Separation of variables for some systems with a fourth-order integral of motion

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Outline

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Phase Space

Physical coordinates:

$$x = (x_1, x_2, x_3) \quad \text{and} \quad J = (J_1, J_2, J_3)$$

Lie-Poisson bracket:

$$\big\{ J_i \,, J_j \,\big\} = \varepsilon_{ijk} J_k \,, \qquad \big\{ J_i \,, x_j \,\big\} = \varepsilon_{ijk} x_k \,, \qquad \big\{ x_i \,, x_j \,\big\} = 0$$

Casimir functions:

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = (x, J) \equiv \sum_{k=1}^{3} x_k J_k$$

Systems in Question

Defined by the Hamilton functions

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2ax_1 - \frac{\lambda C_1}{x_3^2} + \frac{c}{\sqrt{x_1^2 + x_2^2}} + \frac{2C_1 - x_3^2}{x_2^2}\left(d + \frac{ex_1}{\sqrt{x_1^2 + x_2^2}}\right)$$

and

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + c\left(\frac{1}{x_3^4} - \frac{1}{x_3^6}\right) - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right).$$

The first system becomes the Kovalevskaya top in the case $\lambda = c = d = e = 0$.
The second system was discussed by Chaplygin and Goryachev if $\lambda \neq 0$.

Integrals of Motion

For these two systems there are additional integrals of motion (Yehia 2006).
For instance, for the generalized Kovalevskaya system it looks like

$$H_2 = \left(J_1^2 - J_2^2 - 2ax_1 + \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - ax_2 + \frac{2\lambda x_1 x_2}{x_3^2}\right)^2$$
$$+ \frac{1}{x_2^4}\left(dx_3^2 + \frac{cx_2^2 + ex_3^2x_1}{\sqrt{x_1^2 + x_2^2}}\right)\left(2x_2^2(J_1^2 + J_2^2) + dx_3^2 + \frac{cx_2^2 + ex_3^2x_1}{\sqrt{x_1^2 + x_2^2}}\right)$$
$$- \frac{4ax_3^2(dx_1 + e\sqrt{x_1^2 + x_2^2})}{x_2^2} - \frac{2\lambda}{x_2^2}\left(\frac{\sqrt{x_1^2 + x_2^2)}(cx_2^2 - ex_3^2x_1)}{x_3^2} - dx_1^2\right)$$

We are going to build the separation variables and separated equations using only these integrals of motion.

Hamiltonian Mechanics

Consider a dynamical system on a smooth manifold with coordinates $x_1, \ldots, x_m$ defined by equations of motion

$$\dot{x}_i = X_i, \qquad i = 1, \ldots, m$$

From these equations we can switch to a vector field

$$X = \sum X_i \frac{\partial}{\partial x_i}$$

The Hamiltonian of the system can be introduced, defining all the dynamics together with the Poisson bivector P

$$X = PdH$$

Bi-Hamiltonian Manifolds

Bi-Hamiltonian manifolds (Magri 1978)

- Two Poisson bi-vectors

$$P, P'$$

satisfying compatibility condition

$$[P, P] = 0, \qquad [P, P'] = 0, \qquad [P', P'] = 0$$

- Can be used to find integrals

$$X = PdH_1 = P'dH_2$$

- Or if we know integrals in involution

$$X = g_1 X_1 + \cdots + g_n X_n$$

$$X_k = P'dH_k$$

Separation of variables (Falqui & Pedroni 2003)

Separation of variables

Recursion operator

$$N = P' P^{-1}$$

Eigenvalues are separation variables

Most of known additional Poisson bi-vectors are

$$P' = \mathcal{L}_Y P$$

Finding Y

$$(dH_k, \mathcal{L}_Y P dH_m) = 0, \qquad [\mathcal{L}_Y P, \mathcal{L}_Y P] = 0$$

Written in separation variables the solutions are

$$Y_j = 0, \quad Y_{n+j} = f_j(q_j, p_j), \quad j = 1..n$$

Need to narrow the solutions set by imposing constraints

Ansatze for P'

Let us make some assumptions about Y

- Introduce vector field A

$$Y = AX = AP\,dH$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} \Pi & 0 \\ 0 & \Lambda \end{pmatrix}$$

