Separation of variables for some systems with a fourth-order integral of motion

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1 Introduction

Yehia found several integrable deformations of the Kovalevskaya top and the Chaplygin system on a sphere[1]. Here, following the method proposed in [2, 3], we construct separation variables for these deformations.

The phase space with physical coordinates $x = (x_1, x_2, x_3)$ and $J = (J_1, J_2, J_3)$ and a Lie-Poisson bracket

$$\{J_i\,, J_j\} = \varepsilon_{ijk} J_k\,, \qquad \{J_i\,, x_j\} = \varepsilon_{ijk} x_k\,, \qquad \{x_i\,, x_j\} = 0$$

having two Casimir functions $C_1 = |x|^2 \equiv \sum_{k=1}^3 x_k^2$, $C_2 = (x, J) \equiv \sum_{k=1}^3 x_k J_k$ is where we will work.

Systems in question defined by the Hamilton functions

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2ax_1 - \frac{\lambda C_1}{x_3^2} + \frac{c}{\sqrt{x_1^2+x_2^2}} + \frac{2C_1 - x_3^2}{x_2^2}\left(d + \frac{ex_1}{\sqrt{x_1^2+x_2^2}}\right)$$

and

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right).$$

The first system is a generalisation of the Kovalevskaya top, and the second is a generalisation of a system discussed by Chaplygin and Goryachev. We will consider them on a sphere, i. e. at $(x, J) = 0$.

If $C_1 = 1$ and $C_2 = 0$, the Hamiltonians of these two systems commute with respect to Poisson brackets with the intergrals of motion, respectively:

$$\begin{aligned} H_2 = &\left(J_1^2 - J_2^2 - 2ax_1 + \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - ax_2 + \frac{2\lambda x_1 x_2}{x_3^2}\right)^2 \\ &+ \frac{1}{x_2^4}\left(dx_3^2 + \frac{cx_2^2 + ex_3^2 x_1}{\sqrt{x_1^2+x_2^2}}\right)\left(2x_2^2(J_1^2+J_2^2) + dx_3^2 + \frac{cx_2^2 + ex_3^2x_1}{\sqrt{x_1^2+x_2^2}}\right) \\ &- \frac{4ax_3^2(dx_1 + e\sqrt{x_1^2+x_2^2})}{x_2^2} - \frac{2\lambda}{x_2^2}\left(\frac{\sqrt{x_1^2+x_2^2)}(cx_2^2 - ex_3^2x_1)}{x_3^2} - dx_1^2\right) \end{aligned}$$

and

$$\begin{aligned} H_2 = &\left(J_1^2 - J_2^2 - 2bx_3^2 - \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - \frac{2\lambda x_1x_2}{x_3^2}\right)^2 \\ &- 2(J_1^2 + J_2^2)\left(\frac{c\,(x_1^2+x_2^2)}{x_3^6} + \frac{dx_3^2}{x_1^2} + \frac{ex_3^2}{x_2^2}\right) + x_3^4\left(\frac{c(x_1^2+x_2^2)}{x_3^8} + \frac{d}{x_1^2} + \frac{e}{x_2^2}\right)^2 \\ &- 4b\left(\frac{c(x_1^2 - x_2^2)}{x_3^4} + x_3^4\left(\frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\right) - 2\lambda\left(\frac{c(x_1^2+x_2^2)^2}{x_3^8} - \frac{dx_2^2}{x_1^2} - \frac{ex_1^2}{x_2^2}\right) \end{aligned}$$

These integrals of motion were found in [1]. Here we construct separation variables and separated equations using only the Hamilton functions and the integrals of motion for these systems.

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2 Separation of variables

If we consider a dynamical system on a smooth manifold with coordinates $x_1, \dots, x_m$ defined by equations of motion $\dot{x_i} = X_i, \quad i = 1, \dots, m$ we can say these equations yield a vector field $X = \sum X_i \dfrac{\partial}{\partial x_i}$.

