Separation of variables for some systems with a fourth-order integral of motion

Introduction

Yehia found several integrable deformations of the Kovalevskaya top and the Chaplygin system on a sphere, i.e. at the zero value of the surface integral [1]. Here, following the method proposed in [2, 3], we construct separation variables for these deformations.

Phase space with physical coordinates $x = (x_1, x_2, x_3)$ and $J = (J_1, J_2, J_3)$. Lie-Poisson bracket:

$$\{J_i, J_j\} = \varepsilon_{ijk}J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk}x_k, \qquad \{x_i, x_j\} =$$

Casimir functions:

$$C_1 = |x|^2 \equiv \sum_{k=1}^3 x_k^2, \qquad C_2 = (x, J) \equiv \sum_{k=1}^3 x_k J_k$$

Systems in question defined by the Hamilton functions

$$H_{1} = J_{1}^{2} + J_{2}^{2} + 2J_{3}^{2} + 2ax_{1} - \frac{\lambda C_{1}}{x_{3}^{2}} + \frac{c}{\sqrt{x_{1}^{2} + x_{2}^{2}}} + \frac{2C_{1} - x_{3}^{2}}{x_{2}^{2}} \left(d + \frac{c}{\sqrt{x_{1}^{2} + x_{2}^{2}}} \right)$$

and

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)$$

The first system becomes the Kovalevskaya top in the case $\lambda = c = d = e = 0$. The second system was discussed by Chaplygin and Goryachev if $\lambda \neq 0$. For these two systems there are additional integrals of motion (Yehia 2006). We build the separation variables and separated equations from these integrals of motion.

Separation of variables

Consider a dynamical system on a smooth manifold with coordinates x_1, \ldots, x_m defined by equations of motion

$$\dot{x}_i = X_i, \qquad i = 1, \ldots, m$$

From these equations we can switch to a vector field

$$X = \sum X_i \frac{\partial}{\partial x_i}$$

The Hamiltonian of the system can be introduced, defining all the dynamics together with the Poisson bivector P

$$X = PdH$$

Bi-Hamiltonian manifolds (Magri 1978)

- Two Poisson bi-vectors P, P' satisfying compatibility condition [P, P'] = 0
- Can be used to find integrals $X = PdH_1 = P'dH_2$
- Or if we know integrals in involution $X = g_1 X_1 + \cdots + g_n X_n$, $X_k = P' dH_k$ then we can do the separation of variables (Falqui & Pedroni 2003)

Separation of variables

Recursion operator

$$N = P'P^{-1}$$

Eigenvalues are separation variables

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Most of known additional Poisson bi-vectors have the form $P' = \mathcal{L}_Y P$

Finding *Y*

$$(dH_k, \mathcal{L}_Y P dH_m) = 0, \qquad [\mathcal{L}_Y P dH_m]$$

Written in separation variables the solutions are

$$Y_j=0, \quad Y_{n+j}=f_j(q_j,p_j),$$

We need to narrow the solutions set by imposing constra Let us make some assumptions about *Y*

• Introduce vector field A which leads to Y = AX = A

$$A = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \quad \text{or} \quad A$$

• Hamiltonian of natural form (H = T + V) leads to Overview of the method

We start with H_1 and H_2

- Write the equations for the components of P'
- Output Use ansatze and solve for components of Y
- **3** Calculate the Poisson bivector P'
- Build separation variables and separated equations

Application example

Consider the generalized Chaplygin system We start with the Hamiltonian

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right)$$

and the integral of motion

$$H_{2} = \left(J_{1}^{2} - J_{2}^{2} - 2bx_{3}^{2} - \frac{\lambda(x_{1}^{2} - x_{2}^{2})}{x_{3}^{2}}\right)^{2} + \left(2J_{1}J_{2} - \frac{2\lambda x_{1}x_{2}}{x_{3}^{2}}\right)^{2}$$
$$-2(J_{1}^{2} + J_{2}^{2})\left(\frac{c(x_{1}^{2} + x_{2}^{2})}{x_{3}^{6}} + \frac{dx_{3}^{2}}{x_{1}^{2}} + \frac{ex_{3}^{2}}{x_{2}^{2}}\right) + x_{3}^{4}\left(\frac{c(x_{1}^{2} + x_{2}^{2})}{x_{3}^{8}} + \frac{d}{x_{1}^{2}} + \frac{e}{x_{2}^{2}}\right)^{2}$$
$$-4b\left(\frac{c(x_{1}^{2} - x_{2}^{2})}{x_{3}^{4}} + x_{3}^{4}\left(\frac{d}{x_{1}^{2}} - \frac{e}{x_{2}^{2}}\right)\right) - 2\lambda\left(\frac{c(x_{1}^{2} + x_{2}^{2})^{2}}{x_{3}^{8}} - \frac{dx_{2}^{2}}{x_{1}^{2}} - \frac{ex_{1}^{2}}{x_{2}^{2}}\right)$$

