Separation of variables for some systems with a fourth-order integral of motion

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Introduction

Yehia found several integrable deformations of the Kovalevskaya top and the Chaplygin system on a sphere, i. e. at the zero value of the surface integral [1].
Here, following the method proposed in [2, 3], we construct separation variables for these deformations.

Phase space with physical coordinates $x = (x_1, x_2, x_3)$ and $J = (J_1, J_2, J_3)$.
Lie-Poisson bracket:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0$$

Casimir functions:

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = (x, J) \equiv \sum_{k=1}^{3} x_k J_k$$

Systems in question defined by the Hamilton functions

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2ax_1 - \frac{\lambda C_1}{x_3^2} + \frac{c}{\sqrt{x_1^2 + x_2^2}} + \frac{2C_1 - x_3^2}{x_2^2}\left(d + \frac{ex_1}{\sqrt{x_1^2 + x_2^2}}\right)$$

and

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right).$$

The first system becomes the Kovalevskaya top in the case $\lambda = c = d = e = 0$.
The second system was discussed by Chaplygin and Goryachev if $\lambda \neq 0$.
For these two systems there are additional integrals of motion (Yehia 2006).
We build the separation variables and separated equations from these integrals of motion.

Separation of variables

Consider a dynamical system on a smooth manifold with coordinates $x_1, \ldots, x_m$ defined by equations of motion

$$\dot{x}_i = X_i, \qquad i = 1, \ldots, m$$

From these equations we can switch to a vector field

$$X = \sum X_i \frac{\partial}{\partial x_i}$$

The Hamiltonian of the system can be introduced, defining all the dynamics together with the Poisson bivector P

$$X = PdH$$

Bi-Hamiltonian manifolds (Magri 1978)

- Two Poisson bi-vectors P, P' satisfying compatibility condition $[P, P'] = 0$
- Can be used to find integrals $X = PdH_1 = P'dH_2$
- Or if we know integrals in involution $X = g_1 X_1 + \cdots + g_n X_n$, $\quad X_k = P' dH_k$ then we can do the separation of variables (Falqui & Pedroni 2003)

Separation of variables

Recursion operator

$$N = P' P^{-1}$$

Eigenvalues are separation variables

Most of known additional Poisson bi-vectors have the form

$$P' = \mathcal{L}_Y P$$

Finding Y

$$(dH_k, \mathcal{L}_Y P dH_m) = 0, \qquad [\mathcal{L}_Y P, \mathcal{L}_Y P] = 0$$

Written in separation variables the solutions are

$$Y_j = 0, \quad Y_{n+j} = f_j(q_j, p_j), \quad j = 1..n$$

We need to narrow the solutions set by imposing constraints

Let us make some assumptions about Y

- Introduce vector field A which leads to $Y = AX = APdH$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} \Pi & 0 \\ 0 & \Lambda \end{pmatrix}$$

- Hamiltonian of natural form ($H = T + V$) leads to $Y = Y_T + Y_V$

Overview of the method

We start with H_1 and H_2

1. Write the equations for the components of P'
2. Use ansatze and solve for components of Y
3. Calculate the Poisson bivector P'
4. Build separation variables and separated equations

Application example

Consider the generalized Chaplygin system

We start with the Hamiltonian

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right)$$

and the integral of motion

$$\begin{aligned} H_2 = {} & \left(J_1^2 - J_2^2 - 2bx_3^2 - \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - \frac{2\lambda x_1 x_2}{x_3^2}\right)^2 \\ & - 2(J_1^2 + J_2^2)\left(\frac{c(x_1^2 + x_2^2)}{x_3^6} + \frac{dx_3^2}{x_1^2} + \frac{ex_3^2}{x_2^2}\right) + x_3^4\left(\frac{c(x_1^2 + x_2^2)}{x_3^8} + \frac{d}{x_1^2} + \frac{e}{x_2^2}\right)^2 \\ & - 4b\left(\frac{c(x_1^2 - x_2^2)}{x_3^4} + x_3^4\left(\frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\right) - 2\lambda\left(\frac{c(x_1^2 + x_2^2)^2}{x_3^8} - \frac{dx_2^2}{x_1^2} - \frac{ex_1^2}{x_2^2}\right) \end{aligned}$$

The case $c = 0$ is easy to address.

