

Spectral analysis for some multifractional Gaussian processes

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*Dedicated to the memory of Ya. Yu. Nikitin,
our colleague and friend*

Abstract

We study the small ball asymptotics problem in L_2 for two generalizations of the fractional Brownian motion with variable Hurst parameter. To this end, we perform careful analysis of the singular values asymptotics for associated integral operators.

1 Introduction

The spectral asymptotics for Gaussian processes are intensively investigated in the last two decades, closely connected with the problem of small deviation asymptotics for such processes in the Hilbert norm. Namely, see [17], to obtain the *logarithmic* L_2 -small ball asymptotics of a Gaussian process X , it is sufficient to know one-term asymptotics of the eigenvalue counting function of its covariance operator.

In the most elaborated case of the so-called *Green Gaussian processes*, i.e. the processes the covariance functions \mathcal{G}_X of which are the Green functions for the ordinary differential operators (ODO), one can obtain even two-term asymptotics of the eigenvalues with the remainder estimate and thus manage the *exact* small deviation asymptotics (up to a constant). This approach was developed in [20], [18], see more references in [22].

The case of *fractional Gaussian processes* is more complicated. In the pioneer paper [8] the one-term spectral asymptotics was calculated for the fractional Brownian motion (FBM) W^H , i.e. the zero mean-value Gaussian process with covariance function

$$\mathcal{G}_{W^H}(x, y) = \frac{1}{2} (x^{2H} + y^{2H} - |x - y|^{2H})$$

(here $H \in (0, 1)$ is the so-called *Hurst index*, the case $H = \frac{1}{2}$ corresponds to the standard Wiener process).

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A more general approach was suggested in [21]. This approach is based on the powerful theorems on spectral asymptotics of integral operators [3], see also [4, Appendix 7], and covers many fractional processes. Now the problem of logarithmic L_2 -small ball asymptotics for such processes is also well-studied. We mention also the breakthrough paper [9] where the two-term asymptotics of the eigenvalues with the remainder estimate was obtained for the FBM in the full range of the Hurst index. Some generalization of the seminal idea of [9] was given in [19] where the reader also can find an extensive bibliography.

In this paper we consider some more sophisticated Gaussian processes.

The *multifractional Brownian Motion* (mBM) was introduced in [23] and [2] and was investigated in several papers, see, e.g., [11], [1], and [10]. There are some different definitions of mBM equivalent up to a multiplicative deterministic function. We choose the so-called harmonizable representation [2]

$$W^{H(\cdot)}(x) = C_*(H(x)) \int_{-\infty}^{\infty} \frac{e^{ix\xi} - 1}{|\xi|^{H(x)+\frac{1}{2}}} dW(\xi), \quad (1)$$

where $W(\xi)$ is a conventional Wiener process and the functional Hurst parameter $H(x)$ satisfies $0 < H(x) < 1$. The choice of normalizing factor

$$C_*(H) = \left(\frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \right)^{\frac{1}{2}}$$

ensures that $EX^2(1) = 1$.

A different process of the same structure is the *multifractal Brownian Motion* (mfBM) introduced in [24], see also [25]. It can be constructed as

$$X^{H(\cdot)}(x) = \int_0^x K(x, y, H(x)) dW(y), \quad (2)$$

where

$$K(x, y, H) = c_*(H) y^{\frac{1}{2}-H} \int_y^x (z - y)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz \mathbb{I}_{[0,x]}(y), \quad (3)$$

while the variable Hurst parameter $H(x)$ satisfies $\frac{1}{2} < H(x) < 1$. The normalizing factor is defined by the formula

$$c_*(H) = \left(\frac{H(2H - 1)\Gamma(\frac{3}{2} - H)}{\Gamma(2 - 2H)\Gamma(H - \frac{1}{2})} \right)^{\frac{1}{2}},$$

where Γ is Euler's gamma-function.

Both processes (1) and (2) obviously have zero mean, their covariance functions were derived in [1] and [25], respectively. For $H(x) \equiv H = \text{const}$ they both coincide with conventional FBM.

We derive one-term spectral asymptotics for the processes (1) and (2) under some regularity assumptions on the functional parameter $H(x)$. Then, using the results of [14]

we obtain the logarithmic L_2 -small ball asymptotics for these processes. Despite the fact that the behavior of covariances of mBM and mfBM is significantly different, it turns out that under the assumption $H(x) > \frac{1}{2}$ these logarithmic asymptotics coincide.

The structure of our paper is as follows. In Section 2 we introduce operators associated with processes under consideration and formulate the result concerning the asymptotics of their singular values. This result is proved in Section 3. Section 4 is devoted to L_2 -small ball behavior of mBM and mfBM. Some auxiliary estimates and asymptotics of singular values of compact operators are collected in Appendix.

We use the letter C to denote various positive constants. To indicate that C depends on some parameters, we list them in parentheses: $C(\dots)$.

2 Operators associated with mBM and mfBM

Here we define integral operators associated with processes (1) and (2), see [16, §3.2]:

$$\mathbb{T} : L_2(\mathbb{R}) \rightarrow L_2(0, 1), \quad (\mathbb{T}f)(x) := C_*(H(x)) \int_{-\infty}^{\infty} \frac{e^{ix\xi} - 1}{|\xi|^{H(x)+\frac{1}{2}}} f(\xi) d\xi, \quad (4)$$

$$\mathbb{S} : L_2(0, 1) \rightarrow L_2(0, 1), \quad (\mathbb{S}f)(x) := \int_0^x K(x, y, H(x))f(y)dy \quad (5)$$

(the function K is defined in (3)). It is easy to see that the covariance functions

$$\mathcal{G}_{W^{H(\cdot)}}(x, y) := \mathbb{E}W^{H(\cdot)}(x)W^{H(\cdot)}(y), \quad \mathcal{G}_{X^{H(\cdot)}}(x, y) := \mathbb{E}X^{H(\cdot)}(x)X^{H(\cdot)}(y)$$

are the kernels of integral operators $\mathbb{T}\mathbb{T}^*$ and $\mathbb{S}\mathbb{S}^*$, respectively.

