On bi-Hamiltonian formulation of the perturbed Kepler problem

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Introduction

Even for completely integrable systems the existence of bi-Hamiltonian structure is not always satisfied. Fernandes and Olver announced that the perturbed Kepler problem is a completely integrable system without a bi-Hamiltonian formulation with respect to non-degenerate compatible Poisson structures. Here the perturbed Kepler problem is shown to be a bi-Hamiltonian system despite the fact that the graph of the Hamilton function is not a hypersurface of translation.

Bi-Hamiltonian formulation

Consider a dynamical system on a smooth manifold with coordinates $x_1,\dots,x_m$ defined by equations of motion

$$\dot{x}_i = X_i, \qquad i = 1,\dots,m$$

From these equations we can switch to a vector field

$$X = \sum X_i \frac{\partial}{\partial x_i}$$

The Hamiltonian of the system can be introduced, defining all the dynamics together with the Poisson bivector P

$$X = PdH$$

Bi-Hamiltonian manifolds (Magri 1978)

- Two Poisson bi-vectors P, P' satisfying compatibility condition

$$[P,P] = [P,P'] = [P',P'] = 0$$

- Can be used to find integrals of motion $X = PdH_1 = P'dH_2$
- Or if we know integrals in involution $X = g_1X_1 + \dots + g_nX_n, \quad X_k = P'dH_k$, then we can do the separation of variables (Falqui & Pedroni 2003)

Perturbed Kepler problem

Perturbed Kepler system has the Hamiltonian

$$H = \frac{p_x^2+p_y^2+p_z^2}{2} - \frac{1}{r} + \frac{\varepsilon}{2r^2}, \qquad r = \sqrt{x^2+y^2+z^2},$$

and canonical Poisson bivector

$$P = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}.$$

Let us find non-degenerate compatible Poisson structures for it. We will look for a bi-Hamiltonian formulation in the domain of definition of the action-angle variables.

Bogoyavlenskij construction

The vector field X is called non-degenerate or anisochronous if the Kolmogorov condition

$$\det \left| \frac{\partial^2 H(J_1,\dots,J_n)}{\partial J_i \partial J_k} \right| \neq 0$$

is met almost everywhere in the given action-angle coordinates.

For the degenerate or isochronous systems, if in the domain of definition of the action-angle variables we have some nonzero derivative

$$a = \frac{\partial H}{\partial J_m} \neq 0,$$

we can make the following canonical transformation

$$\tilde{J}_k = J_k, \qquad \tilde{\omega}_k = \omega_k - \frac{\partial H}{\partial J_k} a^{-1} \omega_m, \qquad k \neq m$$

$$\tilde{J}_m = H, \qquad \tilde{\omega}_m = a^{-1}\omega_m.$$

This canonical transformation does not add new singularities to the initial action-angle variables and reduces the Hamiltonian to the simplest form

$$H = \tilde{J}_m.$$

It allows us to construct bi-Hamiltonian formulation of the initial vector field X with two functionally dependent Hamiltonians

$$H = \tilde{J}_m \qquad \text{and} \qquad K = g(\tilde{J}_m),$$

but with the non-degenerate second Poisson bivector

$$P' = \sum_{k \neq m}^{n} \beta_k(\tilde{J}_k) \frac{\partial}{\partial \tilde{J}_k} \wedge \frac{\partial}{\partial \tilde{\omega}_k} + \left(\frac{dg}{d\tilde{J}_m}\right)^{-1} \frac{\partial}{\partial \tilde{J}_m} \wedge \frac{\partial}{\partial \tilde{\omega}_m},$$

where $\beta_k(\tilde{J}_k)$ are arbitrary nonzero functions and $g(\tilde{J}_m)$ is such that $g' \neq 0$. In this case the eigenvalues of the corresponding recursion operator $N = P'P^{-1}$ are integrals of motion only, see examples of this type bi-Hamiltonian formulations of the Kepler problem in [3].

