COMMUTATORS OF ELEMENTARY SUBGROUPS: CURIOUSER AND CURIOUSER

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Abstract. Let R be any associative ring with 1, $n \geq 3$, and let A, B be two-sided ideals of R. In our previous joint works with Roozbeh Hazrat [17], [15], we have found a generating set for the mixed commutator subgroup $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$. Later in [29], [34] we noticed that our previous results can be drastically improved and that $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$ is generated by

(1) the elementary conjugates $z_{ij}(ab,c) = t_{ij}(c)t_{ji}(ab)t_{ij}(-c)$ and $z_{ij}(ba,c)$, and

(2) the elementary commutators $[t_{ij}(a), t_{ji}(b)]$,

where $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$, $c \in R$. Later in [33], [35] we noticed that for the second type of generators, it even suffices to fix one pair of indices (i,j). Here we improve the above result in yet another completely unexpected direction and prove that $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$ is generated by the elementary commutators $[t_{ij}(a), t_{hk}(b)]$ alone, where $1 \leq i \neq j \leq n$, $1 \leq h \neq k \leq n$, $a \in A$, $b \in B$. This allows us to revise the technology of relative localisation and, in particular, to give very short proofs for a number of recent results, such as the generation of partially relativised elementary groups $\mathrm{E}(n,A)^{\mathrm{E}(n,B)}$, multiple commutator formulas, commutator width, and the like.

Introduction

In the present note, we generalize the results by Roozbeh Hazrat and the authors [17], [15], [29] on generation of mixed commutator subgroups of relative and unrelative elementary subgroups in the general linear group in a completely unexpected direction.

Let R be an associative ring with 1, and let $\mathrm{GL}(n,R)$ be the general linear group of degree $n \geq 3$ over R. As usual, e denotes the identity matrix, whereas e_{ij} denotes a standard matrix unit. For $c \in R$ and $1 \leq i \neq j \leq n$, we denote by $t_{ij}(c) = e + ce_{ij}$ the corresponding *elementary transvection*. To an ideal $A \trianglelefteq R$, we assign the elementary subgroup

$$\mathrm{E}(n,A) = \langle t_{ij}(a),\ a \in A,\ 1 \leq i \neq j \leq n \rangle.$$

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The corresponding *relative* elementary subgroup $\mathrm{E}(n, R, A)$ is defined as the normal closure of $\mathrm{E}(n, A)$ in the absolute elementary subgroup $\mathrm{E}(n, R)$. From the work of Michael Stein, Jacques Tits, and Leonid Vaserstein it is classically known that *as a group* $\mathrm{E}(n, R, A)$ is generated by the *elementary conjugates*

$$z_{ij}(a, c) = t_{ji}(c)t_{ij}(a)t_{ji}(-c),$$

where $1 \leq i \neq j \leq n$, $a \in A$, $c \in R$.

For $\mathrm{GL}(n, R)$ and $A, B \trianglelefteq R$, the study of the mixed commutator subgroups such as

$$[\mathrm{GL}(n, R, A), \mathrm{GL}(n, R, B)], \quad [\mathrm{GL}(n, R, A), \mathrm{E}(n, R, B)], \quad [\mathrm{E}(n, R, A), \mathrm{E}(n, R, B)]$$

and other related birelative groups goes back to the ground-breaking work of Hyman Bass [4], and was then continued, again at the stable level, by Alec Mason and Wilson Stothers [20], [19], etc. For rings subject to commutativity conditions, this was then resumed and expanded in several directions first by Hong You [39], and then in our joint papers with Roozbeh Hazrat and Alexei Stepanov — see, for instance, [31], [16], [32], [7], [11], [12], [17], [9], [13], [14], [15]. Those papers relied on the very powerful methods proposed by Andrei Suslin, Zenon Borewicz and ourselves, Leonid Vaserstein, Tony Bak, and others to prove standard commutator formulas in the absolute case. See [27], [5], [28], [25], [2], etc. and consult [10] for a detailed survey.

A first version of following result was discovered (in a slightly less precise form) by Roozbeh Hazrat and the second author (see [17, Lem. 12]). In exactly this form it is stated in our paper [15, Thm. 3A]. The second type of generators below are the *elementary commutators*

$$y_{ij}(a, b) = [t_{ij}(a), t_{ji}(b)]$$

$1 \leq i \neq j \leq n$, $a \in A$, $b \in B$. They already belong to the mixed commutator of the corresponding *unrelativised* subgroups $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$.

Recall that a ring R is called *module-finite* if it is finitely generated as a module over its center. *Quasi-finite* rings are direct limits of inductive systems of module-finite rings (see [2] for details).

Theorem A. *Let R be a quasi-finite ring with* 1*, let $n \geq 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup $[\mathrm{E}(n, R, A), \mathrm{E}(n, R, B)]$ is generated as a group by the elements of the form*

- $z_{ij}(ab, c)$ *and* $z_{ij}(ba, c)$,
- $y_{ij}(a, b)$,
- $[t_{ij}(a), z_{ij}(b, c)]$,

where $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$, $c \in R$.

Here, the elementary conjugates generate $\mathrm{E}(n, R, A \circ B)$, where $A \circ B = AB + BA$ is the symmetrised product of ideals. Subsequently, we proved the following result, which is both *terribly* much stronger, and much more general than Theorem A and which completely solves [15, Problem 1] for the case of $\mathrm{GL}(n, R)$. First, in [29], [34] we noticed that the third type of generators are redundant. Then in [30], [33], [35] we observed that modulo $\mathrm{E}(n, R, A \circ B)$ suffices to take elementary commutators for a single position, and that everything works over *arbitrary* associative rings.

Theorem B. *Let R be any associative ring with 1, let $n \geq 3$, and let A, B be two-sided ideals of R. Then the mixed commutator subgroup $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$ is generated as a group by the elements of the form*
- $z_{ij}(ab, c)$ *and* $z_{ij}(ba, c)$,
- $y_{ij}(a, b)$,

where $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$, $c \in R$. Moreover, for the second type of generators, it suffices to fix one pair of indices (i, j).

In particular, this last result implies that for $n \geq 3$ one has

$$[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)] = [\mathrm{E}(n,A), \mathrm{E}(n,B)]$$

and in the body of the paper we usually prefer the shorter notation.

In the present paper, we generalise both Theorem A, and the first claim of Theorem B, in a completely different *unexpected* direction. Namely, we prove that instead of limiting the stock of elementary commutators, one can limit the stock of elementary conjugates! The following theorem asserts that $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$ is generated by the elementary commutators not just over $\mathrm{E}(n,R,A\circ B)$, as in our previous papers [33], [34], [35], but already over the unrelativised elementary subgroup $\mathrm{E}(n,A\circ B)$.

Theorem 1. *Let A and B be two ideals of an associative ring R and let $n \geq 3$. Then the mixed commutator subgroup $[\mathrm{E}(n,R,A), \mathrm{E}(n,R,B)]$ is generated by the elementary commutators $[t_{ij}(a), t_{hk}(b)]$, where $1 \leq i \neq j \leq n$, $1 \leq h \neq k \leq n$, $a \in A$ and $b \in B$.*

We find this result truly astounding. In fact, it is well known that the relative elementary subgroups $\mathrm{E}(n,R,A)$ themselves are not generated by the elementary transvections $t_{ij}(a)$, where $1 \leq i \neq j \leq n$, $a \in A$. Why is it that their mixed commutator subgroups are?