In what follows we denote by $\{\lambda_k(\mathbb{K})\}$ the nonincreasing sequence of eigenvalues of a compact selfadjoint positive operator \mathbb{K} in a Hilbert space \mathcal{H} , enumerated with multiplicities.

Recall that for a compact operator $\mathbb{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we have $\lambda_k(\mathbb{A}^*\mathbb{A}) = \lambda_k(\mathbb{A}\mathbb{A}^*)$, and square roots of these eigenvalues $s_k(\mathbb{A}) := (\lambda_k(\mathbb{A}^*\mathbb{A}))^{\frac{1}{2}}$ are called *the singular values* of the operator \mathbb{A} .

We denote by $\mathcal{N}(t, \mathbb{A})$, $t > 0$, the *distribution function* of singular values,

$$\mathcal{N}(t, \mathbb{A}) := \#\{k \mid s_k(\mathbb{A}) > t^{-1}\}.$$

Notice that the function $t \mapsto \mathcal{N}(t, \mathbb{A})$ is conceptually inverse to the function $k \mapsto s_k^{-1}(\mathbb{A})$. Thus, if the singular values have moderate speed of decay then the one-term asymptotic of $s_k(\mathbb{A})$ as $k \rightarrow \infty$ is uniquely defined by the one-term asymptotic of $\mathcal{N}(t, \mathbb{A})$ as $t \rightarrow \infty$.

We suppose that the functional parameter $H(x)$ is a Hölder continuous function. Then it turns out that the singular value asymptotics for operators (4), (5) heavily depend on the behavior of $H(x)$ in a neighborhood of the set where it attains its minimal value. We set

$$H_{min} := \min_{x \in [0, 1]} H(x), \quad \mathbf{D} := \{x \in [0, 1] \mid H(x) = H_{min}\}.$$

In what follows we use the notation $\mathbf{m} = H_{min} + \frac{1}{2}$.

Next, we introduce the *regularized distance* to \mathbf{D} , see, e.g., [15], that is a function $d(x)$, $x \in [0, 1]$, such that $d(x) \asymp \text{dist}(x, \mathbf{D})$ and

$$d \in \mathcal{C}^\infty([0, 1] \setminus \mathbf{D}), \quad |d^{(n)}(x)| \leq C(n)d^{1-n}(x), \quad x \in [0, 1] \setminus \mathbf{D}, \quad n \in \mathbb{N}.$$

We describe the behavior of $H(x)$ in a neighborhood of \mathbf{D} by the following assumptions:

1. The function $h(x) := H(x) - H_{min}$ is bounded by a power of the distance $d(x)$:

$$h(x) \leq Cd^\kappa(x) \quad \text{for some } \kappa > 0.$$

2. The function $h(x)$ admits an asymptotic representation with the smooth main term in a neighborhood of \mathbf{D} . More precisely, $h(x) = h_0(x) + h_1(x)$, where $h_0, h_1 \in \mathcal{C}^\beta[0, 1]$ with $\beta > 0$, and

$$h_0 \in \mathcal{C}^\infty([0, 1] \setminus \mathbf{D}), \quad |h_0^{(n)}(x)| \leq C(n)h_0(x)d^{-n}(x), \quad x \in [0, 1] \setminus \mathbf{D},$$

$$h_1(x) = O(h_0^{1+\tau}(x)) \quad \text{as } h_0(x) \rightarrow 0 \quad \text{for some } \tau > 0.$$

3. The measure of the small values set for the function h is a regularly varying function of the level:

$$\text{meas} \{x \in [0, 1] \mid 0 < h(x) < s\} = s^\sigma \varphi(s^{-1}), \quad 0 < s < s_0, \quad (6)$$

where $\sigma, s_0 > 0$ and φ is a *slowly varying function* (SVF), see [26].

Theorem 1 *Let the functional parameter $H(x)$ satisfy the assumptions 1–3. Assume in addition that*

$$\tau > \begin{cases} \max\{0, \frac{1}{2}(1 - \sigma)\}, & \text{if } \text{meas } \mathbf{D} > 0; \\ \sigma\mathbf{m} + \frac{1}{2}(1 - \sigma), & \text{if } \text{meas } \mathbf{D} = 0. \end{cases} \quad (7)$$

Then, as $t \rightarrow \infty$,

$$\mathcal{N}(t, \mathbb{T}) = \frac{1}{\pi} \left((2\pi)^{\frac{1}{2}} C_*(H_{min}) \right)^{\frac{1}{\mathbf{m}}} \int_0^1 t^{\frac{1}{H(x) + \frac{1}{2}}} dx (1 + O(\log^{-\nu} t)); \quad (8)$$

$$\mathcal{N}(t, \mathbb{S}) = \frac{1}{\pi} \left(c_*(H_{min}) \Gamma(H_{min} - \frac{1}{2}) \right)^{\frac{1}{\mathbf{m}}} \int_0^1 t^{\frac{1}{H(x) + \frac{1}{2}}} dx (1 + O(\log^{-\nu} t)), \quad (9)$$

Here $\nu = \nu(H_{min}, \sigma, \tau) > 0$. If $\text{meas } \mathbf{D} > 0$ then ν is arbitrary exponent less than $\frac{\tau - \frac{1}{2}(1 - \sigma)}{\mathbf{m} + 1}$.