Action-angle variables

In the spherical variables Hamiltonian H takes the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{1}{r} + \frac{\varepsilon}{2r^2},$$

and we can introduce two commuting integrals

$$l^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}, \qquad m = p_\phi.$$

Then, the action variables can be explicitly calculated and the corresponding angle variables can be obtained from the Jacobi equations.

In the action-angle variables the Hamiltonian takes the form

$$H = -\frac{1}{2\left(J_r + \sqrt{(J_\theta+J_\phi)^2+\varepsilon}\right)^2}.$$

that is not a hypersurface of translation in the action variables in contrast with the initial Kepler problem at $\varepsilon = 0$.

Both the perturbed Kepler problem and the unperturbed Kepler problem are degenerate or isochronous systems with well-defined derivative

$$a = \frac{\partial H}{\partial J_r} = -(-2h)^{3/2}.$$

According to the Bogoyavlenskij theorem it allows us to have bi-Hamiltonian formulation of these systems in the action-angle variables.

Delaunay type variables

Instead of action-angle variables, we can address the system in the Delaunay type variables

$$J_1 = J_\phi, \qquad J_2 = J_\phi + J_\theta, \qquad J_3 = J_r + \sqrt{(J_\phi+J_\theta)^2+\varepsilon},$$

$$\omega_1 = \omega_\phi - \omega_\theta, \qquad \omega_2 = \omega_\theta - \frac{J_\phi + J_\theta}{\sqrt{(J_\phi+J_\theta)^2+\varepsilon}}\omega_r, \qquad \omega_3 = \omega_r.$$

The Delaunay variables, which at $\varepsilon = 0$ coincide with the classical Delaunay elements, have a geometrical meaning directly related to the description of the orbits and their variations are more significant for astronomers than those of Cartesian or spherical variables.

In these variables the Hamiltonian takes the form

$$H = -\frac{1}{2J_3^2}$$

and we can construct bi-Hamiltonian formulation with the second bivector P' given by the previously used formula. For instance, if

$$\beta_1(J_1) = J_1, \qquad \beta_2(J_2) = J_2, \qquad \text{and} \qquad K = -\frac{1}{3J_3^3},$$

second bivector is equal to

$$P' = \sum_{k=1}^{3} J_k \frac{\partial}{\partial J_k} \wedge \frac{\partial}{\partial \omega_k} = \begin{pmatrix} 0 & 0 & 0 & J_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_3 \\ -J_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -J_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J_3 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding recursion operator has three functionally independent eigenvalues which are the first integrals.

In the initial action-angle variables $(\omega_r, \omega_\theta, \omega_\phi, J_r, J_\theta, J_\phi)$ this bivector translates to a more complicated form

$$P' = \begin{pmatrix} 0 & 0 & 0 & J_r + \sqrt{(J_\phi+J_\theta)^2+\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & -\eta & J_\theta + J_\phi & 0 \\ 0 & 0 & 0 & -\eta & J_\theta & J_\phi \\ -J_r - \sqrt{(J_\phi+J_\theta)^2+\varepsilon} & \eta & \eta & 0 & 0 & 0 \\ 0 & -J_\theta - J_\phi & -J_\theta & 0 & 0 & 0 \\ 0 & 0 & -J_\phi & 0 & 0 & 0 \end{pmatrix}$$

where

$$\eta = \frac{\ell(\ell - J_r - \sqrt{\ell^2+\varepsilon})}{\sqrt{\ell^2+\varepsilon}}, \qquad \ell = J_\theta + J_\phi.$$

In much the same way we can obtain other bi-Hamiltonian formulations associated with the two families of the Poincaré type action-angle variables or with other known types of action-angle variables for the perturbed Kepler problem.

Summary

- Both initial and perturbed Kepler problem are degenerate and, therefore, the Fernandes theorem cannot be applied to them.
- The construction used here to obtain bi-Hamiltonian structure for perturbed Kepler system can be applied to other degenerate systems.

References

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