On the separation of variables for the two-dimensional integrable system with velocity-dependent potential

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Outline

Quadratures

In classical mechanics, one of the oldest notions of integrability is related to quadratures

$$I = \int R(x,y)dx$$

- x — independent variable
- R — rational function of x and y
- y — algebraic function

Function $y = y(x)$ is defined by an equation of the form

$$f(x,y) = y^n + A_1 y^{n-1} + \ldots + A_n = 0$$

where A_n are polynomials in x.

In modern notions, we may say that we have a plane curve

$$\mathcal{C} : \qquad f(x, y) = 0,$$

rational differential

$$\omega = R(x, y)dx|_c$$

and an integral

$$I = \int_{(x_0, y_0)}^{(x, y)} \omega$$

on a compact Riemann surface.

Such integrals are generally difficult to study directly.

Abel's idea was to consider a sum of integrals

$$I(t) = \sum_{k=1}^{n} \int^{p_k(t)} \omega$$

where $p_k(t) = (x_k(t), y_k(t))$ are points of intersection of $\mathcal{C}$ with a family of plane curves defined by equation

$$h(x, y, t) = 0$$

depending rationally on t.

Abel Theorem

If ω is a regular differential with no poles on $\mathcal{C}$, then

$$I(t) = \sum_{k=1}^{n} \int^{p_k(t)} \omega = const$$

leading to

$$\frac{d}{dt} \sum_{k=1}^{n} \int^{p_k(t)} \omega = \omega(p_1) + \omega(p_2) + \ldots + \omega(p_n) = 0$$

where

$$\omega(p_k) = R(x_k(t), y_k(t)) x_k'(t) dt$$

Equation in this form is called an Abel differential equation.

Systems in Question

Defined by the Hamilton functions

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - V(q_1, q_2)$$

and the canonical Poisson bracket

$$\{q_i, p_j\} = \delta_{ij}$$

According to Bertrand and Darboux, Hamilton–Jacobi equation can be integrated in quadratures if the potential V has special form.

Plain coordinates

For example,

$$V = \frac{U(u_1) - U(u_2)}{u_1 - u_2},$$

$$\int^{u_1} \frac{du_1}{\sqrt{f(u_1)}} + \int^{u_2} \frac{du_2}{\sqrt{f(u_2)}} = t + \beta_1, \qquad \int^{u_2} \frac{u_1 du_1}{\sqrt{f(u_1)}} + \int^{u_2} \frac{u_2 du_2}{\sqrt{f(u_2)}} = \beta_2 .$$

There are explicit solutions if

$$f(u) = a_6 u^6 + a_5 u^5 + a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0, \qquad a_i \in \mathbb{R} .$$

Bäcklund transformations

Consider a family

$$\frac{dz_i}{dt_k} = \{z_i, H_k\}, \qquad \{H_i, H_k\} = 0$$

Suppose that Hamilton-Jacobi equations

$$H_k = E_k$$

are integrable by quadratures.

Auto Bäcklund transformation is a change of variables

$$z \to \tilde{z}$$

which preserves Hamilton and Hamilton-Jacobi equations.

Bäcklund transformations

Some of applications are:

- Discretize and numerically solve equations of motions
- Construct integrable multivalued alebraic maps
- Construct new integrable system (hetero Bäcklund)
- Classify Poisson brackets compatible with canonical one

The question is: how to construct them for a given system?

Let us consider two Hamiltonian flows

$$\frac{dx_1}{dt_1} = \{x_1, H_1\} = \frac{y_1}{x_1 - x_2}\,, \qquad \frac{dx_2}{dt_1} = \{x_2, H_1\} = -\frac{y_2}{x_1 - x_2},$$

$$\frac{dx_1}{dt_2} = \{x_1, H_2\} = -\frac{x_2 y_1}{x_1 - x_2}\,, \qquad \frac{dx_2}{dt_2} = \{x_2, H_2\} = \frac{x_1 y_2}{x_1 - x_2},$$

with $H_{1,2}$ in involution

$$\{H_1, H_2\} = 0$$

Then $x_{1,2}$ satisfy

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} = dt_1\,, \qquad \frac{x_1 dx_1}{y_1} + \frac{x_2 dx_2}{y_1} = dt_2,$$

where $y_{1,2}$ can be found from the Jacobi equations

$$H_{1,2}(x_1, x_2, y_1, y_2) = \alpha_{1,2}.$$

These equations can be integrated in quadratures if

$$W(x_1, x_2, \alpha_1, \alpha_2) = W_1(x_1, \alpha_1, \alpha_2) + W_2(x_2, \alpha_1, \alpha_2)$$

and the Jacobi equations have the form

$$y_i = \frac{\partial W_i(x_1, \alpha_1, \alpha_2)}{\partial x_i} \equiv \sqrt{f_i(x_i)}\,.$$