The case
$$c = 0$$
 is easy to address.
Using ansatze for the components of *Y* we find that it i

$$T_T = \begin{pmatrix} -\frac{2\cos 2\theta}{\sin 2\theta} p_{\phi} p_{\theta} \\ -\cot \theta p_{\phi}^2 + \tan \theta p_{\theta}^2 \end{pmatrix}$$

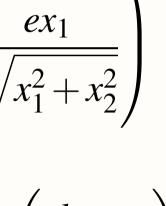
and
$$Y_V$$
 has a term for each of the constants

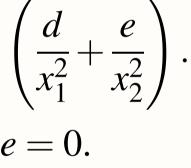
$$Y_{V} = b \begin{pmatrix} \cos 2\phi p_{\phi} + \sin 2\phi \tan \theta p_{\theta} \\ \sin 2\phi \cot \theta p_{\phi} - \cos 2\phi p_{\theta} \end{pmatrix} - \frac{d p_{\theta}}{\sin^{2} \phi} \begin{pmatrix} \frac{2 \cot \phi}{\sin 2\theta} \\ \frac{1}{\sin^{2} \theta} \end{pmatrix} + \frac{e p_{\theta}}{\cos^{2} \phi} \begin{pmatrix} \frac{2 \tan \phi}{\sin 2\theta} \\ -\frac{1}{\sin^{2} \theta} \end{pmatrix}$$

we case $c = 0$ separation variables are the roots of the characteristic polynomial $det(N - \mu I) = P^{2}(\mu)$

In th $det(N - \mu I) = B^{2}(\mu)$

GDIS 2014, 16 - 27 June 2014, Trieste, Italy





$$[\mathcal{L}_Y P] = 0$$

$$j = 1..n$$

$$AP \, dH = \begin{pmatrix} \Pi & 0 \\ 0 & \Lambda \end{pmatrix}$$
$$Y = Y_T + Y_V$$

is a sum of Y_T and Y_V :

where

$$B(\mu) = \mu^2 - \left(\frac{J_1^2 + J_2^2}{x_3^2} - \frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\mu + \frac{b(J_1^2 - J_2^2)}{x_3^2} + \left(\frac{d}{x_1^2} - \frac{e}{x_2^2}\right)b - b^2 = (\mu - q_1)(\mu - q_2).$$

The momenta can be constructed from the polynomial

$$A(\mu) = \left(\frac{x_2J_1 - x_1J_2}{x_3}\right) \frac{\mu}{2} - \frac{b}{2} \left(\frac{x_2J_1 + x_1J_2}{x_3}\right)$$
$$= -\frac{\mu}{2} \tan \theta \, p_\theta - \frac{b}{2} \left(\sin 2\phi \, p_\phi - \cos 2\phi \tan \theta \, p_\theta\right).$$
quations for separation variables in H_1 and H_2 we get
$$\Phi(q_j, p_j) = 0, \qquad j = 1, 2$$

Substituting the eq

where

$$\begin{split} \Phi(q,p) &= \left(8(q^2 - b^2)p^2 - 2q + H_1 - \sqrt{H_2} - \frac{4b\left(q(d-e) + b(d+e)\right)}{q^2 - b^2}\right) \times \\ &\times \left(8(q^2 - b^2)p^2 - 2q + H_1 + \sqrt{H_2} - \frac{4b\left(q(d-e) + b(d+e)\right)}{q^2 - b^2}\right) + 4\lambda q = 0 \end{split}$$

This is a genus two hyperelliptic curve; if $\lambda = 0$, it splits into two elliptic curves. In the case $c \neq 0$ we need a canonical transformation

 $p_{\boldsymbol{\theta}}$

after which we introduce a new function

$$\widetilde{\Phi}(q,p) = \Phi(q,p) - 16\sqrt{c}(q^2 - b^2)p$$

Equation $\Phi(q, p) = 0$ defines a genus three algebraic curve. the Chaplygin system.

- integrals of motion.
- separation variables is straighforward.
- Where to head next
- How to find the ansatze for a given system?
- What to do with the separated equations?
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$$\rightarrow p_{\theta} + \frac{\sqrt{c}\sin\theta}{\cos^{3}\theta}$$

So, we have separation variables and separated equations for the integrable deformation of

Summary

• This method still needs the ansatze to work and how to find it is yet to be determined. • We produce separation variables and separated equations starting strating just from

• The results agree with those of Kovalevskaya but the procedure for constructing

References

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