Remark 1 1. We recall that the definition of the operator \mathbb{T} admits $0 < H(x) < 1$ while the definition of the operator \mathbb{S} admits only $\frac{1}{2} < H_{min} < 1$. However, for $H_{min} > \frac{1}{2}$ asymptotic formulae (8) and (9) coincide since

$$(2\pi)^{\frac{1}{2}} C_*(H) = c_*(H) \Gamma(H - \frac{1}{2}) = (\Gamma(2H + 1) \sin(\pi H))^{\frac{1}{2}}.$$

2. If $h_1(x) \equiv 0$, then formulae (8) and (9) are valid with $O(t^{-r})$ remainder estimate for some $r > 0$.
3. Applying Laplace method for the integral in (8) and (9), we obtain, as $t \rightarrow \infty$,

$$\int_0^1 t^{\frac{1}{H(x)+\frac{1}{2}}} dx = t^{\frac{1}{\mathfrak{m}}} (\text{meas } \mathbf{D} + \mathfrak{m}^{2\sigma} \Gamma(\sigma + 1) (\log t)^{-\sigma} \varphi(\log t) (1 + o(1))). \quad (10)$$

Notice that if $\text{meas } \mathbf{D} > 0$ then the main term of the asymptotics is purely power. If in addition $\sigma < \nu$ then we have even two-term asymptotics. In the case $\nu \leq \sigma$ we obtain only one-term power asymptotics with logarithmic remainder term.

4. We stress that the asymptotic (10) does not change if we replace the function φ with an equivalent SVF.

3 Proof of Theorem 1

The idea of the proof is as follows. We separate the principal terms in operators \mathbb{T} , \mathbb{S} . These terms are compact pseudodifferential operators of variable order. The singular values asymptotic for such operators is known, see [12], [13]. Then, using the asymptotic perturbation theory, we verify that remainder terms do not influence upon the obtained spectral asymptotics of principal terms.

3.1 Operator \mathbb{T} (mBM)

Since a symbol of pseudodifferential operator should be smooth with respect to the dual variable ξ , we introduce an even smooth positive function $\mathbf{p}(\xi)$ such that

$$\mathbf{p}(\xi) = |\xi| \quad \text{for } |\xi| \geq 2; \quad \mathbf{p}(\xi) = 1 \quad \text{for } |\xi| \leq 1.$$

Now we consider the pseudodifferential operator of variable order

$$(\mathbb{A}f)(x) := C_*(H(x)) \int_{-\infty}^{\infty} e^{ix\xi} \mathbf{p}(\xi)^{-(H(x)+\frac{1}{2})} f(\xi) d\xi.$$

The singular value asymptotics for \mathbb{A} is given by part 2 in Corollary 1, see Sec.5.1. Thus, formula (8) is ensured by part 3 in Proposition 1 if we prove the following lemma.

Lemma 1 *The following estimate holds:*

$$\mathcal{N}(t, \mathbb{T} - \mathbb{A}) \leq Ct^{\frac{1}{\mathfrak{m}} - \mu}, \quad t > 1, \quad (11)$$

with some $\mu > 0$.

Proof. The kernels of \mathbb{T} and \mathbb{A} have the same bounded multiplier $C_*(H(x))$, so we need to estimate singular values for the operator with the kernel

$$\begin{aligned} \frac{e^{ix\xi} - 1}{|\xi|^{H(x)+\frac{1}{2}}} - \frac{e^{ix\xi}}{\mathbf{p}(\xi)^{H(x)+\frac{1}{2}}} &= q_1(x, \xi) - q_2(x, \xi) + q_3(x, \xi) \\ &:= \frac{\zeta(\xi) - 1}{|\xi|^{H(x)+\frac{1}{2}}} - \zeta(\xi) \frac{e^{ix\xi}}{\mathbf{p}(\xi)^{H(x)+\frac{1}{2}}} + \zeta(\xi) \frac{e^{ix\xi} - 1}{|\xi|^{H(x)+\frac{1}{2}}}. \end{aligned} \quad (12)$$

Here $\zeta(\xi)$ is a fixed even cut-off function,

$$\zeta(\xi) = 0 \quad \text{for } |\xi| \geq 3; \quad \zeta(\xi) = 1 \quad \text{for } |\xi| \leq 2$$

(notice that $\mathbf{p}(\xi) = |\xi|$ for $\zeta(\xi) \neq 1$).

According to (12), we need to estimate singular values of the operator $\mathbb{Q}_1 - \mathbb{Q}_2 + \mathbb{Q}_3$, where $\mathbb{Q}_j : L_2(\mathbb{R}) \rightarrow L_2(0, 1)$ are the integral operators with kernels $q_j(x, \xi)$, $j = 1, 2, 3$. Due to the part 2 of Proposition 1 the estimate (11) follows from similar estimates for operators \mathbb{Q}_j .

The estimate $\mathcal{N}(t, \mathbb{Q}_1) \leq C(\varepsilon) t^{\frac{1}{m+\lambda} + \varepsilon}$ for any $\varepsilon > 0$ follows from Lemma 3.

The kernel $q_2(x, \xi)$ is bounded in x , smooth and compactly supported in ξ . Therefore, by Proposition 2, $\mathcal{N}(t, \mathbb{Q}_2) \leq C(\varepsilon) t^\varepsilon$ for any $\varepsilon > 0$.

The kernel q_3 is singular at the point $\xi = 0$. We separate the principal part of this singularity

$$q_3(x, \xi) = q_{3,0}(x, \xi) + q_{3,1}(x, \xi), \quad q_{3,0}(x, \xi) := \zeta(\xi) \frac{ix\xi}{|\xi|^{H(x) + \frac{1}{2}}}.$$

So, we can write $\mathbb{Q}_3 = \mathbb{Q}_{3,0} + \mathbb{Q}_{3,1}$.