We only consider the case of f being a sixth order polynomial

$$f(x) = a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0\,.$$

The next step is to rewrite the equations for the systems admitting Bäcklund transformations in the form of Abel differential equation.

Let the change of variables

$$B: \qquad (x_1, y_1, x_2, y_2) \to (x_3, y_3, x_4, y_4)$$

be an auto Bäcklund transform. Then,

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} = dt_1 = \frac{dx_3}{y_3} + \frac{dx_4}{y_4}\,,$$

$$\frac{x_1dx_1}{y_1} + \frac{x_2dx_2}{y_1} = dt_2 = \frac{x_3dx_3}{y_3} + \frac{x_4dx_4}{y_4}\,,$$

or

$$\frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} + \frac{dx_3}{-\sqrt{f(x_3)}} + \frac{dx_4}{-\sqrt{f(x_4)}} = 0\,,$$

$$\frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} + \frac{x_3dx_3}{-\sqrt{f(x_3)}} + \frac{x_4dx_4}{-\sqrt{f(x_4)}} = 0\,.$$

- These equations coincide with the Abel equations

$$\mathcal{C}: \qquad y^2 = f(x)$$

due to the hyperelliptic involution $\sigma : (y, x) \to (-y, x)$.

- In practice, composition of Bäcklund transformation and hyperelliptic involution yields us an Abel equation.
- Its solutions are points of intersection of two plane curves, in our case, the hyperelliptic $\mathcal{C}$ and

$$g(x, y) = y - P(x)\,, \qquad P(x) = \sqrt{a_6}x^3 + b_2x^2 + b_1x + b_0\,.$$

According to Abel, we can eliminate the variable y from these equations and get a polynomial linking all the abscissas of the points of intersection.

The explicit formula for the Abel polynomial

$$\psi(x) = \left(a_5 - 2\sqrt{a_6}b_2\right)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-\lambda)\,,$$

linking the coordinates of five points of intersection

$$p_{1,2} = (x_{1,2}, y_{1,2})\,, \qquad p_{3,4} = (x_{3,4}, -y_{3,4}) \quad \text{and} \quad p_5 = (\lambda, \mu)\,.$$

We can divide the roots into two parts

$$(x-x_3)(x-x_4) = \frac{\psi(x)}{\left(a_5 - 2\sqrt{a_6}b_2\right)(x-x_1)(x-x_2)}$$

or add the y coordinates and write an integral of motion.

Since we know that

$$y_1 - P(x_1) = 0\,, \qquad y_2 - P(x_2) = 0 \quad \text{and} \quad \mu - P(\lambda) = 0\,,$$

we can do Lagrange interpolation for $P(x)$:

$$P(x) = \sqrt{a_6}x^3 + b_2x^2 + b_1x + b_0 = \sqrt{a_6}(x - x_1)(x - x_2)(x - \lambda)$$

$$+ \frac{y_1(x - x_2)(x - \lambda)}{(x_1 - x_2)(x_1 - \lambda)} + \frac{y_2(x - x_1)(x - \lambda)}{(x_2 - x_1)(x_2 - \lambda)} + \frac{\mu(x - x_1)(x - x_2)}{(\lambda - x_1)(\lambda - x_2)}\,.$$

and we can explicitly find b_2 in the Abel polynomial

$$\psi(x) = \left(a_5 - 2\sqrt{a_6}b_2\right)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - \lambda)\,,$$

So, we started from the Bäcklund transformation

$$B: \qquad (x_1, y_1, x_2, y_2) \to (x_3, y_3, x_4, y_4),$$

and now we can get x_3, x_4 from Abel polynomial

$$(x - x_3)(x - x_4) = \frac{f(x) - P(x)^2}{\left(a_5 - 2\sqrt{a_6}\, b_2\right)(x - x_1)(x - x_2)(x - \lambda)}\,.$$

and y_3, y_4, also as functions of (x_1, y_1), (x_2, y_2) and (λ, μ), from

$$-y_i = P(x_i)\,, \qquad i = 3, 4\,.$$

These equations determine Bäcklund transformations for the two-dimensional Jacobi systems.