Actually, Theorem 1 immediately implies many remarkable corollaries. In particular, it allows us to consider elementary commutators $y_{ij}(a,b)$, $a \in A$, $b \in B$, modulo the *true* (= unrelative) elementary subgroup $\mathrm{E}(n,A\circ B)$ of level $A \circ B$, and not modulo the corresponding relative elementary subgroup $\mathrm{E}(n,R,A\circ B)$ as we were doing in [33], [35], [36].

The balance of the paper is organised as follows. In §1 we recall some basic facts concerning elementary subgroups and their mixed commutator subgroups. In §2 we prove Lemma 6 that allows us to move around elementary commutators. It is essentially a slight improvement of our results from [33], [35], [36] that is not directly needed to prove Theorem 1 itself, but serves as a model, and is crucial to establish Theorem 4. After that, in §3 we prove Lemma 7, which is essentially a slightly more precise form of both of level computations and Theorem 1. The rest of the paper are refinements and applications.

In §4 we derive a generation result for partially relativised elementary groups $\mathrm{E}(n,B,A) = \mathrm{E}(n,A)^{\mathrm{E}(n,B)}$, where A and B are two ideals (Theorem 2).

In §5 we prove that already the true elementary subgroup $\mathrm{E}(n,A\circ B)$ is normal in the mix commutator subgroup $[\mathrm{E}(n,A), \mathrm{E}(n,B)]$, with abelian quotient group $[\mathrm{E}(n,A), \mathrm{E}(n,B)]/\,\mathrm{E}(n,A\circ B)$ (Theorem 3).

In §6 we notice that Theorem 1 admits a remarkable generalisation. Namely the generating set therein can be further substantially reduced, allowing only elementary commutators corresponding to some positions in the unipotent radical of a maximal parabolic subgroup plus the elementary commutators in *one more* position (Theorem 4).

In §7 we apply the above results to the generation of mixed commutator subgroups. Modulo the results of our previous paper [35], and we are now in a position to exhibit very economical generating sets for the multiple commutator subgroups $[\![\mathrm{E}(n, I_1), \ldots, \mathrm{E}(n, I_m)]\!]$ (Theorem 5).

Finally, in §8 we briefly mention some further applications and unsolved problems.

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1. Relative subgroups

Let $G = \mathrm{GL}(n, R)$ be the general linear group of degree n over an associative ring R with 1. In the sequel for a matrix $g \in G$, we denote by g_{ij} its matrix entry in the position (i, j), so that $g = (g_{ij})$, $1 \le i, j \le n$. The inverse of g will be denoted by $g^{-1} = (g'_{ij})$, $1 \le i, j \le n$.

Let A be a two-sided ideal of R. We consider the corresponding reduction homomorphism:

$$\pi_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A), \quad (g_{ij}) \mapsto (g_{ij} + A).$$

Now, the *principal congruence subgroup* $\mathrm{GL}(n, R, A)$ of level A is the kernel of π_A.

The *unrelative elementary subgroup* $\mathrm{E}(n, A)$ of level A in $\mathrm{GL}(n, R)$ is generated by all elementary matrices of level A. In other words,

$$\mathrm{E}(n, A) = \langle t_{ij}(a),\ 1 \le i \ne j \le n,\ a \in A \rangle.$$

In general, $\mathrm{E}(n, A)$ has little chance to be normal in $\mathrm{GL}(n, R)$. The *relative elementary subgroup* $\mathrm{E}(n, R, A)$ of level A is defined as the normal closure of $\mathrm{E}(n, A)$ in the absolute elementary subgroup $\mathrm{E}(n, R)$:

$$\mathrm{E}(n, R, A) = \langle t_{ij}(a),\ 1 \le i \ne j \le n,\ a \in A \rangle^{\mathrm{E}(n,R)}.$$

The following lemma on generation of relative elementary subgroups $\mathrm{E}(n, R, A)$ is a classical result discovered in various contexts by Stein, Tits and Vaserstein (see, for instance, [28] (or [15, Lem. 3] for a complete elementary proof). It is stated in terms of the *Stein–Tits–Vaserstein generators*:

$$z_{ij}(a, c) = t_{ij}(c) t_{ji}(a) t_{ij}(-c), \quad 1 \le i \ne j \le n, \quad a \in A, \quad c \in R.$$

Lemma 1. *Let R be an associative ring with* 1*, let $n \geq 3$, and let A be a two-sided ideal of R. Then,*

$$\mathrm{E}(n,R,A) = \big\langle z_{ij}(a,c),\ 1 \leq i \neq j \leq n, a \in A, c \in R \big\rangle.$$

We also need the following results on the mixed commutators $[\mathrm{E}(n,A),\mathrm{E}(n,B)]$. Everywhere below the commutators are left-normed so that for two elements x, y of a group G one has $[x,y] = {}^{x}y \cdot y^{-1} = x \cdot {}^{y}(x^{-1}) = xyx^{-1}y^{-1}$, where ${}^{x}y = xyx^{-1}$ is the left conjugate of y by x. In the sequel, we repeatedly use standard commutator identities such as $[xy,z] = {}^{x}[y,z] \cdot [x,z]$ or $[x,yz] = [x,y] \cdot {}^{y}[x,z]$ without any explicit reference.

Denote by $A \circ B = AB + BA$ the symmetrised product of two-sided ideals A and B. For commutative rings, $A \circ B = AB = BA$ is the usual product of ideals A and B. However, in general, the symmetrised product is not associative. Thus, when writing something like $A \circ B \circ C$, we have to specify the order in which products are formed. The following level computation is standard — see, for instance, [31], [32], [15] and references there.

Lemma 2. *R be an associative ring with* 1*, $n \geq 3$, and let A and B be two-sided ideals of R. Then,*

$$\mathrm{E}(n,R,A\circ B) \leq \big[\,\mathrm{E}(n,A),\mathrm{E}(n,B)\big] \leq \big[\,\mathrm{E}(n,R,A),\mathrm{E}(n,R,B)\big] \leq \mathrm{GL}(n,R,A\circ B).$$

Since all generators listed in Theorem B already belong to the commutator subgroup of unrelative elementary subgroups, we get the following corollary, [34, Thm. 2].

Lemma 3. *Let R be any associative ring with* 1*, let $n \geq 3$, and let A, B be two-sided ideals of R. Then one has*

$$\big[\,\mathrm{E}(n,R,A),\mathrm{E}(n,R,B)\big] = \big[\,\mathrm{E}(n,R,A),\mathrm{E}(n,B)\big] = \big[\,\mathrm{E}(n,A),\mathrm{E}(n,B)\big].$$

In particular, it follows that $[\mathrm{E}(n,A),\mathrm{E}(n,B)]$ is normal in $\mathrm{E}(n,R)$. Our proof of Theorem 1 is based on the following computation, which is contained within the proof of [33, Lem. 3] or [35, Lem. 9].