- Hamiltonian of natural form $(H = T + V)$

$$Y = Y_T + Y_V$$

Overview of the Method

We start with H_1 and H_2

1. Write the equations for the components of P'
2. Use ansatze and solve for components of Y
3. Calculate the Poisson bivector P'
4. Build separation variables and separated equations

Outline

Application example

Generalized Kovalevskaya system

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2ax_1 - \frac{\lambda C_1}{x_3^2} + \frac{c}{\sqrt{x_1^2 + x_2^2}} + \frac{2C_1 - x_3^2}{x_2^2}\left(d + \frac{ex_1}{\sqrt{x_1^2 + x_2^2}}\right)$$

$$H_2 = \left(J_1^2 - J_2^2 - 2ax_1 + \frac{\lambda(x_1^2 - x_2^2)}{x_3^2}\right)^2 + \left(2J_1J_2 - ax_2 + \frac{2\lambda x_1 x_2}{x_3^2}\right)^2$$
$$+ \frac{1}{x_2^4}\left(dx_3^2 + \frac{cx_2^2 + ex_3^2 x_1}{\sqrt{x_1^2 + x_2^2}}\right)\left(2x_2^2(J_1^2 + J_2^2) + dx_3^2 + \frac{cx_2^2 + ex_3^2 x_1}{\sqrt{x_1^2 + x_2^2}}\right)$$
$$- \frac{4ax_3^2(dx_1 + e\sqrt{x_1^2 + x_2^2})}{x_2^2} - \frac{2\lambda}{x_2^2}\left(\frac{\sqrt{x_1^2 + x_2^2}(cx_2^2 - ex_3^2 x_1)}{x_3^2} - dx_1^2\right)$$

Coordinate Transformation

Switch to spherical coordinates ϕ, θ and momenta p_ϕ, p_θ

$$\{\phi, p_\phi\} = \{\theta, p_\theta\} = 1\,, \qquad \{\phi, \theta\} = \{p_\phi, p_\theta\} = \{\phi, p_\theta\} = \{\theta, p_\phi\} = 0$$

Casimir functions

$$C_1 = (x, x) = 1\,, \qquad C_2 = (x, J) = 0$$

Canonical Poisson bi-vector

$$P = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}$$

Components of vector field Y

Solution of the equations for compatibility condition

$$Y = \begin{pmatrix} Y_T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Y_V \end{pmatrix} = \begin{pmatrix} Y_T \\ Y_V \end{pmatrix},$$

Splits into two parts

$$Y_T = \begin{pmatrix} \left(\dfrac{1}{\cos\theta} + \ln\dfrac{1-\cos\theta}{\sin\theta}\right) p_\phi p_\theta \\ \\ \dfrac{1}{2}\ln\dfrac{1-\cos\theta}{\sin\theta}\, p_\phi^2 + \dfrac{1}{2\cos\theta} p_\theta^2 \end{pmatrix}$$

One term for each constant

$$Y_V = \frac{a}{2}\begin{pmatrix} \sin\phi\, p_\phi + \cos\phi\tan\theta\, p_\theta \\ \\ -\cos\phi\cot\theta\, p_\phi - \sin\phi\, p_\theta \end{pmatrix} - \frac{d\,p_\theta}{\cos^2\phi}\begin{pmatrix} \dfrac{\tan\phi}{\cos\theta} \\ \\ -\dfrac{1}{2\sin\theta} \end{pmatrix} + \frac{e\,p_\theta}{\cos^2\phi}\begin{pmatrix} \dfrac{cos^2\phi - 2}{\cos\phi\cos\theta} \\ \\ \dfrac{\sin\phi}{\sin\theta} \end{pmatrix}$$

Separation Variables

Recursion operator N constructed from P and P'

- Separation variables — roots of polynomial $B(\mu)$

$$\det(N-\mu I)=B^2(\mu)$$

$$\begin{aligned}B(\mu)&=\mu^2+\left(\frac{\sqrt{x_1^2+x_2^2}(J_1^2+J_2^2)}{x_3^2}+\frac{ex_1+d\sqrt{x_1^2+x_2^2}}{x_2^2}\right)\mu\\&\quad-\frac{a(ax_3^2+2x_1J_1^2-2x_1J_2^2+4x_2J_1J_2)}{2x_3^2}-\frac{a(dx_1+e\sqrt{x_1^2+x_2^2})}{x_2^2}\\&=(\mu-q_1)(\mu-q_2)\end{aligned}$$