Using ansatze for the components of Y we find that it is a sum of Y_T and Y_V:

$$Y_T = \begin{pmatrix} -\dfrac{2\cos 2\theta}{\sin 2\theta} p_\phi p_\theta \\ -\cot\theta\, p_\phi^2 + \tan\theta\, p_\theta^2 \end{pmatrix}$$

and Y_V has a term for each of the constants

$$Y_V = b\begin{pmatrix} \cos 2\phi\, p_\phi + \sin 2\phi \tan\theta\, p_\theta \\ \sin 2\phi \cot\theta\, p_\phi - \cos 2\phi\, p_\theta \end{pmatrix} - \frac{d\, p_\theta}{\sin^2\phi}\begin{pmatrix} \dfrac{2\cot\phi}{\sin 2\theta} \\ \dfrac{1}{\sin^2\theta} \end{pmatrix} + \frac{e\, p_\theta}{\cos^2\phi}\begin{pmatrix} \dfrac{2\tan\phi}{\sin 2\theta} \\ -\dfrac{1}{\sin^2\theta} \end{pmatrix}$$

In the case $c = 0$ separation variables are the roots of the characteristic polynomial

$$\det(N - \mu I) = B^2(\mu)$$

where

$$B(\mu) = \mu^2 - \left(\frac{J_1^2 + J_2^2}{x_3^2} - \frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\mu + \frac{b(J_1^2 - J_2^2)}{x_3^2} + \left(\frac{d}{x_1^2} - \frac{e}{x_2^2}\right)b - b^2 = (\mu - q_1)(\mu - q_2).$$

The momenta can be constructed from the polynomial

$$\begin{aligned} A(\mu) &= \left(\frac{x_2J_1 - x_1J_2}{x_3}\right)\frac{\mu}{2} - \frac{b}{2}\left(\frac{x_2J_1 + x_1J_2}{x_3}\right) \\ &= -\frac{\mu}{2}\tan\theta\, p_\theta - \frac{b}{2}\left(\sin 2\phi\, p_\phi - \cos 2\phi \tan\theta\, p_\theta\right). \end{aligned}$$

Substituting the equations for separation variables in H_1 and H_2 we get

$$\Phi(q_j, p_j) = 0, \qquad j = 1, 2$$

where

$$\begin{aligned} \Phi(q,p) = {} & \left(8(q^2 - b^2)p^2 - 2q + H_1 - \sqrt{H_2} - \frac{4b\big(q(d-e) + b(d+e)\big)}{q^2 - b^2}\right) \times \\ & \times \left(8(q^2 - b^2)p^2 - 2q + H_1 + \sqrt{H_2} - \frac{4b\big(q(d-e) + b(d+e)\big)}{q^2 - b^2}\right) + 4\lambda q = 0 \end{aligned}$$

This is a genus two hyperelliptic curve; if $\lambda = 0$, it splits into two elliptic curves.

In the case $c \neq 0$ we need a canonical transformation

$$p_\theta \to p_\theta + \frac{\sqrt{c}\sin\theta}{\cos^3\theta}$$

after which we introduce a new function

$$\widetilde{\Phi}(q,p) = \Phi(q,p) - 16\sqrt{c}(q^2 - b^2)p$$

Equation $\widetilde{\Phi}(q,p) = 0$ defines a genus three algebraic curve.

So, we have separation variables and separated equations for the integrable deformation of the Chaplygin system.

Summary

- This method still needs the ansatze to work and how to find it is yet to be determined.
- We produce separation variables and separated equations starting strating just from integrals of motion.
- The results agree with those of Kovalevskaya but the procedure for constructing separation variables is straighforward.
- Where to head next
  - How to find the ansatze for a given system?
  - What to do with the separated equations?

References

[1] H. M. Yehia, *The master integrable two-dimensional system with a quartic second integral*, J. Phys. A: Math. Gen., 39, 5807 - 5824, 2006.

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[3] A. V. Tsiganov, *On natural Poisson bivectors on the sphere*, J. Phys. A: Math. Theor., 44, 105203 (15pp), 2011.

[4] Yu. A. Grigoryev, V. A. Khudobakhshov, A. V. Tsiganov, *Separation of variables for some systems with a fourth-order integral of motion*, Theoretical and Mathematical Physics, v. 177, n. 3, pp. 1680-1692, 2013

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