Lemma 4. *Let R be an associative ring with* 1*, $n \geq 3$. For any three pair-wise distinct indices i, j, h and any $a, b, c \in R$ one has*

$$\begin{aligned}
&[t_{ih}(c), y_{ij}(a,b)] = t_{ih}(-abc - ababc)t_{jh}(-babc),\\
&[t_{jh}(c), y_{ij}(a,b)] = t_{ih}(abac)t_{jh}(bac),\\
&[t_{hi}(c), y_{ij}(a,b)] = t_{hi}(cab)t_{hj}(-caba),\\
&[t_{hj}(c), y_{ij}(a,b)] = t_{hi}(cbab)t_{hj}(-cba - cbaba).
\end{aligned}$$

When $a \in A$ and $b \in B$ it follows that

$${}^{t_{kl}(c)}y_{ij}(a,b) \in \mathrm{E}(n, A\circ B)y_{ij}(a,b),$$

unless $(k,l) = (i,j), (j,i)$.

2. Rolling over elementary commutators

Our proof is yet another variation on the following theme. The result was observed by Wilberd van der Kallen, as part of the proof of [18, Lem. 2.2] that modulo the elementary transvections of a given level elementary conjugates are connected by a triple identity that allows one of them to express in terms of two other ones in different positions. It is reproduced in this precise form with a detailed proof in [36, Lem. 12].

Lemma 5. *Let $A \trianglelefteq R$ be an ideal of an associative ring, let $n \geq 3$, and let i, j, h be three pair-wise distinct indices. If a subgroup $\mathrm{E}(n, A) \leq H \leq \mathrm{GL}(n, R)$ contains*

- *either $z_{ih}(a, c)$ and $z_{jh}(a, c)$,*
- *or $z_{hi}(a, c)$ and $z_{hj}(a, c)$,*

for all $a \in A$, $c \in R$, then it also contains $z_{ij}(a, c)$ and $z_{ji}(a, c)$, for all such a and c.

In this section and the next one, we establish its counterparts for elementary commutators in $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$. The following result is *essentially* [33, Lem. 5] or [35, Lem. 11]. Of course, there it was stated in a weaker form, as a congruence between elementary commutators modulo the relative elementary subgroup $\mathrm{E}(n, R, A \circ B)$, without specifying that we actually only need elementary conjugates in *one* position. As a result, the calculations in [33], [35] were not residing at the level in elementaries. At the moment a factor from $\mathrm{E}(n, R, A \circ B)$ occurred, it was immediately discarded. Here, we have to come up with a genuine calculation of all terms.

Lemma 6. *Let $A, B \trianglelefteq R$ be two-sided ideals of an associative ring, let $n \geq 3$, and let i, j, h be three pair-wise distinct indices. If a subgroup*

$$\mathrm{E}(n, A \circ B) \leq H \leq \mathrm{GL}(n, R)$$

contains

- *either $y_{ih}(a, b)$ and $z_{jh}(ba, c)$,*
- *or $z_{hi}(ab, c)$ and $y_{hj}(a, b)$,*

for all $a \in A$, $b \in B$ and $c \in R$, then it also contains $y_{ij}(a, b)$, for all such a and b.

Proof. Take any $h \neq i, j$ and rewrite the elementary commutator

$$z = y_{ij}(a, b) = \big[t_{ij}(a), t_{ji}(b)\big]$$

in the following form

$$z = t_{ij}(a) \cdot {}^{t_{ji}(b)}t_{ij}(-a) = t_{ij}(a) \cdot {}^{t_{ji}(b)}\big[t_{ih}(a), t_{hj}(-1)\big].$$

Expanding the conjugation by $t_{ji}(b)$, we see that

$$z = t_{ij}(a) \cdot \big[{}^{t_{ji}(b)}t_{ih}(a), {}^{t_{ji}(b)}t_{hj}(-1)\big] = t_{ij}(a) \cdot \big[t_{jh}(ba)t_{ih}(a), t_{hj}(-1)t_{hi}(b)\big].$$

Using multiplicativity of the commutator with respect to the first argument we get

$$z = t_{ij}(a) \cdot {}^{t_{jh}(ba)}\big[t_{ih}(a), t_{hj}(-1)t_{hi}(b)\big] \cdot \big[t_{jh}(ba), t_{hj}(-1)t_{hi}(b)\big].$$

Using multiplicativity of the commutator with respect to the second argument,

$$z = t_{ij}(a) \cdot {}^{t_{jh}(ba)}[t_{ih}(a), t_{hj}(-1)] \cdot {}^{t_{jh}(ba)t_{hj}(-1)}[t_{ih}(a), t_{hi}(b)] \cdot [t_{jh}(ba), t_{hj}(-1)] \cdot {}^{t_{hj}(-1)}[t_{jh}(ba), t_{hi}(b)].$$

Let us look at the factors separately.

The product of the first two factors equals $t_{ih}(aba) \in \mathrm{E}(n, A \circ B) \leq H$.

In the third factor, one has $t_{jh}(ba) \in \mathrm{E}(n, A \circ B)$, so that the corresponding conjugation can be discarded, whereas

$$^{t_{hj}(-1)}y_{ih}(a, b) \in \mathrm{E}(n, A \circ B) \cdot y_{ih}(a, b)$$

belongs to H.

The fourth factor equals $t_{jh}(ba)z_{jh}(-ba, -1) \in H$.

Finally, the last factor equals $t_{ji}(bab)t_{hi}(-bab) \in \mathrm{E}(n, A \circ B) \leq H$. $\square$

3. Proof of Theorem 1

The calculation in the preceding section can be reversed to express elementary conjugates as products of elementary commutators. This is accomplished in the following lemma. Again, *essentially* this lemma is based on the same calculation that was used in level calculations to prove the leftmost inclusion in Lemma 2 — see, for instance, [15, Lem. 1A]. But it was never stated in this precise form. This lemma together with Theorem B immediately implies Theorem 1, but its full force will be revealed in §6, where it will be used to establish much more precise results.

Lemma 7. *Let $A, B \trianglelefteq R$ be two-sided ideals of an associative ring, let $n \geq 3$, and let i, j, h be three pair-wise distinct indices. If a subgroup H,*

$$\mathrm{E}(n, A \circ B) \leq H \leq \mathrm{GL}(n, R),$$

contains

- *either $y_{ih}(a, b)$ and $y_{jh}(a, b)$,*
- *or $y_{hi}(b, a)$ and $y_{hj}(b, a)$,*

for all $a \in A$, $b \in B$, then it also contains $z_{ij}(ab, c)$ for all such a and b, and all $c \in R$. Similarly for the elementary conjugates $z_{ij}(ba, c)$ with A and B interchanged.

Proof. Since the condition is symmetric with respect to A and B, and $y_{ij}(b, a)^{-1} = y_{ji}(a, b) \in H$ implies that $y_{ij}(b, a) \in H$, it suffices to consider one of the four occurring cases. Thus, we only have to verify that $z_{ij}(ab, c) \in H$ provided that $y_{ih}(a, b), y_{jh}(a, b) \in H$.