- Conjugate momenta — can be calculated from $A(\mu)$

$$\{B(\nu),A(\mu)\}=\frac{1}{\mu-\nu}\Big((\mu^2-b^2)B(\nu)-(\nu^2-b^2)B(\mu)\Big),\qquad \{A(\nu),A(\mu)\}=0$$

$$A(\mu)=\frac{(\mu-q_1)p_2(q_2^2-a^2)}{q_2-q_1}+\frac{(\mu-q_2)p_1(q_1^2-b^2)}{q_1-q_2}=-\frac{x_1J_2-x_2J_1}{x_3}\mu-\frac{a\sqrt{x_1^2+x_2^2}\,J_2}{x_3}$$

Separated Equations

Substituting separation variables into H_1, H_2 we get

$$\Phi(q_j, p_j) = 0, \qquad j = 1, 2$$

$$\Phi(q,p) = \left(2(q^2 - a^2)p^2 + H_1 + \sqrt{H_2} + 2a\frac{da - eq}{u^2 - a^2}\right) \times \\ \times \left(2(q^2 - a^2)p^2 + H_1 - \sqrt{H_2} + 2a\frac{da - eq}{u^2 - a^2}\right) - 4q^2 + 4cq$$

$\lambda \neq 0$ case: additional transformation

$$p_\theta \to p_\theta + \frac{\sqrt{\lambda}}{\cos\theta}\,.$$

$$\widetilde{\Phi}(q,p) = \Phi(q,p) - 8\sqrt{\lambda}(q^2 - a^2)p = 0$$

Separated Equations

The equations of motion in separated variables are

$$\frac{\dot{q}_1}{p_1\left((a^2-q_1^2)\big(2(q_1^2-a^2)p_1^2+H_1\big)-2a(ad-eq_1)\right)-\sqrt{\lambda}(a^2-q_1^2)}+$$

$$+\frac{\dot{q}_2}{p_2\left((a^2-q_2^2)\big(2(q_2^2-a^2)p_2^2+H_1\big)-2a(ad-eq_2)\right)-\sqrt{\lambda}(a^2-q_2^2)}=0\,,$$

$$\frac{\left(a(ad-eq_1)+\left(a^2-q_1^2\right)^2p_1^2\right)\dot{q}_1}{(a^2-q_1^2)\left[p_1\left((a^2-q_1^2)\big(2(q_1^2-a^2)p_1^2+H_1\big)-2a(ad-eq_1)\right)-\sqrt{\lambda}(a^2-q_1^2)\right]}+$$

$$+\frac{\left(a(ad-eq_2)+\left(a^2-q_2^2\right)^2p_2^2\right)\dot{q}_2}{(a^2-q_2^2)\left[p_1\left((a^2-q_2^2)\big(2(q_2^2-a^2)p_2^2+H_2\big)-2a(ad-eq_2)\right)-\sqrt{\lambda}(a^2-q_2^2)\right]}=-2$$

Outline

Generalized Chaplygin System

As before, we start with the Hamiltonian

$$H_1 = J_1^2+J_2^2+2J_3^2-2b(x_1^2-x_2^2)+\frac{\lambda C_1}{x_3^2}+c\left(\frac{1}{x_3^4}-\frac{1}{x_3^6}\right)-(2C_1-x_3^2)\left(\frac{d}{x_1^2}+\frac{e}{x_2^2}\right)$$

and the integral of motion

$$\begin{aligned} H_2 = &\left(J_1^2-J_2^2-2bx_3^2-\frac{\lambda(x_1^2-x_2^2)}{x_3^2}\right)^2+\left(2J_1J_2-\frac{2\lambda x_1x_2}{x_3^2}\right)^2 \\ &-2(J_1^2+J_2^2)\left(\frac{c\,(x_1^2+x_2^2)}{x_3^6}+\frac{dx_3^2}{x_1^2}+\frac{ex_3^2}{x_2^2}\right)+x_3^4\left(\frac{c(x_1^2+x_2^2)}{x_3^8}+\frac{d}{x_1^2}+\frac{e}{x_2^2}\right)^2 \\ &-4b\left(\frac{c(x_1^2-x_2^2)}{x_3^4}+x_3^4\left(\frac{d}{x_1^2}-\frac{e}{x_2^2}\right)\right)-2\lambda\left(\frac{c(x_1^2+x_2^2)^2}{x_3^8}-\frac{dx_2^2}{x_1^2}-\frac{ex_1^2}{x_2^2}\right) \end{aligned}$$