The function $q_{3,1}(x, \cdot)$ is compactly supported and belongs to the Sobolev space $W_2^1(\mathbb{R})$ uniformly with respect to x . Therefore, Proposition 2 gives $\mathcal{N}(t, \mathbb{Q}_1) = O(t^{\frac{2}{3}})$. The required estimate follows from the inequality $\frac{1}{m} > \frac{2}{3}$.

It remains to estimate singular values of the operator $\mathbb{Q}_{3,0}$. Its kernel is nonzero only for $|\xi| < 3$, and we can consider $\mathbb{Q}_{3,0}$ as the operator from $L_2(-3, 3)$ to $L_2(0, 1)$.

We introduce the isometry

$$\mathbb{U} : L_2(-3, 3) \rightarrow L_2(\mathbb{R}); \quad (\mathbb{U}f)(z) := \sqrt{3}e^{-\frac{|z|}{2}} f(3 \operatorname{sign}(z)e^{-|z|}).$$

Then singular values of $\mathbb{Q}_{3,0}$ coincide with ones of $\mathbb{Q}_{3,0}\mathbb{U}^{-1} : L_2(0, \infty) \rightarrow L_2(0, 1)$. Since the kernel $q_{3,0}$ is odd with respect to ξ , we have

$$(\mathbb{Q}_{3,0}\mathbb{U}^{-1}g)(x) = 3^{1-H(x)}ix \int_0^\infty \zeta(3e^{-z})e^{-(1-H(x))z} (g(z) - g(-z)) dz. \quad (13)$$

The function $1 - H(\cdot)$ belongs to $\mathcal{C}^\beta[0, 1]$ and is bounded away from zero. Therefore, the integrand in (13) belongs to \mathcal{C}^β in x and decays exponentially as $z \rightarrow \infty$. Lemma 3 yields the estimate $\mathcal{N}(t, \mathbb{Q}_{3,0}) \leq C(\varepsilon)t^\varepsilon$ for any $\varepsilon > 0$.

Summing up the obtained estimates we arrive at (11). \square

3.2 Operator \mathbb{S} (mfBM)

First, we change the variable $z = y(1 + w)$ in integral (3) and rewrite the kernel of \mathbb{S} as

$$K(x, y, H) = c_*(H)y^{H-\frac{1}{2}} \int_0^{\frac{x-y}{y}} w^{H-\frac{3}{2}}(1+w)^{H-\frac{1}{2}} dw \mathbb{I}_{[0,x]}(y). \quad (14)$$

Next, we separate the principal homogeneous term in (14) and write

$$K(x, y, H) = \tilde{q}_1(x, y, H) + \tilde{q}_2(x, y, H),$$

where

$$\begin{aligned}\tilde{q}_1(x, y, H) &:= c_*(H)y^{H-\frac{1}{2}} \int_0^{\frac{x-y}{y}} w^{H-\frac{3}{2}} dw \chi_{[0,x]}(y) = \frac{c_*(H)}{H-\frac{1}{2}} (x-y)^{H-\frac{1}{2}} \mathbb{I}_{[0,x]}(y); \\ \tilde{q}_2(x, y, H) &= c_*(H)y^{H-\frac{1}{2}} \Phi\left(\frac{x-y}{y}, H\right) \mathbb{I}_{[0,x]}(y), \\ \Phi(s, H) &:= \int_0^s w^{H-\frac{3}{2}} ((1+w)^{H-\frac{1}{2}} - 1) dw.\end{aligned}$$

Since $x, y \in (0, 1)$, we can assume that \tilde{q}_1 is multiplied by a cut-off function $\theta(x-y)$,

$$\theta \in \mathcal{C}^\infty(\mathbb{R}), \quad \theta(w) = 1 \quad \text{for } |w| \leq 1, \quad \theta(w) = 0 \quad \text{for } |w| \geq 2. \quad (15)$$

Let $\tilde{\mathbb{Q}}_1$ and $\tilde{\mathbb{Q}}_2$ be operators in $L_2(0, 1)$ with kernels $\tilde{q}_1(x, y, H(x))$ and $\tilde{q}_2(x, y, H(x))$ respectively. We claim that $\tilde{\mathbb{Q}}_1$ is in fact a pseudodifferential operator of variable order. Indeed, we have

$$(\tilde{\mathbb{Q}}_1 f)(x) = (2\pi)^{-1} c_*(H(x)) \int_{-\infty}^{\infty} \int_0^1 e^{i(x-y)\xi} R\left(\xi, H(x) - \frac{1}{2}\right) f(y) dy d\xi,$$

where

$$R(\xi, \gamma) := \int_0^{\infty} e^{-iz\xi} \theta(z) \frac{z^\gamma}{\gamma} dz.$$

For any $\gamma > -1$, $\gamma \neq 0$, we have $R(\cdot, \gamma) \in \mathcal{C}^\infty(\mathbb{R})$. Moreover, up to a function of the Schwartz class \mathcal{S} , the function $R(\cdot, \gamma)$ coincides at infinity with the Fourier transform of z_+^γ/γ :

$$R(\xi, \gamma) = \Gamma(\gamma) |\xi|^{-(1+\gamma)} \exp(i \operatorname{sign}(\xi)(1+\gamma)\pi/4) + O(|\xi|^{-n}) \quad \text{for any } n \in \mathbb{N}.$$

Thus, $R(\xi, \gamma)$ is a classical symbol of order $-(1+\gamma)$. Therefore, $\tilde{\mathbb{Q}}_1$ can be considered as a pseudodifferential operator of variable order, and the claim follows.