Obviously, $t_{ij}(ab) = [t_{ih}(a), t_{hj}(b)]$. Decompose $z = z_{ij}(ab, c)$ accordingly:

$$z = z_{ij}(ab, c) = {}^{t_{ji}(c)}t_{ij}(ab) = {}^{t_{ji}(c)}[t_{ih}(a), t_{hj}(b)] = [t_{jh}(ca)t_{ih}(a), t_{hj}(b)t_{hi}(-bc)].$$

Using multiplicativity of commutators with respect to the first argument we see that

$$z = {}^{t_{jh}(ca)}[t_{ih}(a), t_{hj}(b)t_{hi}(-bc)] \cdot [t_{jh}(ca), t_{hj}(b)t_{hi}(-bc)].$$

Expanding both factors in the above expression for z with respect to the second argument, we get

$$z = {}^{t_{jh}(ca)}[t_{ih}(a), t_{hj}(b)] \cdot {}^{t_{jh}(ca)t_{hj}(b)}[t_{ih}(a), t_{hi}(-bc)] \\ \cdot [t_{jh}(ca), t_{hj}(b)] \cdot {}^{t_{hj}(b)}[t_{jh}(ca), t_{hi}(-bc)].$$

Consider the four factors separately. Clearly,

$$\begin{aligned} {}^{t_{jh}(ca)}[t_{ih}(a), t_{hj}(b)] &= {}^{t_{jh}(ca)}t_{ij}(ab) = t_{ij}(ab)t_{ih}(-abca) \in H. \\ [t_{jh}(ca), t_{hj}(b)] &= y_{jh}(ca, b) \in H. \\ {}^{t_{hj}(b)}[t_{jh}(ca), t_{hi}(-bc)] &= {}^{t_{hj}(b)}t_{ji}(-cabc) = t_{hi}(-bcabc)t_{ji}(-cabc) \in H. \end{aligned}$$

Thus, the only slightly problematic factor is the second one,

$$w = {}^{t_{jh}(ca)t_{hj}(b)}[t_{ih}(a), t_{hi}(-bc)].$$

However,

$$w = {}^{t_{jh}(ca)t_{hj}(b)}y_{ih}(a, -bc) = {}^{t_{jh}(ca)}\big([t_{hj}(b), y_{ih}(a, -bc)] \cdot y_{ih}(a, -bc)\big).$$

By Lemma 4 one has

$$[t_{hj}(b), y_{ih}(a, -bc)] = t_{ij}(-abcab)t_{hj}(-bcac).$$

Plugging this into the above expression for w we get

$$\begin{aligned} w &= {}^{t_{jh}(ca)}\big(t_{ij}(-abcab) \cdot t_{hj}(-bcac) \cdot y_{ih}(a, -bc)\big) \\ &= {}^{t_{jh}(ca)}t_{ij}(-abcab) \cdot {}^{t_{jh}(ca)}t_{hj}(-bcac) \cdot {}^{t_{jh}(ca)}y_{ih}(a, -bc). \end{aligned}$$

Clearly, the first two factors

$${}^{t_{jh}(ca)}t_{ij}(-abcab) = t_{ij}(-abcab)t_{ih}(abcabca),$$

$${}^{t_{jh}(ca)}t_{hj}(-bcac) = [t_{jh}(ca), t_{hj}(-bcac)] \cdot t_{hj}(-bcac) = y_{jh}(ca, -bcac) \cdot t_{hj}(-bcac),$$

both belong to H. On the other hand, using Lemma 4 once more we get

$$\begin{aligned} {}^{t_{jh}(ca)}y_{ih}(a, -bc) &= [t_{jh}(ca), y_{ih}(a, -bc)] \cdot y_{ih}(a, -bc) \\ &= t_{ji}(cabcabc) \cdot t_{jh}(cabca - cabcabca) \cdot y_{ih}(a, -bc) \in H. \end{aligned}$$

This means that the second factor in the above expression of z also belongs to H and we are done. □

Theorem 1 immediately follows from Theorem B and Lemma 7.

4. Partially relativised subgroups

For two ideals $A, B \trianglelefteq R$ we denote by $\mathrm{E}(n,B,A)$ the *partially relativised elementary group*, which is the smallest subgroup containing $\mathrm{E}(n,A)$ and normalised by $\mathrm{E}(n,B)$:

$$\mathrm{E}(n,B,A) = \mathrm{E}(n,A)^{\mathrm{E}(n,B)}.$$

In particular, when $B = R$ we get the usual relative group $\mathrm{E}(n,R,A)$, as defined above.

Remark. In a special case, the groups $\mathrm{E}(n,B,A)$ were first systematically considered by Roozbeh Hazrat and the second author in their works on relative localisation [16], [17]. It soon became clear that $\mathrm{E}(n,R,s^m I)$ were too large to serve as a convenient system of neighborhoods of 1, whereas $\mathrm{E}(n,s^m I)$ were way too small. The partially relativised groups $\mathrm{E}(n,s^m R,s^m I)$ turned out to be a much smarter choice. This important technical innovation then proved extremely useful in our joint work with Roozbeh Hazrat — see, in particular, [11], [12], [14]. Later, partially relativised elementary groups figured prominently in the universal localisation by Alexei Stepanov [23], [24], [1].

The following obvious observation relates partially relativised subgroups with double commutators of elementary subgroups.

Lemma 8. *Let R be any associative ring with* 1*, and let A, B be two-sided ideals of R. Then for any $n \geq 2$ one has*

$$\mathrm{E}(n,B,A) = [\mathrm{E}(n,A),\mathrm{E}(n,B)] \cdot \mathrm{E}(n,A).$$

Proof. By the very definition $\mathrm{E}(n,A) \leq \mathrm{E}(n,B,A)$. The mixed commutator subgroup $[\mathrm{E}(n,A),\mathrm{E}(n,B)]$ is generated by the commutators $[x,y] = x(yx^{-1}y^{-1})$, where $x \in \mathrm{E}(n,A)$, $y \in \mathrm{E}(n,B)$. Thus, $[\mathrm{E}(n,A),\mathrm{E}(n,B)] \leq \mathrm{E}(n,B,A)$. This means that the right-hand side is contained in the left-hand side.

Conversely, observe that the product on the right-hand side is a subgroup. Indeed, ${}^{x}[y,z] = [xy,z] \cdot [x,z]^{-1}$, where $x, y \in \mathrm{E}(n,A)$, $z \in \mathrm{E}(n,B)$. It follows that $\mathrm{E}(n,A)$ normalises $[\mathrm{E}(n,A),\mathrm{E}(n,B)]$. Finally, for all such x and z one has ${}^{z}x = [z,x] \cdot x$, so that the right-hand side contains all generators of $\mathrm{E}(n,B,A)$. □

The following result is [36, Thm. 2]. There it was derived from Theorem B by another calculation in the style of level calculations, and with Theorem 1 on deck, it becomes immediate.

Theorem 2. *Let R be an associative ring with identity* 1*, $n \geq 3$, and let A and B be two-sided ideals of R. Then*

$$\mathrm{E}(n,B,A) = \big\langle z_{ij}(a,b),\ 1 \leq i \neq j \leq n, a \in A, b \in B \big\rangle.$$

Proof. Consider the subgroup

$$H = \big\langle z_{ij}(a,b),\ 1 \leq i \neq j \leq n, a \in A, b \in B \big\rangle,$$

generated by the elementary conjugates contained in $\mathrm{E}(n,B,A)$. By the very definition, $z_{ij}(a,b)\in \mathrm{E}(n,B,A)$ so that $H\le \mathrm{E}(n,B,A)$.

To prove the converse inclusion, it suffices to verify that the generators of $\mathrm{E}(n,B,A)$ listed in Lemma 8 are in fact contained already in H.