Example of application

Let us look at a particular case

$$a_0 = 0\,, \qquad \lambda = 0\,, \qquad \mu = \sqrt{f(0)} = \sqrt{a_0} = 0$$

Then, $b_2 = \dfrac{y_1}{x_1(x_1 - x_2)} + \dfrac{y_2}{x_2(x_2 - x_1)} - \sqrt{a_6}(x_1 + x_2)$

And the equations for the coordinates $(x_{3,4}, -y_{3,4})$ are

$$x_3 + x_4 = \frac{1}{(x_1 - x_2)^2(a_5 - 2\sqrt{a_6}\,b_2)}\Big(2a_6x_1x_2(x_1^2 - x_1x_2 + x_2^2) + a_5x_1x_2(x_1 + x_2) + 2a_4x_1x_2$$
$$+ a_3(x_1 + x_2) + 2a_2 + \frac{a_1(x_1 + x_2) - 2\sqrt{a_6}(x_1 - x_2)(y_1x_2^2 - y_2x_1^2) - 2y_1y_2}{x_1x_2}\Big)\,,$$
$$x_3x_4 = \frac{a_1}{x_1x_2(a_5 - 2\sqrt{a_6}\,b_2)}\,,$$
$$y_k = -\sqrt{a_6}(x_k - x_1)(x_k - x_2)x_k - \frac{y_1(x_k - x_2)x_k}{(x_1 - x_2)x_1} - \frac{y_2(x_k - x_1)x_k}{(x_2 - x_1)x_2}\,, \quad k = 3, 4.$$

Parabolic coordinates on a plane

If q_1, q_2 — cartesian coordinates on a plane, we can define parabolic coordinates by setting

$$u - 2q_2 - \frac{q_1^2}{u} = \frac{(u-u_1)(u-u_2)}{u}\,.$$

Sunstituting the cartesian coordinates and momenta

$$q_1 = \sqrt{-u_1 u_2}, \quad q_2 = \frac{u_1+u_2}{2}\,,$$
$$p_1 = \frac{2\sqrt{-u_1 u_2}(p_{u_1} - p_{u_2})}{u_1 - u_2}, \quad p_2 = \frac{2(p_{u_1}u_1 - p_{u_2}u_2)}{u_1 - u_2}$$

into the Hamilton function, we get

$$H_1 = \frac{2(p_{u_1}^2 u_1 - p_{u_2}^2 u_2)}{u_1 - u_2} - V(u_1, u_2)\,.$$

This system is integrable, if the potential

$$V(u_1, u_2) = \frac{u_1 U_1(u_1) - u_2 U_2(u_2)}{u_1 - u_2}\,.$$

In this case

$$H_2 = \frac{2u_1 u_2 (p_{u_1} - p_{u_2})}{u_2 - u_1} - \frac{u_1 u_2 \Big(U_1(u_1) - U_2(u_2)\Big)}{u_2 - u_1}\,,$$

and the separated equations are

$$p_{u_i}^2 = \left(\frac{\partial W_i(q_i, \alpha_1, \alpha_2)}{\partial u_i}\right)^2 = \frac{1}{2}\Big(U_i(u_i) + H_1 + \frac{H_2}{u_i}\Big)\,, \qquad i = 1, 2.$$

Setting

$$H_{1,2} = 2\alpha_{1,2}\,, \qquad \text{and} \qquad U = 2u(au^3 + bu^2 + cu + d)$$

one can get equations

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 4dt_1\,, \qquad \frac{u_1 du_1}{\sqrt{f(u_1)}} + \frac{u_2 du_2}{\sqrt{f(u_2)}} = 4dt_2,$$

where $f(x)$ is a polynomial of degree 6 with zero as a tailing coefficient

$$f(u) = u(au^5 + bu^4 + cu^3 + du^2 + \alpha_1 u + \alpha_2)\,.$$

In the original coordinates,

$$V = 2a(q_1^4 + 12q_1^2q_2^2 + 16q_2^4) + 8bq_2(q_1^2 + 2q_2^2)b + 2(q_1^2 + 4q_2^2)c + 4dq_2\,.$$