Generalized Chaplygin System

The case $c = 0$ is easy to address.
Using ansatze for the components of Y we find

$$Y = \begin{pmatrix} Y_T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Y_V \end{pmatrix} = \begin{pmatrix} Y_T \\ Y_V \end{pmatrix}$$

where

$$Y_T = \begin{pmatrix} -\dfrac{2\cos 2\theta}{\sin 2\theta} p_\phi p_\theta \\ \\ -\cot\theta p_\phi^2 + \tan\theta p_\theta^2 \end{pmatrix}$$

and there is a term for each of the constants

$$Y_V = b \begin{pmatrix} \cos 2\phi p_\phi + \sin 2\phi \tan\theta p_\theta \\ \\ \sin 2\phi \cot\theta p_\phi - \cos 2\phi p_\theta \end{pmatrix} - \frac{d\, p_\theta}{\sin^2\phi} \begin{pmatrix} \dfrac{2\cot\phi}{\sin 2\theta} \\ \\ \dfrac{1}{\sin^2\theta} \end{pmatrix} + \frac{e\, p_\theta}{\cos^2\phi} \begin{pmatrix} \dfrac{2\tan\phi}{\sin 2\theta} \\ \\ -\dfrac{1}{\sin^2\theta} \end{pmatrix}$$

Poisson Brackets

$$\{\phi,\theta\}' = \frac{p_\phi}{\cos\theta\sin\theta}\,, \qquad \{\phi,p_\phi\}' = -2b(\cos^2\phi - 2)\,,$$

$$\{\phi,p_\theta\}' = -\frac{p_\phi\,p_\theta}{\cos^2\theta\sin^2\theta} - \frac{b\sin 2\phi\cos\theta}{\sin\theta} - \frac{2\sqrt{c}(\cos^2\theta - 2)\,p_\phi}{\sin\theta\cos^5\theta}\,,$$

$$\{\theta,p_\phi\}' = -\frac{b\sin 2\phi\sin\theta}{\cos\theta} + \frac{1}{\sin\theta\cos\theta}\left(\frac{d\cos\phi}{\sin^3\phi} - \frac{e\sin\phi}{\cos^3\phi}\right)\,,$$

$$\{\theta,p_\theta\}' = -\frac{p_\theta^2}{\cos^2\theta} + 2b(\cos^2\phi + 1) - \frac{1}{\sin^2\theta}\left(p_\phi^2 - \frac{d}{\sin^2\phi} - \frac{e}{\cos^2\phi}\right)$$

$$+\frac{2\sqrt{c}\sin\theta\,p_\theta}{\cos^5\theta} - \frac{c\sin^2\theta}{\cos^8\theta},$$

$$\{p_\phi,p_\theta\}' = b\left(\frac{2\cos 2\phi\cos\theta\,p_\phi}{\sin\theta} + \frac{\sin 2\phi\cos 2\theta\,p_\theta}{\cos^2\theta}\right) + \frac{p_\theta}{\cos^2\theta\sin^2\theta}\left(\frac{d\cos\phi}{\sin^3\phi} - \frac{e\sin\phi}{\cos^3\phi}\right)$$

$$+\frac{\sqrt{c}}{\cos^5\theta}\left(4b\sin 2\phi\sin^3\theta + \frac{2(\cos^2\theta - 2)}{\sin\theta}\left(\frac{d\cos\phi}{\sin^3\phi} - \frac{e\sin\phi}{\cos^3\phi}\right)\right).$$

Separation Variables

In the case $c = 0$ separation variables are the roots of the characteristic polynomial

$$\det(N - \mu I) = B^2(\mu)$$

where

$$B(\mu) = \mu^2 - \left(\frac{J_1^2 + J_2^2}{x_3^2} - \frac{d}{x_1^2} - \frac{e}{x_2^2}\right)\mu + \frac{b(J_1^2 - J_2^2)}{x_3^2} + \left(\frac{d}{x_1^2} - \frac{e}{x_2^2})\right) b - b^2$$
$$= (\mu - q_1)(\mu - q_2).$$

