + \frac{I_z \dot{\mathbf{k}}^2}{2},$$

here  $M$  is the mass of the body,  $I_x = \int_\tau x^2 \mu d\tau$ ,  $I_y = \int_\tau y^2 \mu d\tau$ ,  $I_z = \int_\tau z^2 \mu d\tau$ .

Now using [1, 2] the vector Lagrange equations of the first kind can be written in the form

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\rho}} - \frac{\partial T}{\partial \rho} &= \mathbf{Q}_\rho, & \varkappa &= \overline{1, 6}, \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{i}}} - \frac{\partial T}{\partial \mathbf{i}} &= \mathbf{Q}_i + \Lambda_\varkappa \frac{\partial f^\varkappa}{\partial \mathbf{i}} \equiv \mathbf{Q}_i + 2\Lambda_1 \mathbf{i} + \Lambda_4 \mathbf{j} + \Lambda_6 \mathbf{k}, \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{j}}} - \frac{\partial T}{\partial \mathbf{j}} &= \mathbf{Q}_j + \Lambda_\varkappa \frac{\partial f^\varkappa}{\partial \mathbf{j}} \equiv \mathbf{Q}_j + 2\Lambda_2 \mathbf{j} + \Lambda_5 \mathbf{k} + \Lambda_4 \mathbf{i}, \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{k}}} - \frac{\partial T}{\partial \mathbf{k}} &= \mathbf{Q}_k + \Lambda_\varkappa \frac{\partial f^\varkappa}{\partial \mathbf{k}} \equiv \mathbf{Q}_k + 2\Lambda_3 \mathbf{k} + \Lambda_6 \mathbf{i} + \Lambda_5 \mathbf{j}. \end{aligned} \tag{1.3}$$

Next, we use the constraint equations (1.2) to exclude the Lagrange multipliers from (1.3).

The resulting vector equations

$$M\ddot{\rho} = \sum_\nu \mathbf{F}_\nu, \tag{1.4}$$

$$\begin{aligned} \ddot{\mathbf{i}} &= -\dot{\mathbf{i}}^2 \mathbf{i} - \frac{2I_y}{I_x + I_y} (\mathbf{i} \cdot \dot{\mathbf{j}}) \dot{\mathbf{j}} - \frac{2I_z}{I_z + I_x} (\dot{\mathbf{k}} \cdot \mathbf{i}) \dot{\mathbf{k}} + \frac{L_z}{I_x + I_y} \dot{\mathbf{j}} - \frac{L_y}{I_z + I_x} \dot{\mathbf{k}}, \\ \ddot{\mathbf{j}} &= -\dot{\mathbf{j}}^2 \mathbf{j} - \frac{2I_z}{I_y + I_z} (\dot{\mathbf{j}} \cdot \dot{\mathbf{k}}) \dot{\mathbf{k}} - \frac{2I_x}{I_x + I_y} (\mathbf{i} \cdot \dot{\mathbf{j}}) \dot{\mathbf{i}} + \frac{L_x}{I_y + I_z} \dot{\mathbf{k}} - \frac{L_z}{I_x + I_y} \dot{\mathbf{i}}, \\ \ddot{\mathbf{k}} &= -\dot{\mathbf{k}}^2 \mathbf{k} - \frac{2I_x}{I_z + I_x} (\dot{\mathbf{k}} \cdot \mathbf{i}) \dot{\mathbf{i}} - \frac{2I_y}{I_y + I_z} (\dot{\mathbf{j}} \cdot \dot{\mathbf{k}}) \dot{\mathbf{j}} + \frac{L_y}{I_z + I_x} \dot{\mathbf{i}} - \frac{L_x}{I_y + I_z} \dot{\mathbf{j}}, \end{aligned} \tag{1.5}$$

where  $L_x, L_y, L_z$  are the projections of the main moment of external forces relative to the center of mass

$$\mathbf{L} = \sum_{\nu} (x_{\nu}\mathbf{i} + y_{\nu}\mathbf{j} + z_{\nu}\mathbf{k}) \times \mathbf{F}_{\nu}, \quad (1.6)$$

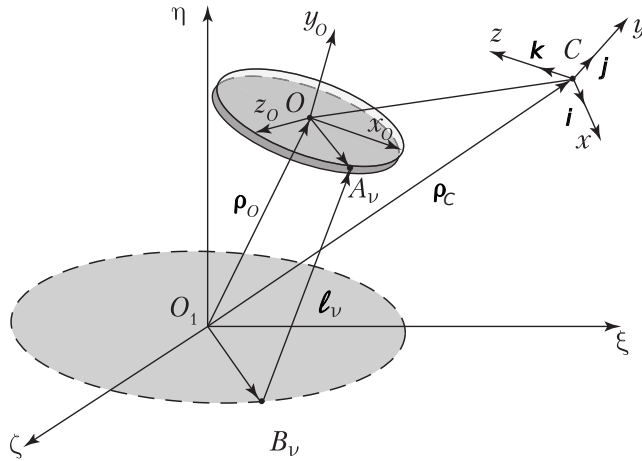
are known as the differential equations of motion of a rigid body in redundant coordinates, because they define 12 projections of the vectors from (1.1), which is twice the number of degrees of freedom of the rigid body.