We define a pseudodifferential operator

$$(\tilde{\mathbb{A}}f)(x) := (2\pi)^{-1} c_*(H(x)) \Gamma(H(x) - \frac{1}{2}) \int_{-\infty}^{\infty} \int_0^1 e^{i(x-y)\xi} (\tilde{\mathbf{p}}(\xi))^{-(m+h(x))} f(y) dy d\xi,$$

where $h(x) = H(x) - H_{\min}$ while $\tilde{\mathbf{p}}(\xi)$ is a smooth complex-valued function such that

$$\tilde{\mathbf{p}}(\xi) \neq 0; \quad \tilde{\mathbf{p}}(\xi) = |\xi| \exp(i \operatorname{sign}(\xi)\pi/4) \quad \text{for } |\xi| > 1.$$

The kernel of $\tilde{\mathbb{Q}}_1 - \tilde{\mathbb{A}}$ is smooth in y and bounded in x . By Proposition 2 we have $\mathcal{N}(t, \tilde{\mathbb{Q}}_1 - \tilde{\mathbb{A}}) = O(t^\varepsilon)$ as $t \rightarrow \infty$ for any $\varepsilon > 0$, and part 3 of Proposition 1 gives $\mathcal{N}(t, \tilde{\mathbb{Q}}_1) = \mathcal{N}(t, \tilde{\mathbb{A}}) + O(t^\varepsilon)$.

Part 1 in Corollary 1, see Sec.5.1, gives the singular value asymptotics for the operator $\tilde{\mathbb{A}}$ and therefore for the operator $\tilde{\mathbb{Q}}_1$. Thus, formula (9) is ensured by part 3 of Proposition 1 if we prove the following lemma.

Lemma 2 *The following estimate holds:*

$$\mathcal{N}(t, \tilde{\mathbb{Q}}_2) \leq Ct^{\frac{2}{3}}, \quad t > 1. \quad (16)$$

Proof. The kernel \tilde{q}_2 has singularities on the diagonal $y = x$ and at the point $y = 0$, i.e. for $s \equiv \frac{x-y}{y} = 0$ and $s = \infty$ respectively. We consider the influence of these singularities upon the singular value asymptotic separately.

We consider two functions

$$\Phi_0(s, H) = \theta(s)\Phi(s, H), \quad \Phi_1(s, H) = (1 - \theta(s))\Phi(s, H),$$

where the cut-off function θ is defined in (15), and denote by $\tilde{\mathbb{Q}}_{2,j}$, $j = 0, 1$, the operators in $L_2(0, 1)$ with kernels

$$\tilde{q}_{2,j}(x, y, H(x)) = c_*(H(x))y^{H(x)-\frac{1}{2}}\Phi_j\left(\frac{x-y}{y}, H(x)\right)\chi_{[0,x]}(y).$$

Since $\tilde{\mathbb{Q}}_2 = \tilde{\mathbb{Q}}_{2,0} + \tilde{\mathbb{Q}}_{2,1}$, the estimate (16) follows from similar estimates for operators $\tilde{\mathbb{Q}}_{2,j}$.

We begin with the operator $\tilde{\mathbb{Q}}_{2,0}$. The kernel $\tilde{q}_{2,0}(x, y, H(x))$ does not vanish only for $\frac{x}{3} < y < x$. Since $\Phi(s, H) = O(s^{H+\frac{1}{2}})$ and $\partial_s\Phi(s, H) = O(s^{H-\frac{1}{2}})$ as $s \rightarrow 0$, we evidently have the estimate

$$|\tilde{q}_{2,0}(x, y, H)| \leq C \frac{(x-y)^{H+\frac{1}{2}}}{y}; \quad |\partial_y \tilde{q}_{2,0}(x, y, H)| \leq C \left(\frac{(x-y)^{H-\frac{1}{2}}}{y} + \frac{(x-y)^{H+\frac{1}{2}}}{y^2} \right).$$

We recall that the functional parameter satisfies $H(x) > \frac{1}{2}$. Therefore, for a fixed x the function $\tilde{q}_{2,0}(x, \cdot, H(x))$ is \mathcal{C}^1 -smooth. Moreover, the following estimate holds:

$$\|\tilde{q}_{2,0}(x, \cdot, H(x))\|_{W_2^1(0,1)}^2 \leq Cx^2 \int_{x/3}^x \left(\frac{(x-y)^{H(x)-\frac{1}{2}}}{y^2} \right)^2 dy \leq Cx^{2H(x)-2},$$

so, the integral

$$\int_0^1 \|\tilde{q}_{2,0}(x, \cdot, H(x))\|_{W_2^1(0,1)}^2 dx$$

converges. Now Proposition 2 yields the estimate

$$\mathcal{N}(t, \tilde{\mathbb{Q}}_{2,0}) \leq Ct^{\frac{2}{3}}, \quad t > 1.$$

Further, the kernel $\tilde{q}_{2,1}(x, y, H(x))$ does not vanish only for $0 \leq y \leq x/2$ and has singularity at the point $y = 0$.

Similarly to the estimate for the operator $\mathbb{Q}_{3,0}$ in the previous subsection, we introduce the isometry

$$\tilde{\mathbb{U}} : L_2(0, 1) \rightarrow L_2(0, \infty); \quad (\tilde{\mathbb{U}}f)(z) := e^{-\frac{z}{2}}f(e^{-z}).$$

Then singular values of the operator $\tilde{\mathbb{Q}}_{2,1}$ in $L_2(0, 1)$ coincide with ones of the operator $\tilde{\mathbb{Q}}_{2,1}\tilde{\mathbb{U}}^{-1} : L_2(0, \infty) \rightarrow L_2(0, 1)$. Changing the variable we obtain that the kernel of $\tilde{\mathbb{Q}}_{2,1}\tilde{\mathbb{U}}^{-1}$ is

$$r(x, z, H(x)) := c_*(H(x))x^{H(x)}(1+s)^{-H(x)}\Phi_1(s)|_{s=xe^z-1}.$$

The following estimates for $n \in \mathbb{N} \cup \{0\}$ are obvious:

$$((1+s)\partial_s)^n \Phi_1(s, H) = O(s^{2H-1}), \quad \text{as } s \rightarrow \infty.$$

Since $\partial_z(g(xe^z - 1)) = (1+s)\partial_s g(s)|_{s=xe^z-1}$, we obtain for any $n \in \mathbb{N} \cup \{0\}$

$$|\partial_z^n r(x, z, H(x))| \leq C(n)x^{H(x)}(1+s)^{-(1-H(x))}|_{s=xe^z-1} \stackrel{*}{\leq} C(n)e^{-(1-H(x))z}$$

(the inequality $(*)$ follows from $H(x) > \frac{1}{2}$).