If we approach the system from a physical point of view, we need to introduce the Hamiltonian function, which defines all the dynamics together with the Poisson bivector P, which we assume known beforehand from the kinematic considerations:

$$X = PdH$$

2.1 Bi-Hamiltonian systems

Moving further, we consider bi-Hamiltonian systems (due to Magri 1978) that have two Poisson bi-vectors P, P' satisfying the compatibility condition with respect to the Schouten bracket

$$[P, P'] = 0.$$

Studying such systems we can take two different approaches:

- We can use the bi-Hamiltonian structure to find integrals of motion from the construction

$$X = PdH_1 = P'dH_2$$

- Or if we beforehand know the integrals in involution, then we can construct the decomposition of the original vector field $X = g_1 X_1 + \cdots + g_n X_n, \quad X_k = P'dH_k$ and do the separation of variables (Falqui & Pedroni 2003).

The separation of variables for the latter case is based on constructing the recursion operator $N = P'P^{-1}$ whose eigenvalues are separation variables for the system. So, eventually the key to this method is finding the second Poisson bivector P'.

2.2 Ansatze for P'

Most of known additional Poisson bi-vectors have the form

$$P' = \mathcal{L}_Y \, P,$$

where $\mathcal{L}_Y$ is a Lie derivative along some vector field Y. But finding Y from this equation coupled with the compatibility condition between the two Poisson bivectors is still difficult, because the solutions we obtain from there are too common and too gereral to be of use. To get practical results we need to introduce yet another constraints on Y to narrow the set of solutions or, which is the same, the set of systems in question.

These assumptions about Y can be make, e. g., by introducing proportionality A between Y and X

$$Y = AX = AP\,dH, \quad \text{where} \quad A = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \Pi & 0 \\ 0 & \Lambda \end{pmatrix}$$

In our situation we do not need to write A explicitly, but instead declare that for the Hamiltonian of natural form ($H = T + V$) vector field Y splits likewise into two components corresponding to two parts of the Hamiltonian: $Y = Y_T + Y_V$.

2.3 Overview of the method

We start with H_1 and H_2

1. Write the equations for the components of the second Poisson bivector P'
2. Use ansatze and solve compatibility condition for the components of Y
3. From the obtained P' calculate the recursion operatior N
4. Build separation variables and separated equations

And then we can introduce additional terms to the system in question and repeat this again.

3 Application example

In this handout what follows is a brief overview of the procedure applied to the generalized Chaplygin system, and for more details and also for the results for the generalized Kovalevskaya top you can see the full paper [4].

We start with the Hamiltonian

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right)$$

and the integral of motion

$$\begin{aligned} H_2 = &\left(J_1^2 - J_2^2 - 2bx_3^2 - \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - \frac{2\lambda x_1 x_2}{x_3^2}\right)^2 \\ &- 2(J_1^2 + J_2^2)\left(\frac{c\,(x_1^2 + x_2^2)}{x_3^6} + \frac{dx_3^2}{x_1^2} + \frac{ex_3^2}{x_2^2}\right) + x_3^4\left(\frac{c(x_1^2 + x_2^2)}{x_3^8} + \frac{d}{x_1^2} + \frac{e}{x_2^2}\right)^2 \\ &- 4b\left(\frac{c(x_1^2 - x_2^2)}{x_3^4} + x_3^4\left(\frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\right) - 2\lambda\left(\frac{c(x_1^2 + x_2^2)^2}{x_3^8} - \frac{dx_2^2}{x_1^2} - \frac{ex_1^2}{x_2^2}\right) \end{aligned}$$