Using the system of differential equations (1.4), (1.5), we solve the direct and inverse problems of dynamics for the study of the motion of the hexapod.

## 2. The direct and inverse problems of dynamics

Let us find the loads in the rods ( $\mathbf{F}_{\nu}, \nu = \overline{1,6}$ ) from the given motion law of the platform, that is, we solve the direct dynamics problem.

We will assume that the forces  $\mathbf{F}_{\nu}, \nu = \overline{1,6}$ , act along the directions of the vectors  $\boldsymbol{\ell}_{\nu}, \nu = \overline{1,6}$ , which define the upper endpoints of the rods  $A_{\nu}, \nu = \overline{1,6}$  (Figure. 2).



**Figure 2.** Mathematical model of a loaded Stewart platform.

The forces  $\mathbf{F}_{\nu}, \nu = \overline{1,6}$ , can be written as

$$\mathbf{F}_{\nu} = \frac{u_{\nu}\boldsymbol{\ell}_{\nu}}{\ell_{\nu}} \equiv U_{\nu}\boldsymbol{\ell}_{\nu}, \quad \ell_{\nu} = |\boldsymbol{\ell}_{\nu}|, \quad \nu = \overline{1,6}, \quad (2.1)$$

where  $u_{\nu}, \nu = \overline{1,6}$ , are the control parameters securing the required motion of the hexapod.

Equation (1.4) for the hexapod can be written in the form

$$\mathbf{F} \equiv \sum_{\nu=1}^6 \mathbf{F}_{\nu} = M\ddot{\boldsymbol{\rho}} - M\mathbf{g}. \quad (2.2)$$

From the vector function  $\boldsymbol{\rho}(t)$  one can determine the time variation of the principal vector  $\mathbf{F}$  of the system of forces  $\mathbf{F}_{\nu}, \nu = \overline{1,6}$ . Rewriting equation (2.2) in the scalar form, we have

$$\begin{cases} F_x(t) = M(\ddot{\rho} - \mathbf{g}) \cdot \mathbf{i}(t) = \\ \quad = M[\ddot{\xi}(t)\beta_{11}(t) + \ddot{\eta}(t)\beta_{12}(t) + \ddot{\zeta}(t)\beta_{13}(t) - g\beta_{13}(t)], \\ F_y(t) = M[\ddot{\xi}(t)\beta_{21}(t) + \ddot{\eta}(t)\beta_{22}(t) + \ddot{\zeta}(t)\beta_{23}(t) - g\beta_{23}(t)], \\ F_z(t) = M[\ddot{\xi}(t)\beta_{31}(t) + \ddot{\eta}(t)\beta_{32}(t) + \ddot{\zeta}(t)\beta_{33}(t) - g\beta_{33}(t)], \end{cases} \quad (2.3)$$

where  $\beta_{\sigma\tau}(t)$ ,  $\sigma, \tau = \overline{1,3}$ , are the cosines of the direction angles of the unit vectors of the body-axes coordinate system. For a given motion of the hexapod, the cosines  $\beta_{\sigma\tau}(t)$ ,  $\sigma, \tau = \overline{1,3}$ , are unknown functions (see [1]).

Projecting equations (1.5) to the axes of  $Cxyz$ , we get

$$\begin{aligned} \ddot{\mathbf{j}} \cdot \mathbf{k} &= -\frac{2I_z}{I_y + I_z}(\dot{\mathbf{j}} \cdot \dot{\mathbf{k}}) + \frac{L_x}{I_y + I_z}, \\ \ddot{\mathbf{k}} \cdot \mathbf{i} &= -\frac{2I_x}{I_z + I_x}(\dot{\mathbf{k}} \cdot \dot{\mathbf{i}}) + \frac{L_y}{I_z + I_x}, \\ \ddot{\mathbf{i}} \cdot \mathbf{j} &= -\frac{2I_y}{I_x + I_y}(\dot{\mathbf{i}} \cdot \dot{\mathbf{j}}) + \frac{L_z}{I_x + I_y}. \end{aligned} \tag{2.4}$$