The momenta can be constructed from the polynomial

$$A(\mu) = \left(\frac{x_2 J_1 - x_1 J_2}{x_3}\right)\frac{\mu}{2} - \frac{b}{2}\left(\frac{x_2 J_1 + x_1 J_2}{x_3}\right)$$
$$= -\frac{\mu}{2}\tan\theta\, p_\theta - \frac{b}{2}\Big(\sin 2\phi\, p_\phi - \cos 2\phi \tan\theta\, p_\theta\Big).$$

like

$$p_j = \frac{1}{q_j^2 - b^2}\, A(\mu = q_j), \qquad j = 1, 2.$$

Separation Variables

Original variables written in separation variables look like

$$
\begin{aligned}
x_1 &= \sqrt{\frac{2(q_2-b)(b-q_1)\big(p_1(q_1+b)-p_2(q_2+b)\big)^2}{b(q_2-q_1)^2}-\frac{2bd}{(q_2-b)(b-q_1)}}\,,\\
x_2 &= \sqrt{\frac{2(q_2+b)(b+q_1)\big(p_1(q_1-b)-p_2(q_2-b)\big)^2}{b(q_2-q_1)^2}-\frac{2be}{(q_2+b)(b+q_1)}}\,,\\
x_3 &= \sqrt{1-x_1^2-x_2^2}\,,\\
J_1 &= x_3\sqrt{\frac{(b+q_1)(q_2+b)}{2b}+\frac{e}{x_2^2}}\,,\\
J_2 &= x_3\sqrt{\frac{(b-q_1)(q_2-b)}{2b}+\frac{d}{x_1^2}}\,,\\
J_3 &= -\frac{x_1J_1+x_2J_2}{x_3}\,.
\end{aligned}
$$

Separated Equations

Substituting the equation for separation variables in H_1 and H_2 we get

$$\Phi(q_j, p_j) = 0, \qquad j = 1, 2$$

where

$$\Phi(q,p) = \left(8(q^2-b^2)p^2 - 2q + H_1 - \sqrt{H_2} - \frac{4b\big(q(d-e)+b(d+e)\big)}{q^2-b^2}\right)\times$$
$$\times\left(8(q^2-b^2)p^2 - 2q + H_1 + \sqrt{H_2} - \frac{4b\big(q(d-e)+b(d+e)\big)}{q^2-b^2}\right) + 4\lambda q = 0$$

This is a genus two hyperelliptic curve.
It $\lambda = 0$, it splits into two elliptic curves.

$c \neq 0$ case

In the case $c \neq 0$ we need a canonical transformation

$$p_\theta \to p_\theta + \frac{\sqrt{c}\sin\theta}{\cos^3\theta}$$

Substituting the separation variables into equation

$$\widetilde{\Phi}(q,p) = \Phi(q,p) - 16\sqrt{c}\,(q^2 - b^2)p$$

we get a Hamiltonian

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2b(x_1^2 - x_2^2) + \frac{\lambda C_1}{x_3^2} + \frac{2(x_2 J_1 - x_1 J_2)\sqrt{c}}{x_3^3} - (2C_1 - x_3^2)\left(\frac{d}{x_1^2} + \frac{e}{x_2^2}\right),$$

which matches the original Hamiltonian after a transformation

$$J_1 = J_1 - \frac{\sqrt{c}\,x_2}{x_3^3}, \qquad J_1 = J_2 + \frac{\sqrt{c}\,x_1}{x_3^3}.$$

Separated Equations

Equation

$$\widetilde{\Phi}(q,p)=0$$

defines a genus three algebraic curve.
Quadratures can be written in integral form

$$\int_{q_0}^{q_1}\Omega_1+\int_{q_0}^{q_2}\Omega_1=\beta_1\,,\qquad \int_{q_0}^{q_1}\Omega_2+2\Omega_3+\int_{q_0}^{q_2}\Omega_2+2\Omega_3=-4t+\beta_2$$

So, we have separation variables and separated equations for the integrable deformation of the Chaplygin system.

Summary

- This method still needs the ansatze to work and how to find it is yet to be determined.
- We build separation variables and separated equations starting just from integrals of motion.
- The results agree with the Kovalevskaya top but building separation variables is straighforward.

- Where to head next
  - How to find the ansatze for a given system?
  - What to do with the separated equations?