Bäcklund transformation for this system

Now let us find the Bäcklund transformation

$$B: \qquad (u_1, p_{u_1}, u_2, p_{u_2}) \to (\tilde{u}_1, \tilde{p}_{u_1}, \tilde{u}_2, \tilde{p}_{u_2}),$$

For that, we just put

$$x_{1,2} = u_{1,2}\,, \quad y_{1,2} = u_{1,2} p_{u_{1,2}}$$
$$x_{3,4} = \tilde{u}_{1,2}\,, \quad y_{3,4} = \tilde{u}_{1,2} \tilde{p}_{u_{1,2}}$$

into our equations for $x_{3,4}, y_{3,4}$ expressed in $x_{1,2}, y_{1,2}$.

Then, if $\lambda = 0$, we can use simplified equations to find the coordinates $\tilde{u}_{1,2}$

$$\tilde{u}_1 + \tilde{u}_2 = \frac{(u_1 - u_2)^2\big(a(u_1^2 + u_2^2) + b(u_1 + u_2)\big) + (u_1 - u_2)\big(2\sqrt{a}(p_{u_1}u_2 - p_{u_2}u_1) + c\big) - (p_{u_1} - p_{u_2})^2}{(u_2 - u_1)\big(2a(u_1^2 - u_2^2) - 2\sqrt{a}(p_{u_1} - p_{u_2}) + b(u_1 - u_2)\big)},$$

$$\tilde{u}_1\tilde{u}_2 = \frac{a(u_1^4 - u_2^4) + b(u_1^3 - u_2^3) + c(u_1^2 - u_2^2) + d(u_1 - u_2) - p_{u_1}^2 + p_{u_2}^2}{2a(u_1^2 - u_2^2) - 2\sqrt{a}(p_{u_1} - p_{u_2}) + b(u_1 - u_2)}$$

and momenta $\tilde{p}_{u_{1,2}}$

$$\tilde{p_{u_i}} = \frac{p_{u_1}(u_2 - \tilde{u}_i) - p_{u_2}(u_1 - \tilde{u}_i)}{u_1 - u_2} - \sqrt{a}(u_2 - \tilde{u}_i)(u_1 - \tilde{u}_i)\,,$$

which is the Bäcklund transformation for this system.

We can also rewrite it in the original variables:

$$\tilde{q}_1^2 = \frac{8q_1^2q_2(q_1^2+2q_2^2)a + 2bq_1^2(q_1^2+4q_2^2) + 4cq_1^2q_2 + 2dq_1^2 + p_1(p_1q_2 - p_2q_1)}{2q_1(p_1\sqrt{a} - 4aq_1q_2 - bq_1)},$$

$$\tilde{q}_2 = \frac{8q_1^2(q_1^2+2q_2^2)a + 4q_1(2p_1q_2 - p_2q_1)\sqrt{a} + 8bq_1^2q_2 + 4cq_1^2 - p_1^2}{8q_1(p_1\sqrt{a} - 4aq_1q_2 - bq_1)},$$

$$\tilde{p}_1 = \tilde{q}_1\,\frac{8q_1^2(q_1^2+6q_2^2)a^{3/2} - 4q_1(p_1q_2+p_2q_1)a + (16bq_1^2q_2 + 4cq_1^2 + p_1^2)\sqrt{a} - 2bp_1q_1}{2q_1(p_1\sqrt{a} - 4aq_1q_2 - bq_1)},$$

$$\tilde{p}_2 = -\frac{A}{8q_1^2(p_1\sqrt{a} - 4aq_1q_2 - bq_1)},$$

Using these equations we can explicitly check that this transformation is, indeed, a Bäcklund transformation.

$$\tilde{p}_{u_i}^2 - u_i(au_i^3 + bu_i^2 + cu_i + d) = \frac{H_1}{2} + \frac{H_2}{2\tilde{u}_i}, \qquad i = 1, 2.$$

Integrals of motion

Adding and substracting the separated equations in the new variables we get a new pair of integrals

$$\tilde{H}_1 = \sum_{i=1}^{2} \tilde{p}_{u_i}^2 - u_i(au_i^3 + bu_i^2 + cu_i + d) = H_1 + \frac{H_2}{2}\frac{\tilde{u}_1 + \tilde{u}_2}{\tilde{u}_1\tilde{u}_2}$$

and

$$\sqrt{\tilde{H}_2} = \sum_{i=1}^{2} (-1)^i \Big(\tilde{p}_{u_i}^2 - u_i(au_i^3 + bu_i^2 + cu_i + d)\Big) = H_2 \frac{\tilde{u}_2 - \tilde{u}_1}{\tilde{u}_1\tilde{u}_2}\,.$$

that can also be rewritten in the original variables.