Let us now determine the moment of forces relative to the center of mass  $\mathbf{L}$ . By (2.1), the projections  $\mathbf{F}_\nu$  have the form

$$F_x(t) = \sum_{\nu=1}^6 U_\nu \ell_{\nu x}, \quad F_y(t) = \sum_{\nu=1}^6 U_\nu \ell_{\nu y}, \quad F_z(t) = \sum_{\nu=1}^6 U_\nu \ell_{\nu z}. \tag{2.5}$$

By definition of the principal moment of the forces of the rods (1.6) and using (2.5), we get

$$\begin{aligned} \mathbf{L} &= \sum_{\nu=1}^6 (x_\nu \mathbf{i} + y_\nu \mathbf{j} + z_\nu \mathbf{k}) \times \mathbf{F}_\nu = \sum_{\nu=1}^6 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_\nu & y_\nu & z_\nu \\ U_\nu \ell_{\nu x} & U_\nu \ell_{\nu y} & U_\nu \ell_{\nu z} \end{vmatrix} = \\ &= \sum_{\nu=1}^6 \left( \begin{vmatrix} y_\nu & z_\nu \\ U_\nu \ell_{\nu y} & U_\nu \ell_{\nu z} \end{vmatrix} \cdot \mathbf{i} - \begin{vmatrix} x_\nu & z_\nu \\ U_\nu \ell_{\nu x} & U_\nu \ell_{\nu z} \end{vmatrix} \cdot \mathbf{j} + \begin{vmatrix} x_\nu & y_\nu \\ U_\nu \ell_{\nu x} & U_\nu \ell_{\nu y} \end{vmatrix} \cdot \mathbf{k} \right) = \\ &= \sum_{\nu=1}^6 U_\nu (y_\nu \ell_{\nu z} - z_\nu \ell_{\nu y}) \mathbf{i} + \sum_{\nu=1}^6 U_\nu (z_\nu \ell_{\nu x} - x_\nu \ell_{\nu z}) \mathbf{j} + \sum_{\nu=1}^6 U_\nu (x_\nu \ell_{\nu y} - y_\nu \ell_{\nu x}) \mathbf{k}. \end{aligned}$$

This gives us the projections of  $\mathbf{L}$ :

$$\left. \begin{aligned} L_x &= \sum_{\nu=1}^6 U_\nu (y_\nu \ell_{\nu z} - z_\nu \ell_{\nu y}), \\ L_y &= \sum_{\nu=1}^6 U_\nu (z_\nu \ell_{\nu x} - x_\nu \ell_{\nu z}), \\ L_z &= \sum_{\nu=1}^6 U_\nu (x_\nu \ell_{\nu y} - y_\nu \ell_{\nu x}), \end{aligned} \right\} \tag{2.6}$$

where

$$\begin{cases} \ell_{\nu x} = \xi(t)\beta_{11}(t) + \eta(t)\beta_{12}(t) + \zeta(t)\beta_{13}(t) + x_\nu - \xi_\nu\beta_{11}(t) - \zeta_\nu\beta_{13}(t), \\ \ell_{\nu y} = \xi(t)\beta_{21}(t) + \eta(t)\beta_{22}(t) + \zeta(t)\beta_{23}(t) + y_\nu - \xi_\nu\beta_{21}(t) - \zeta_\nu\beta_{23}(t), \\ \ell_{\nu z} = \xi(t)\beta_{31}(t) + \eta(t)\beta_{32}(t) + \zeta(t)\beta_{33}(t) + z_\nu - \xi_\nu\beta_{31}(t) - \zeta_\nu\beta_{33}(t), \\ \nu = \overline{1,6}. \end{cases} \tag{2.7}$$

Considering the formulas (2.5) and (2.6) as a system of linear algebraic equations with respect to  $U_\nu$ ,  $\nu = \overline{1,6}$ , we find the control parameters  $u_\nu(t) = \ell_\nu(t)U_\nu(t)$ ,  $\nu = \overline{1,6}$ , which control the given motion of the hexapod.

Let us now solve the inverse problem of dynamics, that is, we determine the law of motion of the hexapod from the given forces created in the rods. To this end, we need to find 3 projections of the radius vector  $\boldsymbol{\rho}$  and 9 cosines of the direction angles  $\beta_{\sigma\tau}(t)$ ,  $\sigma, \tau = \overline{1, 3}$ , of the basis vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

To solve this problem, we will use 6 scalar differential equations of motion in redundant coordinates (2.3), (2.4) and 6 constraint equations (1.2). When working with the equations, the formulas (2.5)–(2.7) should be taken into account.

For numerical solution, we need to set the initial conditions

$$\left. \begin{aligned} \xi(0) &= \xi^0, & \eta(0) &= \eta^0, & \zeta(0) &= \zeta^0, \\ \dot{\xi}(0) &= 0, & \dot{\eta}(0) &= 0, & \dot{\zeta}(0) &= 0, \\ \beta_{\sigma\tau}(0) &= \beta_{\sigma\tau}, & \dot{\beta}_{\sigma\tau}(0) &= 0, \\ \sigma, \tau &= \overline{1, 3}, \end{aligned} \right\} \quad (2.8)$$

and so the inverse problem of dynamics can be solved.