By definition, any $x\in \mathrm{E}(n,A)$ is a product of the elementary generators $x_{ij}(a)=z_{ij}(a,0)$. In other words, $\mathrm{E}(n,A)\le H$.

By Theorem 1, modulo $\mathrm{E}(n,R,A\circ B)\le \mathrm{E}(n,B,A)$ the mixed commutator subgroup $[\mathrm{E}(n,A),\mathrm{E}(n,B)]$ is generated by the elementary commutators $y_{ij}(a,b)$, where $1\le i\ne j\le n$, $a\in A$ and $b\in B$. However,

$$y_{ij}(a,b)=[x_{ij}(a),x_{ji}(b)]=x_{ij}(a)\cdot {}^{x_{ji}(b)}x_{ij}(-a)=z_{ij}(a,0)z_{ij}(-a,b)\in H,$$

which finishes the proof. $\square$

5. Elementary commutators modulo $\mathrm{E}(n,A\circ B)$

In [33], [35] Lemma 4 was used to establish that the quotient

$$\big[\mathrm{E}(n,R,A),\mathrm{E}(n,R,B)\big]/\,\mathrm{E}(n,R,A\circ B)=\big[\mathrm{E}(n,A),\mathrm{E}(n,B)\big]/\,\mathrm{E}(n,R,A\circ B)$$

is central in $\mathrm{E}(n,R)/\,\mathrm{E}(n,R,A\circ B)$.

Lemma 9. *Let R be an associative ring with 1, $n\ge 3$, and let A,B be two-sided ideals of R. Then*

$$\big[\big[\mathrm{E}(n,A),\mathrm{E}(n,B)\big],\mathrm{E}(n,R)\big]=\mathrm{E}(n,R,A\circ B).$$

In particular, $\big[\mathrm{E}(n,A),\mathrm{E}(n,B)\big]/\,\mathrm{E}(n,R,A\circ B)$ is itself abelian. In turn, Lemma 9 was itself a key step in the proof of the following more general result on triple commutators. The following result is [35, Lem. 7].

Lemma 10. *Let R be an associative ring with 1, let $n\ge 3$, and let A,B,C be three two-sided ideals of R. Then*

$$\big[\big[\mathrm{E}(n,A),\mathrm{E}(n,B)\big],\mathrm{E}(n,C)\big]=\big[\mathrm{E}(n,A\circ B),\mathrm{E}(n,C)\big].$$

In [35] we succeeded in proving an analogue of Lemma 10 for quadruple commutators only under the stronger assumption that $n\ge 4$, see [35, Lem. 8]. However, it follows from the standard commutator formula that over *quasi-finite* rings the claim holds also for $n=3$ (see [17], [15]).

Lemma 11. *Assume that either R is an arbitrary associative ring with 1 and $n\ge 4$, or $n=3$ and R is quasi-finite. Further, let A,B,C,D be four two-sided ideals of R. Then*

$$\big[\big[\mathrm{E}(n,A),\mathrm{E}(n,B)\big],\big[\mathrm{E}(n,C),\mathrm{E}(n,D)\big]\big]=\big[\mathrm{E}(n,A\circ B),\mathrm{E}(n,C\circ D)\big].$$

When writing [33], [35] and even [36], we failed to notice the following generalisation of Lemma 9.

Theorem 3. *Let R be an associative ring with identity 1, let $n \geq 3$, and let A and B be two-sided ideals of R. Then*

$$\mathrm{E}(n, A \circ B) \trianglelefteq \big[\, \mathrm{E}(n, A), \mathrm{E}(n, B)\big]$$

and the quotient $\mathrm{E}(n, R, A \circ B) / \mathrm{E}(n, A \circ B)$ *is central in*

$$\big[\, \mathrm{E}(n, A), \mathrm{E}(n, B)\big] / \mathrm{E}(n, A \circ B).$$

If, moreover, either $n \geq 4$, or R is quasi-finite, then this last quotient is itself abelian.

Proof. By Lemma 10 one has

$$\big[\big[\, \mathrm{E}(n, A), \mathrm{E}(n, B)\big], \mathrm{E}(n, A \circ B)\big] = \big[\, \mathrm{E}(n, A \circ B), \mathrm{E}(n, A \circ B)\big] \leq \mathrm{E}(n, A \circ B),$$

which establishes the first two claims of lemma. To check the last claim, observe that under these additional assumptions one has

$$\big[\big[\, \mathrm{E}(n, A), \mathrm{E}(n, B)\big], \big[\, \mathrm{E}(n, A), \mathrm{E}(n, B)\big]\big] = \big[\, \mathrm{E}(n, A \circ B), \mathrm{E}(n, A \circ B)\big] \leq \mathrm{E}(n, A \circ B),$$

either by Lemma 11 (when $n \geq 4$) or by [15, Thm. 5A] (when R is quasi-finite). □

This means that instead of studying $[\mathrm{E}(n, A), \mathrm{E}(n, B)] / \mathrm{E}(n, R, A \circ B)$, as we did in [33], [35], [36], we can now study the larger quotient

$$[\mathrm{E}(n, A), \mathrm{E}(n, B)\big] / \mathrm{E}(n, A \circ B).$$

In [33], [35], [36] we retrieved some of the relations among the elementary commutators $y_{ij}(a, b)$ modulo the relative elementary subgroup $\mathrm{E}(n, R, A \circ B)$. However, a naive attempt to generalise these relations by eliminating the elementary conjugates at the cost of introducing further elementary commutators leads to complicated and unsavoury relations.

6. Further reducing the generating set of $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$

Let us state a result which is *essentially* due to Wilberd van der Kallen and Alexei Stepanov. Namely, in [18, Lem. 2.2] this result is established for the unipotent radical of a *terminal* parabolic in $\mathrm{GL}(n, R)$. Subsequently, it was generalised to all maximal parabolics in Chevalley groups in [22], [23], [24], but of course, in these papers R was always assumed to be commutative. In that form, it is stated as corollary to [35, Thm. 3] (of course, it immediately follows already from Lemma 5 above = [35, Lem. 12]). Morally, it is a trickier and mightier version of the classical generation result for $\mathrm{E}(n, R, A)$, Lemma 1, with a smaller set of generators. Actually, one does not even need the whole unipotent radical, just $n-1$ roots in interlaced positions that allow us to repeatedly apply Lemma 5.

Theorem C. *Let $A \trianglelefteq R$ be a two-sided ideal of an associative ring, let $n \geq 3$, and let $1 \leq r \leq n-1$. Then the relative elementary subgroup* $\mathrm{E}(n, R, A)$ *is generated by the true elementary subgroup* $\mathrm{E}(n, A)$ *and the elementary conjugates $z_{ij}(a, c)$, where $a \in A$, $c \in R$, whereas (i, j) is one of the following*:

- *Either (i, h), for all $1 \leq i \leq r$ and a fixed $r+1 \leq h \leq n$,*
- *or (k, j), for a fixed $1 \leq k \leq r$ and all $r+1 \leq j \leq n$.*

Proof. Denote the subgroup generated by $\mathrm{E}(n, A)$ and the above elementary conjugates by H. By Lemma 1, it suffices to verify that $z_{ij}(a, c) \in H$ for all $1 \leq i \neq j \leq n$.