The function $1 - H(\cdot)$ belongs to $\mathcal{C}^\beta[0, 1]$ and is bounded away from zero. Therefore, the kernel $r(x, z, H(x))$ belongs to \mathcal{C}^β in x and decays exponentially as $z \rightarrow \infty$. Lemma 3 yields the estimate $\mathcal{N}(t, \tilde{\mathbb{Q}}_{2,1}) \leq C(\varepsilon)t^\varepsilon$ for any $\varepsilon > 0$.

Summing up the estimates for $\tilde{\mathbb{Q}}_{2,0}$ and $\tilde{\mathbb{Q}}_{2,1}$ we arrive at (16). \square

4 Small ball asymptotics for mBM and mfBM

As explained in the Introduction, one-term asymptotic of eigenvalues for covariance operator provides, under mild assumptions (see [17, Theorem 1]), the logarithmic L_2 -small ball asymptotic for corresponding process.

We begin with the multifractional Brownian motion (1). The case $\text{meas } \mathbf{D} > 0$ is in fact quite elementary. In this case formula (8) and part 3 of Remark 1 give

$$\mathcal{N}(t, \mathbb{T}) \sim \frac{\mathfrak{C}}{\pi} t^{\frac{1}{\mathfrak{m}}}, \quad \text{as } t \rightarrow \infty,$$

where

$$\mathfrak{C} = \left(\Gamma(2\mathfrak{m}) \sin(\pi H_{min}) \right)^{\frac{1}{2\mathfrak{m}}} \text{meas } \mathbf{D}.$$

Since the function $t \mapsto \mathcal{N}(t, \mathbb{T})$ is in essence inverse to the function $k \mapsto s_k^{-1}(\mathbb{T})$, we have, as $k \rightarrow \infty$,

$$s_k(\mathbb{T}) \sim \left(\frac{\mathfrak{C}}{\pi k} \right)^{\mathfrak{m}} \iff \lambda_k(\mathbb{T}\mathbb{T}^*) \sim \left(\frac{\mathfrak{C}}{\pi k} \right)^{2\mathfrak{m}}.$$

Notice that $2\mathfrak{m} > 1$. Applying Proposition 2.1 in [21] we obtain

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{2}{2\mathfrak{m}-1}} \log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} = - \frac{2\mathfrak{m}-1}{2} \left(\frac{\mathfrak{C}}{2\mathfrak{m} \sin(\frac{\pi}{2\mathfrak{m}})} \right)^{\frac{2\mathfrak{m}}{2\mathfrak{m}-1}}.$$

Now we consider the case $\text{meas } \mathbf{D} = 0$. In this case formula (8) and part 3 of Remark 1 give

$$\mathcal{N}(t, \mathbb{T}) \sim \frac{\tilde{\mathfrak{C}}}{\pi} t^{\frac{1}{\mathfrak{m}}} (\log t)^{-\sigma} \varphi(\log t).$$

where

$$\tilde{\mathfrak{C}} = \left(\Gamma(2\mathfrak{m}) \sin(\pi H_{min}) \right)^{\frac{1}{2\mathfrak{m}}} \mathfrak{m}^{2\sigma} \Gamma(\sigma + 1).$$

Therefore, we have, as $k \rightarrow \infty$,

$$s_k(\mathbb{T}) \sim \left(\frac{\tilde{\mathfrak{C}}}{\pi \mathfrak{m}^\sigma} \cdot \frac{\varphi(\log k)}{k \log^\sigma k} \right)^{\mathfrak{m}} \iff \lambda_k(\mathbb{T}\mathbb{T}^*) \sim \left(\frac{\tilde{\mathfrak{C}}}{\pi \mathfrak{m}^\sigma} \cdot \frac{\varphi(\log k)}{k \log^\sigma k} \right)^{2\mathfrak{m}}.$$

Since λ_k is a sequence regularly varying with index $2\mathbf{m} > 1$, we can apply [14, Theorem 4.2] where a general situation was considered. Concretization of formula (4.5) in [14] for our case gives

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{2}{2\mathbf{m}-1}} \left(\frac{\log^\sigma \frac{1}{\varepsilon}}{\varphi(\log \frac{1}{\varepsilon})} \right)^{\frac{2\mathbf{m}}{2\mathbf{m}-1}} \log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \\ = - \frac{2\mathbf{m}-1}{2} \left(\frac{\tilde{\mathfrak{C}}}{2\mathbf{m} \sin(\frac{\pi}{2\mathbf{m}})} \left(\frac{2\mathbf{m}-1}{2\mathbf{m}} \right)^\sigma \right)^{\frac{2\mathbf{m}}{2\mathbf{m}-1}}. \end{aligned}$$

Now we are able to formulate the final statement.