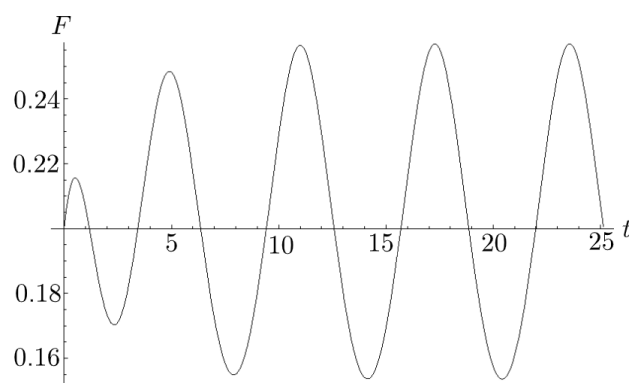
### 3. Numerical examples. Necessity of feedbacks

For numerical integration of the resulting system of differential equations, this system was written in the dimensionless form, where as the unit of length measurement we take the radius  $R_b$  of the circle drawn through the lower hinges of the rods controlling the platform motion, and all the masses were related to the total mass of the mechanical system. The dimensionless time was obtained by multiplying the real time by the value  $\sqrt{g/R_b}$ , where  $g$  is the gravity acceleration. We denote the dimensionless time by  $t$ .

Consider the vertical oscillations of the hexapod by the law

$$\eta(t) = 0.2 (\sin t)(1 - e^{-t/2})^2. \quad (3.1)$$

Using the Wolfram Mathematica software system, we found the forces  $F(t)$  in hydraulic cylinders that provide the simplest given vertical oscillatory movement of a symmetrically loaded platform by law (3.1). The graph of this force is given Figure. 3.



**Figure 3.** The graph of the force in the rod.

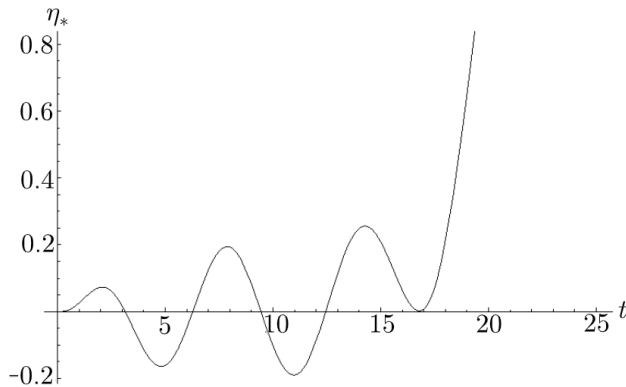
Note that when solving the inverse problem by applying the found forces  $F(t)$  to the loaded platform, we obtain the required vertical oscillations of the platform by the law (3.1).

However, in real situations, it is not possible to specify the exact program force  $F(t)$  providing the necessary motion of the hexapod. Consider the case of a small perturbation of the control  $F(t)$ . We denote by  $\eta_*(t)$  the motion of the platform corresponding to the small perturbation of the control  $F(t)$  given in the form  $0.0001 (\sin 2t)(1 - e^{-t/2})$  obtained as a result of the solution of the direct problem.

So, let

$$F_*(t) = F(t) + 0.0001(\sin 2t)(1 - e^{-t/2}).$$

The graph of the function  $\eta_*(t)$  is shown in Figure. 4.



**Figure 4.** Platform motion under a small perturbation of the vertical force.

The graph shows that during the initial period of time the introduced disturbance has little effect, while in the future there is an intense upward motion of the platform, despite the fact that  $F_*(t)$  differs insignificantly from  $F(t)$ .

To obtain stable vertical vibrations of the platform according to the law (3.1), we follow [3, 4] and introduce the feedbacks by specifying the forces  $F_k(t)$ ,  $k = \overline{1, 6}$  in the form

$$F_k(t) = F_k^p(t) + G(l_k^p(t) - l_k(t)), \quad k = \overline{1, 6},$$

where  $F_k^p(t)$  are the programmed control forces,  $l_k^p(t)$  are the programmed lengths of the hydraulic cylinders  $l_k(t)$  are the measured actual lengths of hydraulic cylinders, and  $G$  is the feedback coefficient.

In [3], it was shown by numerical experiments that by a certain choice of  $G$  it is always possible to achieve a deviation of the true motion from the programmed one with a given accuracy. Moreover, for a sufficiently large feedback coefficient  $G$ , it is possible to implement the programmed motion even for  $F_k^p(t) \equiv 0$ .

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