First, let $r = 1$ or $r = n-1$. In this case, our theorem is precisely [18, Lem. 2.2]. Indeed, let $r = n-1$. Then H contains $z_{in}(a, c)$, for all $1 \leq i \leq n-1$, and thus by Lemma 5 one has $z_{ij}(a, c) \in H$ for all $1 \leq i \neq j \leq n-1$. Now, fix an h, $1 \leq h \leq n-1$. Applying Lemma 5 to $z_{hj}(a, c) \in H$, $j \neq h$, we can conclude that $z_{nj}(a, c) \in H$, $j \neq h$. Since h here is arbitrary and $n-1 \geq 2$, it follows that $z_{nj}(a, c) \in H$ for all $1 \leq j \leq n-1$.

Now, let $2 \leq r \leq n-2$. Without loss of generality, we can assume that $h = r+1$ and $k = 1$. Applying the previous case to the group $\mathrm{GL}(r+1, R)$ embedded in $\mathrm{GL}(n, R)$ to the first $r+1$ rows and columns, we get that $z_{ij}(a, b) \in H$ for all $1 \leq i \neq j \leq r+1$. In particular, $z_{1j}(a, c) \in H$ for all $j \neq 1$. Applying the previous case again, now to the group $\mathrm{GL}(n, R)$ itself, we can conclude that $z_{ij}(a, c) \in H$ for all $1 \leq i \neq j \leq n$, as claimed. □

Combined with Theorem B, it immediately yields the following further sharpening of Theorems A and B. Notice that $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$ is in general *strictly larger* than $\mathrm{E}(n, R, A \circ B)$ — see, for instance, the discussion in [35], § 7. Thus, we need an extra type of generators, for one more position, which gives us n positions.

Theorem D. *Let R be any associative ring with* 1, *let $n \geq 3$ and let $1 \leq r \leq n-1$. Further, let A, B be two-sided ideals of R. Then the mixed commutator subgroup* $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$ *is generated as a group by the true elementary subgroup* $\mathrm{E}(n, A \circ B)$ *and the elements of the two following forms*:

- *$z_{ij}(ab, c)$ and $z_{ij}(ba, c)$,*
- *$y_{st}(a, b)$,*

where $a \in A$, $b \in B$, $c \in R$, the pairs of indices (i, j) are as in Theorem C, *while (s, t), $1 \leq s \neq t \leq n$ is an arbitrary fixed pair of indices.*

Recall that Theorem B itself was based on a version of Lemma 6, while the derivation of Theorem D from Theorem B ultimately depends on Lemma 5. On the other hand, now that we have Lemma 7, expressing elementary conjugates in terms of elementary commutators, we can prove a similar sharper form of Theorem 1. It is another main result of the present paper and a counterpart of Theorems C and D for mixed commutator subgroups of elementary groups.

Theorem 4. *Let $A, B \trianglelefteq R$ be a two-sided ideal of an associative ring, let $n \geq 3$, and let $1 \leq r \leq n-1$. Then the mixed commutator* $[\mathrm{E}(n, A), \mathrm{E}(n, B)]$ *is generated by:*

- *The true elementary subgroup* $\mathrm{E}(n, A \circ B)$.
- *The elementary commutators $y_{ij}(a, b)$, for all $a \in A$, $b \in B$, the pairs of indices (i, j) are as in Theorem* C, *and the elementary conjugates/commutators for one more position.*

• *Either the elementary conjugates* $z_{st}(ab,c)$ *and* $z_{st}(ba,c)$ *for all* $a \in A$, $b \in B$, $c \in R$ *and any position* (s,t) *such that either* $1 \leq s \leq r$ *and* $r+1 \leq t \leq n$, *or* $1 \leq t \leq r$ *and* $r+1 \leq s \leq n$.

• *Or the elementary commutators* $y_{st}(a,b)$, *for all* $a \in A$, $b \in B$, *and any position* (s,t) *such that either* $1 \leq s \neq t \leq r$, *or* $r+1 \leq s \neq t \leq n$.

Proof. Denote the subgroup generated by $\mathrm{E}(n, A \circ B)$ and the elementary commutators $y_{ij}(a,b)$ by H. First, we verify that in that case $z_{ij}(ab,c), z_{ij}(ba,c) \in H$ for all $1 \leq i \neq j \leq r$ or $r+1 \leq i \neq j \leq n$.

As above, first we treat the case where $r=1$ or $r=n-1$. Let $r=n-1$. Then H contains $y_{in}(a,b)$ for all $1 \leq i \leq n-1$, and thus by Lemma 7 one has $z_{ij}(ab,c) \in H$ and $z_{ij}(ba,c) \in H$ for all $1 \leq i \neq j \leq n-1$. As above, fix an h, $1 \leq h \leq n-1$. Applying Lemma 6 to $z_{hj}(ab,c) \in H$, $j \neq h, n$ and $y_{hn}(a,b)$, we can conclude that $y_{nj}(a,b) \in H$, $j \neq h$. Since h here is arbitrary and $n-1 \geq 2$, it follows that $y_{nj}(a,b) \in H$ for all $1 \leq j \leq n-1$.

Now, let $2 \leq r \leq n-2$ and let h and k be as in the statement of Theorem C. Without loss of generality, we can assume that $h=r+1$ and $k=r$. Applying the previous case to the group $\mathrm{GL}(r+1,R)$ embedded in $\mathrm{GL}(n,R)$ on the first $r+1$ rows and columns, we get that $z_{ij}(ab,c), z_{ij}(ba,c) \in H$ for all $1 \leq i \neq j \leq r$. Similarly, applying it to the group $\mathrm{GL}(n-r+1,R)$ embedded in $\mathrm{GL}(n,R)$ on the last $n-r+1$ rows and columns, we get that $z_{ij}(ab,c), z_{ij}(ba,c) \in H$ for all $r+1 \leq i \neq j \leq n$.

Now if $z_{st}(ab,c), z_{st}(ba,c) \in H$ for such an extra position (s,t), then we get inside H the configuration of elementary commutators accounted for by Theorem D, and can conclude that $H = [\mathrm{E}(n,A), \mathrm{E}(n,B)]$.

On the other hand, let $y_{st}(a,b) \in H$, and let $1 \leq s \neq t \leq r$. Then applying Lemma 7 again, now to $y_{st}(a,b), y_{sh}(a,b) \in H$, we can conclude that the inclusions $z_{th}(an,c), z_{th}(ba,c) \in H$ hold and we are in the conditions of the previous item. Again, we can conclude that $H = [\mathrm{E}(n,A), \mathrm{E}(n,B)]$. □

Of course, combining this result with Lemma 7, it is now easy to obtain a sharper form of Theorem 2 in similar spirit. The reason we failed to notice this refinement in [36] was that we were expecting a statement in the style of Theorem C, rather than the one in the style of Theorem 4.

7. Multiple commutators

To state the results of this section, we have to recall some further pieces of notation from [7], [17], [9], [14], [15], [24], [35]. Namely, let $H_1, \dots, H_m \leq G$ be subgroups of G. There are many ways to form higher commutators of these groups, depending on where we put the brackets. Thus, for three subgroups $F, H, K \leq G$ one can form two triple commutators: $[[F,H],K]$ and $[F,[H,K]]$. In the sequel, we denote by $[\![H_1, H_2, \dots, H_m]\!]$ *any* higher mixed commutator of $H_1, \dots, H_m$, with an arbitrary placement of brackets. Thus, for instance, $[\![F,H,K]\!]$ refers to any of the two arrangements above.