3.1 The case $c = 0$

Using ansatze for the components of Y we find that it is a sum of Y_T and Y_V:

$$Y_T = \begin{pmatrix} -\dfrac{2\cos 2\theta}{\sin 2\theta} p_\phi p_\theta \\ \\ -\cot\theta p_\phi^2 + \tan\theta p_\theta^2 \end{pmatrix}$$

and Y_V has a term for each of the constants

$$Y_V = b\begin{pmatrix} \cos 2\phi p_\phi + \sin 2\phi \tan\theta p_\theta \\ \\ \sin 2\phi \cot\theta p_\phi - \cos 2\phi p_\theta \end{pmatrix} - \frac{d\,p_\theta}{\sin^2\phi}\begin{pmatrix} \dfrac{2\cot\phi}{\sin 2\theta} \\ \\ \dfrac{1}{\sin^2\theta} \end{pmatrix} + \frac{e\,p_\theta}{\cos^2\phi}\begin{pmatrix} \dfrac{2\tan\phi}{\sin 2\theta} \\ \\ -\dfrac{1}{\sin^2\theta} \end{pmatrix}$$

In the case $c = 0$ separation variables are the roots of the characteristic polynomial

$$\det(N - \mu I) = B^2(\mu)$$

where

$$B(\mu) = \mu^2 - \left(\frac{J_1^2 + J_2^2}{x_3^2} - \frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\mu + \frac{b(J_1^2 - J_2^2)}{x_3^2} + \left(\frac{d}{x_1^2} - \frac{e}{x_2^2})\right) b - b^2 = (\mu - q_1)(\mu - q_2).$$

The momenta can be constructed from the polynomial

$$A(\mu) = \left(\frac{x_2J_1 - x_1J_2}{x_3}\right)\frac{\mu}{2} - \frac{b}{2}\left(\frac{x_2J_1 + x_1J_2}{x_3}\right) = -\frac{\mu}{2}\tan\theta\, p_\theta - \frac{b}{2}\Big(\sin 2\phi\, p_\phi - \cos 2\phi \tan\theta\, p_\theta\Big).$$

this way:

$$p_j = \frac{1}{q_j^2 - b^2}\, A(\mu = q_j)\,, \qquad j = 1, 2\,.$$

Substituting the equations for separation variables in H_1 and H_2 we get

$$\Phi(q_j, p_j) = 0, \qquad j = 1, 2$$

where

$$\begin{aligned} \Phi(q,p) = &\left(8(q^2 - b^2)p^2 - 2q + H_1 - \sqrt{H_2} - \frac{4b\big(q(d-e) + b(d+e)\big)}{q^2 - b^2}\right) \times \\ &\times \left(8(q^2 - b^2)p^2 - 2q + H_1 + \sqrt{H_2} - \frac{4b\big(q(d-e) + b(d+e)\big)}{q^2 - b^2}\right) + 4\lambda q = 0 \end{aligned}$$

This is a genus two hyperelliptic curve; if $\lambda = 0$, it splits into two elliptic curves.

3.2 The case $c \neq 0$

For the general situation the separation variables differ from those found above by the canonical transformation

$$p_\theta \to p_\theta + \frac{\sqrt{c}\sin\theta}{\cos^3\theta}$$

Substituting the separation variables into equation

$$\widetilde{\Phi}(q,p) = \Phi(q,p) - 16\sqrt{c}\,(q^2 - b^2)p$$

we get a Hamiltonian

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + \frac{2(x_2 J_1 - x_1 J_2)\sqrt{c}}{x_3^3} - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right),$$

which matches the original Hamiltonian after a transformation

$$J_1 = J_1 - \frac{\sqrt{c}\,x_2}{x_3^3}\,, \qquad J_1 = J_2 + \frac{\sqrt{c}\,x_1}{x_3^3}\,.$$

Equation $\widetilde{\Phi}(q,p) = 0$ defines a genus three algebraic curve.

So, this ends the procedure of obtaining separation variables and separated equations for the integrable deformation of the Chaplygin system.

4 Summary

- This method still needs the ansatze to work and how to find it is yet to be determined.
- We produce separation variables and separated equations starting strating just from integrals of motion.
- The results agree with those of Kovalevskaya but the procedure for constructing separation variables is straighforward.
- Where to head next
  - How to find the ansatze for a given system?
  - What to do with the separated equations?

References

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