Theorem 2 *Assume that the variable Hurst parameter $0 < H(x) < 1$ satisfies the assumptions 1–3 before Theorem 1 with τ subject to the condition (7). Then for the mBM $W^{H(\cdot)}$, the following relation holds as $\varepsilon \rightarrow 0$:*

1. *If $\text{meas } \mathbf{D} > 0$ then*

$$\begin{aligned} \log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \\ \times \frac{H_{\min} \text{meas } \mathbf{D}}{(2H_{\min} + 1) \sin(\frac{\pi}{2H_{\min}+1})} \left(\frac{\Gamma(2H_{\min} + 1) \sin(\pi H_{\min}) \text{meas } \mathbf{D}}{(2H_{\min} + 1) \sin(\frac{\pi}{2H_{\min}+1})} \right)^{\frac{1}{2H_{\min}}} \end{aligned} \quad (17)$$

2. *If $\text{meas } \mathbf{D} = 0$ then*

$$\begin{aligned} \log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \left(\frac{\varphi(\log \frac{1}{\varepsilon})}{\log^\sigma \frac{1}{\varepsilon}} \right)^{\frac{2H_{\min}+1}{2H_{\min}}} \\ \times H_{\min} \left(\frac{(\Gamma(2H_{\min} + 1) \sin(\pi H_{\min}))^{\frac{1}{2H_{\min}+1}} \Gamma(\sigma + 1) (H_{\min}(H_{\min} + \frac{1}{2}))^\sigma}{(2H_{\min} + 1) \sin(\frac{\pi}{2H_{\min}+1})} \right)^{\frac{2H_{\min}+1}{2H_{\min}}} \end{aligned} \quad (18)$$

To illustrate this theorem, we give several examples. For simplicity only, we assume that $H_{\min} = \frac{1}{2}$.

Example 1. Let $H(x) = \frac{1}{2} + (x - x_0)_+^\gamma$, $0 < x_0 \leq 1$, $\gamma > 0$. Then we have $\mathbf{D} = [0, x_0]$, and formula (17) reads

$$\log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\frac{x_0^2}{8} \varepsilon^{-2}, \quad \text{as } \varepsilon \rightarrow 0.$$

For $x_0 = 1$ we obtain standard Wiener process on $[0, 1]$, and this result is well known. However, for $x_0 < 1$ even this simplest result seems to be new.

Example 2. Let $H(x) = \frac{1}{2} + |x - x_0|^\gamma$, $\gamma > 0$. In this case we have purely power-like behavior of the measure of the small values set. Namely, formula (6) holds with

$$\sigma = \frac{1}{\gamma}; \quad \varphi(s) \equiv 2 \quad \text{if } 0 < x_0 < 1; \quad \varphi(s) \equiv 1 \quad \text{if } x_0 = 0, 1.$$

Therefore, formula (18) gives so

$$\log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\widehat{C}(x_0)\Gamma^2\left(1 + \frac{1}{\gamma}\right) \cdot \left(\varepsilon \log^{\frac{1}{\gamma}} \frac{1}{\varepsilon}\right)^{-2}, \quad \text{as } \varepsilon \rightarrow 0, \quad (19)$$

where

$$\widehat{C}(x_0) = 2^{-1-\frac{2}{\gamma}} \quad \text{if } 0 < x_0 < 1; \quad \widehat{C}(x_0) = 2^{-3-\frac{2}{\gamma}} \quad \text{if } x_0 = 0, 1.$$

Example 3. Let $H(x) = \frac{1}{2} + \min_{1 \leq k \leq N} |x - x_k|^{\gamma_k}$, with $0 \leq x_k \leq 1$, $\gamma_k > 0$, $k = 1, \dots, N$.

In this case the function H attains minimum at several points but only the point(s) with maximal γ_k affect the asymptotics. For instance, if $\gamma_k = \gamma$ for $k \leq n$, $\gamma_k < \gamma$ for $k > n$, and $0 < x_k < 1$ for $k \leq n$ then formula (19) holds with $\widehat{C}(x_0) = 2^{-1-\frac{2}{\gamma}}n^2$.

Example 4. Let $H(x) = \frac{1}{2} + |x - x_0|^\gamma \log^b(|x - x_0|^{-1})$, with $x_0 \in (0, 1)$, $\gamma > 0$ and $b \in \mathbb{R}$. Then formula (6) holds with

$$\sigma = \frac{1}{\gamma}; \quad \varphi(s^{-1}) = 2\gamma^{\frac{b}{\gamma}} (\log s^{-1})^{-\frac{b}{\gamma}} (1 + o(1)) \quad \text{as } s \rightarrow 0.$$

By part 4 of Remark 1, we can drop $o(1)$ in the latter relation, and formula (18) gives

$$\log \mathbb{P}\{\|W^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\frac{\gamma^{\frac{2b}{\gamma}} \Gamma^2(1 + \frac{1}{\gamma})}{2^{1+\frac{2}{\gamma}}} \cdot \left(\varepsilon \log^{\frac{1}{\gamma}} \frac{1}{\varepsilon} \log^{\frac{b}{\gamma}} \log \frac{1}{\varepsilon}\right)^{-2}, \quad \text{as } \varepsilon \rightarrow 0.$$

Example 5. Let $H(x) = \frac{1}{2} + \text{dist}^\gamma(x, \mathfrak{D})$, where \mathfrak{D} is standard Cantor set. A tedious but simple calculation gives for small $s > 0$

$$\text{meas}\{x \in [0, 1] \mid 0 < h(x) < s\} = s^{1-\frac{\log 2}{\log 3}} \phi(\log s^{-1}),$$

where ϕ is a *periodic* function. Therefore, in this case the assumption **3** before Theorem 1 is not valid, and this case is not covered by our Theorem 2. We are planning to consider corresponding class of processes in a forthcoming paper.

Now we turn to the multifractal Brownian motion (2). By part 1 of Remark 1, for $H_{\min} > \frac{1}{2}$ formulae (8) and (9) coincide, and thus the logarithmic asymptotic for $X^{H(\cdot)}$ coincides with that of $W^{H(\cdot)}$.