Actually, the primary attribute of a bracket arrangement that plays major role in our results is its cut point. Namely, a higher commutator subgroup $[\![H_1,H_2,\dots,H_m]\!]$

can be uniquely written as a double commutator

$$[\![H_1, H_2, \dots, H_m]\!] = \big[[\![H_1, \dots, H_s]\!], [\![H_{s+1}, \dots, H_m]\!]\big],$$

for some $s = 1, \dots, m-1$. This s is called the cut point of our multiple commutator.

For *non-commutative* rings there is another aspect that affects the final answer. Namely, in this case symmetrised product of ideals is not associative. For instance, for three ideals $A, B, C \trianglelefteq R$ one has

$$(A \circ B) \circ C = ABC + BAC + CAB + CBA,$$

whereas

$$A \circ (B \circ C) = ABC + ACB + BCA + CBA,$$

that in general do not coincide.

To account for this, in the sequel we write $(\!|I_1 \circ \cdots \circ I_m|\!)$ to denote the symmetrised product of $I_1, \dots, I_m$ with an arbitrary placement of parenthesis. Thus, for instance, $(\!|A \circ B \circ C|\!)$ may refer either to $(A \circ B) \circ C$, or to $A \circ (B \circ C)$, depending. In the sequel the initial bracketing of higher commutators will be reflected in the parenthesizing of the corresponding multiple symmetrised products.

In this notation [35, Thm. 1] combined with [7, Thm. 8A] and [15, Thm. 5A] can be stated as follows. Similar results in the more general context of Bak's unitary groups[1] were obtained in [14], [38].

Theorem E. *Let R be any associative ring with 1, and let either $n \geq 4$, or $n = 3$ and R is quasi-finite. Further, let $I_i \trianglelefteq R$, $i = 1, \dots, m$, be two-sided ideals of R. Consider an arbitrary arrangement of brackets $[\![\dots]\!]$ with the cut point s. Then one has*

$$[\![\, \mathrm{E}(n, I_1), \mathrm{E}(n, I_2), \dots, \mathrm{E}(n, I_m)]\!] = \Big[E\big(n, (\!|I_1 \circ \dots \circ I_s|\!)\big), E\big(n, (\!|I_{s+1} \circ \dots \circ I_m|\!)\big)\Big],$$

where the parenthesizing of symmetrised products on the right hand side coincides with the bracketing of the commutators on the left-hand side.

Of course, the first question that immediately occurs is whether the above quadruple and multiple elementary commutator formulas also hold for $\mathrm{GL}(3, R)$ over arbitrary associative rings. This question was already stated as [35, Problem 1]. The results of §5 make it even more imperative. Though we are rather more inclined to believe in the positive answer, up to now all our attempts to verify it by a direct calculation in the style of [35, Lem. 7] failed.

Problem 1. *Prove that Lemma* 11 *and Theorem* E *hold also for $n = 3$.*

In conjunction with Theorem 1 the above Theorem C allows a very slick generating set for the multiple commutator subgroups $[\![\mathrm{E}(n, I_1), \dots, \mathrm{E}(n, I_m)]\!]$.

[1] This is indeed a more general context, since $\mathrm{GL}(n, R)$ are interpreted as a special case of unitary groups, those over hyperbolic form rings.

Theorem 5. *Let R be any associative ring with* 1*, and let either $n \geq 4$, or $n = 3$ and R is quasi-finite. Further, let $I_i \trianglelefteq R$, $i = 1, \dots, m$, be two-sided ideals of R. Consider an arbitrary arrangement of brackets $[\![\dots]\!]$ with the cut point s. Then the mixed multiple commutator subgroup*

$$[\![\mathrm{E}(n, I_1), \mathrm{E}(n, I_2), \dots, \mathrm{E}(n, I_m)]\!]$$

is generated by the double elementary commutators

$$[t_{ij}(a), t_{hk}(b)], \qquad a \in (\!|I_1 \circ \dots \circ I_s|\!), \quad b \in (\!|I_{s+1} \circ \dots \circ I_m|\!),$$

where the parenthesizing of the above symmetrised products coincides with the bracketing of the commutators before and after the cut point, respectively.

Of course, these generating sets could be further reduced in the spirit of Theorems D or 4.

8. Final remarks

Below we collect some of the most immediate open problems, related to the results of the present paper.

Modulo $\mathrm{E}(n, R, A \circ B)$ the elementary commutators $y_{ij}(a, b) \in [\mathrm{E}(n, A), \mathrm{E}(n, B)]$ behave as symbols in algebraic K-theory, and are subject to very nice relations (see [33], [35], [36]). Even so, an explicit calculation of the quotient

$$[\mathrm{E}(n, A), \mathrm{E}(n, B)] / \mathrm{E}(n, R, A \circ B)$$

turned out to be quite a challenge and so far we succeeded in getting conclusive results only over Dedekind rings. A similar question for the larger quotient

$$[\mathrm{E}(n, A), \mathrm{E}(n, B)] / \mathrm{E}(n, A \circ B)$$

seems to be much harder.

Problem 2. *Find defining relations among the elementary commutators $y_{ij}(a, b) \in [\mathrm{E}(n, A), \mathrm{E}(n, B)]$ modulo $\mathrm{E}(n, A \circ B)$.*

Theorem 1 suggests to resuscitating the approach to relative localisation developed in our joint papers with Roozbeh Hazrat [16], [11], [12], [17], [14], [15]. The heft of the technical difficulty in applying relative localisation to multiple commutator formulas [17], [14], [15] stemmed from the necessity to develop massive chunks of the conjugation calculus and commutator calculus [7], [9], [15] for the new type of generators listed in Theorem A. We believe that, now that we have Theorems B and 1, these calculations could be reduced to a *fraction* of their initial length.

Problem 3. *Apply Theorem* 1 *to rethink localisation proofs of the multiple commutator formulas.*

As of today, the remarkable approach via universal localisation as developed by Alexei Stepanov [23], [24] only works for algebraic groups over commutative rings. On the other hand, further applications of the initial versions of relative localisation were effectively blocked by technical obstacles.

Such agreeable generating sets as found in Theorems 1 and 5 could be especially advantageous for improving bounds in results on multiple commutator width, in the spirit of Alexei Stepanov (see [21], [26], [24] and a survey in [8]).

Problem 4. *Apply Theorems* 1 *and* 5 *to get results on multirelative commutator width.*

It would be natural to generalise results of the present paper to more general contexts. The following development seems to be immediate.

Problem 5. *Generalise results of the present paper to Chevalley groups.*

We believe that in this context all fragments of the necessary calculations were already elaborated: analogues of Lemma 5 by Alexei Stepanov in [22], [23], analogues of Lemma 6 in our joint papers with Roozbeh Hazrat [12], [13], and finally, analogues of Lemma 7 in our recent works [34], [37]. Of course, they were not stated there this way, but to establish them in the desired forms would only require some moderate processing of the existing proofs.

The following problem is similar, but seems to be somewhat harder.

Problem 6. *Generalise results of the present paper to Bak's unitary groups.*

In fact, here too large fragments of the necessary theory were already developed in our previous joint papers with Anthony Bak and Roozbeh Hazrat [3], [6], [11], [14], [15], and in our recent preprint [38]. But in this case even the analogues of Lemma 5 and Theorem C are lacking, and it would also take considerably more work to bring the results from the above papers to the required form.