Integrals of motion

First integral

$$\tilde{H}_1 = \frac{p_1^2 + p_2^2}{2} - 2aq_2^2(16q_2^2 + 5q_1^2) + 2\sqrt{a}q_1 \left(p_1 q_2 - \frac{p_2 q_1}{4}\right) - bq_2(16q_2^2 + 3q_1^2) - 2c\left(\frac{q_1^2}{4} - 4q_2^2\right) - 4dq_2 \,.$$

Second integral

$$\tilde{H}_2 = H_2^2 \frac{(\tilde{u}_1 + \tilde{u}_2)^2 - 4\tilde{u}_1\tilde{u}_2}{\tilde{u}_1^2 \tilde{u}_2^2}$$

is a fourth-order polynomial in momenta

Second integral

$$
\begin{aligned}
\tilde{H}_2 =& \frac{p_1^4}{4} + 2\sqrt{a}p_1^3 q_1 q_2 - \frac{p_1^2 q_1^2}{2}\Big(aq_1^2 + 12aq_2^2 + 3p_2\sqrt{a} + 6bq_2 + c\Big) + q_1^3(2aq_2 + b)p_1 p_2 + \frac{ap_2^2 q_1^4}{4} \\
&+ \sqrt{a}q_1^3\Big(6aq_2(q_1^2 + 4q_2^2) + b(q_1^2 + 12q_2^2) + 6cq_2 + 2d\Big)p_1 - \frac{\sqrt{a}q_1^4(aq_1^2 + 4aq_2^2 + 2bq_2 + c)p_2}{2} \\
&+ \frac{q_1^4}{4}\Big((q_1^2 + 4q_2^2)(q_1^2 - 28q_2^2)a^2 - 2\big(2q_2(3q_1^2 + 20q_2^2)b - (q_1^2 - 12q_2^2)c + 8dq_2\big)a \\
&- 2(q_1^2 - 6q_2^2)b^2 - 4(cq_2 + d)b + c^2\Big).
\end{aligned}
$$

This system and its integrable deformations is discussed in detail in

- A. P. Sozonov and A. V. Tsiganov, *Theor. Math. Phys.*, 183, 768–781 (2015)
- Tsiganov A.V., *Phys. Letters A*, 379, 2903–2907 (2015)

Overview

- Using this approach we can build Bäcklund transformations for some two-dimensional integrable systems.
  - Parabolic coordinates
  - Elliptic coordinates
- The key is searching for Bäcklund transformations that allow the system to be integrated in Abel quadratures using the standard parabolic and elliptic coordinates.
- The standard method of building auto Bäcklund transformations relies on a transformation of Lax matrix.

The standard way to build Bäcklund transformations

$$z \to \tilde{z}$$

is to use a gauge (or Darboux) transformation of the Lax matrix

$$L(u, z) = ML(u, \tilde{z})M^{-1}$$

The idea [Sklyanin, Kuznetsov] behind this construction is that

- Lax matrix $L(u, \tilde{z})$ has the same algebraic form as the original matrix $L(u, z)$
- gauge matrix M is to have a form of a Lax matrix satisfying the r-matrix Poisson bracket with the same r-matrix as the original Lax matrix

In practice, this procedure is not straightforward and needs experience and expert knowledge to be successfully applied.

Summary

- Using this approach we can build Bäcklund transformations for some two-dimensional integrable systems.
- The key is searching for Bäcklund transformations that allow the system to be integrated in Abel quadratures using the standard parabolic and elliptic coordinates.
- The standard method of building auto Bäcklund tranformations relies on a transformation of Lax matrix.

- Where to head next
  - Other integrable systems?
  - Higher dimensions?