References

[1] H. Apte, A. Stepanov, *Local-global principle for congruence subgroups of Chevalley groups,* Cent. Eur. J. Math. **12** (2014), no. 6, 801–812.

[2] A. Bak, *Non-abelian* K*-theory: The nilpotent class of* K_1 *and general stability*, *K*–Theory **4** (1991), 363–397.

[3] A. Bak, N. Vavilov, *Structure of hyperbolic unitary groups*I: *elementary subgroups*, Algebra Colloquium **7** (2000), no. 2, 159–196.

[4] H. Bass, K*-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 22, 5–60.

[5] З. И. Боревич, Н. А. Вавилов, *Расположение подгрупп в полной линейной группе над коммутативным кольцом*, Тр. МИАН СССР **165** (1984), 24–42. Engl. transl.: Z. I. Borevich, N. A. Vavilov, *Arrangement of subgroups in the general linear group over a commutative ring*, Proc. Steklov Inst. Math. **165** (1985), 27–46.

[6] R. Hazrat, *Dimension theory and nonstable* K_1 *of quadratic modules*, K-Theory **27** (2002), 293–328.

[7] R. Hazrat, A. Stepanov, N. Vavilov, Zuhong Zhang, *The yoga of commutators*, J. Math. Sci. **179** (2011), no. 6, 662–678.

[8] R. Hazrat, A. Stepanov, N. Vavilov, Zuhong Zhang, *Commutator width in Chevalley groups*, Note di Matematica **33** (2013), no. 1, 139–170.

[9] R. Hazrat, A. Stepanov, N. Vavilov, Zuhong Zhang, *The yoga of commutators, further applications*, J. Math. Sci. **200** (2014), no. 6, 742–768.

[10] R. Hazrat, N. Vavilov, *Bak's work on the K-theory of rings, with an appendix by Max Karoubi*, J. K-Theory **4** (2009), 1–65.

[11] R. Hazrat, N. Vavilov, Zuhong Zhang, *Relative unitary commutator calculus and applications*, J. Algebra **343** (2011) 107–137.

[12] R. Hazrat, N. Vavilov, Zuhong Zhang, *Relative commutator calculus in Chevalley groups*, J. Algebra **385** (2013), 262–293.

[13] R. Hazrat, N. Vavilov, Zuhong Zhang *Generation of relative commutator subgroups in Chevalley groups*, Proc. Edinburgh Math. Soc. **59** (2016), 393–410.

[14] R. Hazrat, N. Vavilov, Zuhong Zhang, *Multiple commutator formulas for unitary groups*, Israel J. Math. **219** (2017), 287–330.

[15] R. Hazrat, N. Vavilov, Zuhong Zhang, *The commutators of classical groups,* J. Math. Sci. **222** (2017), no. 4, 466–515.

[16] R. Hazrat, Zuhong Zhang, *Generalized commutator formula,* Commun. Algebra **39** (2011), no. 4, 1441–1454.

[17] R. Hazrat, Zuhong Zhang, *Multiple commutator formula,* Israel J. Math. **195** (2013), 481–505.

[18] W. van der Kallen, *A group structure on certain orbit sets of unimodular rows,* J. Algebra **82** (1983), 363–397.

[19] A. W. Mason, *On subgroup of* $\mathrm{GL}(n, A)$ *which are generated by commutators,* II, J. reine angew. Math. **322** (1981), 118–135.

[20] A. W. Mason, W. W. Stothers, *On subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators,* Invent. Math. **23** (1974), 327–346.

[21] A. Sivatski, A. Stepanov, *On the word length of commutators in* $\mathrm{GL}_n(R)$, K-theory **17** (1999), 295–302.

[22] A. Stepanov, *Elementary calculus in Chevalley groups over rings,* J. Prime Res. Math. **9** (2013), 79–95.

[23] A. V. Stepanov, *Non-abelian* K*-theory for Chevalley groups over rings,* J. Math. Sci. **209** (2015), no. 4, 645–656.

[24] A. Stepanov, *Structure of Chevalley groups over rings via universal localization*, J. Algebra **450** (2016), 522–548.

[25] A. Stepanov, N. Vavilov, *Decomposition of transvections*: *a theme with variations*, K-Theory **19** (2000), no. 2, 109–153.

[26] A. Stepanov, N. Vavilov, *On the length of commutators in Chevalley groups,* Israel J. Math. **185** (2011), 253–276.

[27] А. А. Суслин, *О структуре специальной линейной группы над кольцами многочленов*, Изв. АН СССР. Сер. матем. **41** (1977), вып. 2, 235–252. Engl. transl.: A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izv. **11** (1977), no. 2, 221–238.

[28] L. N. Vaserstein, *On the normal subgroups of the* GL_n *of a ring,* in: *Algebraic* K*-Theory*, Evanston 1980, Lecture Notes in Math., Vol. 854, Springer, Berlin, 1981, pp. 454–465.

[29] N. Vavilov, *Unrelativised standard commutator formula,* J. Math. Sci. **243** (2019), no. 4, 527–534.

[30] N. Vavilov, *Commutators of congruence subgroups in the arithmetic case*, J. Math. Sci. **479** (2019), 5–22.

[31] Н. А. Вавилов, А. В. Степанов, *Стандартная коммутационная формула*, Вестник Санкт-Петерб. унив. Сер. 1. Матем. Мех. Астр. (2008), no. 1, 9–14. Engl. transl.: N. A. Vavilov, A. V. Stepanov, *Standard commutator formula,* Vestnik St. Petersburg Univ.: Mathematics **41** (2008), no. 1, 5–8.

[32] Н. А. Вавилов, А. В. Степанов, *Еще раз о стандартной коммутационной формуле*, Вестник Санкт-Петерб. унив. Сер. 1. Матем. Мех. Астр. (2010), no. 1, 16–22. Engl. transl.: N. A. Vavilov, A. V. Stepanov, *Standard commutator formulae, revisited,* Vestnik St. Petersburg Univ.: Mathematics **43** (2010), no. 1, 12–17.

[33] N. Vavilov, Zuhong Zhang, *Commutators of relative and unrelative elementary groups, revisited,* J. Math. Sci. **485** (2019), 58–71.

[34] N. Vavilov, Zuhong Zhang, *Generation of relative commutator subgroups in Chevalley groups* II*,* Proc. Edinburgh Math. Soc. **63** (2020), no. 2, 497–511.

[35] N. Vavilov, Zuhong Zhang, *Multiple commutators of elementary subgroups*: *end of the line,* Linear Algebra Applications **599** (2020), 1–17.

[36] N. Vavilov, Zuhong Zhang, *Inclusions among commutators of elementary subgroups,* `arXiv:1911.10526` (2019).

[37] N. Vavilov, Zuhong Zhang, *Commutators of relative and unrelative elementary subgroups in Chevalley groups,* `arXiv:2004.00576` (2020).

[38] N. Vavilov, Zuhong Zhang, *Commutators of relative and unrelative elementary unitary groups,* St. Petersburg Math, J., submitted, 1–40.

[39] Hong You, *On subgroups of Chevalley groups which are generated by commutators,* J. Northeast Normal Univ. (1992), no. 2, 9–13.