Theorem 3 *Assume that the variable Hurst parameter $\frac{1}{2} < H(x) < 1$ satisfies the assumptions **1–3** before Theorem 1 with τ subject to the condition (7). Then for the mfBM $X^{H(\cdot)}$, the following relation holds as $\varepsilon \rightarrow 0$:*

1. If $\text{meas } \mathbf{D} > 0$ then

$$\begin{aligned} & \log \mathbb{P}\{\|X^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \\ & \times \frac{H_{\min} \text{meas } \mathbf{D}}{(2H_{\min} + 1) \sin\left(\frac{\pi}{2H_{\min}+1}\right)} \left(\frac{\Gamma(2H_{\min} + 1) \sin(\pi H_{\min}) \text{meas } \mathbf{D}}{(2H_{\min} + 1) \sin\left(\frac{\pi}{2H_{\min}+1}\right)} \right)^{\frac{1}{2H_{\min}}}. \end{aligned}$$

2. If $\text{meas } \mathbf{D} = 0$ then

$$\log \mathbb{P}\{\|X^{H(\cdot)}\|_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \left(\frac{\varphi(\log \frac{1}{\varepsilon})}{\log^\sigma \frac{1}{\varepsilon}} \right)^{\frac{2H_{\min}+1}{2H_{\min}}} \\ \times H_{\min} \left(\frac{(\Gamma(2H_{\min} + 1) \sin(\pi H_{\min}))^{\frac{1}{2H_{\min}+1}} \Gamma(\sigma + 1) (H_{\min}(H_{\min} + \frac{1}{2}))^\sigma}{(2H_{\min} + 1) \sin(\frac{\pi}{2H_{\min}+1})} \right)^{\frac{2H_{\min}+1}{2H_{\min}}}.$$

Remark 2 For both processes (1) and (2), in the case $\text{meas } \mathbf{D} > 0$ we obtain purely power asymptotics. Even in the case $\sigma < \nu$ when Theorem 1 gives a two-term asymptotics, see part 3 in Remark 1, the estimate of the remainder term is not sufficient to obtain exact small ball asymptotics.

The asymptotic coefficient in this case depends on $\text{meas } \mathbf{D}$ and for $\text{meas } \mathbf{D} = 1$ coincides with classical result of Bronski [8], see also [21, Theorem 3.1].

5 Appendix

5.1 Asymptotics of singular values for pseudodifferential operators of variable order

We consider a compact pseudodifferential operator $\mathbb{A} : L_2(\mathbb{R}) \rightarrow L_2(0, 1)$,

$$(\mathbb{A}f)(x) := (2\pi)^{-1} a(h(x)) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-y)\xi} (\mathbf{p}(\xi))^{-(m+h(x))} f(y) dy d\xi. \quad (20)$$

Here $m > \frac{1}{2}$, and $h \in \mathcal{C}^\lambda[0, 1]$ is a non-negative function. The complex-valued non-vanishing function \mathbf{p} is an elliptic symbol in Hörmander class $S_{\rho,0}^1$, $0 < \rho \leq 1$, namely,

$$\mathbf{p} \in \mathcal{C}^\infty(\mathbb{R}), \quad |\mathbf{p}(\xi)| = v|\xi|(1 + O(|\xi|^{-\mu})) \quad \text{as } |\xi| \rightarrow \infty \quad \text{for some } \mu > 0; \quad (21) \\ |\mathbf{p}^{(n)}(\xi)| \leq C(n)(1 + |\xi|)^{1-\rho n} \quad \text{for any } n \in \mathbb{N} \cup 0.$$

The (complex-valued) multiplier $a \in \mathcal{C}^\infty[0, \infty)$ satisfies $a(0) \neq 0$.

The order of decay of the symbol $p(\xi)^{-(m+h(x))}$ as $|\xi| \rightarrow \infty$, which can be interpreted as the local order of operator \mathbb{A} , depends on x . So, we say that \mathbb{A} is an operator of variable order.

Similarly to Section 2, we consider the set $\mathbf{D} := \{x \in [0, 1] \mid h(x) = 0\}$.

Theorem 4 Assume that the assumptions 1–3 before Theorem 1 are fulfilled. Assume in addition that

$$\tau > \begin{cases} \max\{0, \frac{1}{2}(1 - \sigma)\}, & \text{if } \text{meas } \mathbf{D} > 0; \\ \sigma m + \frac{1}{2}(1 - \sigma), & \text{if } \text{meas } \mathbf{D} = 0. \end{cases}$$

Then, as $t \rightarrow \infty$,

$$\mathcal{N}(t, \mathbb{A}) = \frac{1}{\pi v} (a(0))^{\frac{1}{m}} \int_0^1 t^{\frac{1}{m+h(x)}} dx (1 + O(\log^{-\nu} t)). \quad (22)$$

Here ν is defined in (21) while $\nu = \nu(m, \sigma, \tau) > 0$. If $\text{meas } \mathbf{D} > 0$ then ν is arbitrary exponent less than $\frac{\tau - \frac{1}{2}(1-\sigma)}{m+1}$.

Proof. Assumptions 1–3 ensure that the conditions of [13, theorem 1.3] are satisfied. In [13] the statement was proved, even in a multidimensional case, for $a \equiv 1$. The proof runs without changes for the multiplier $a(h_0(x))$. To include the nonsmooth multiplier $a(h(x))$ one needs to study the operator with the symbol

$$(a(h(x)) - a(h_0(x))) (\mathbf{p}(\xi))^{-(m+h(x))}.$$

Repeating the argument of [13, theorem 5.1] we obtain the estimate

$$O(\log^{-\nu} t) \int_0^1 t^{\frac{1}{m+h(x)}} dx$$

for the singular values counting function of this operator. Part 4 in Proposition 1 with regard to (10) completes the proof. \square

Remark 3 1. Theorem 4 in fact claims that Weyl's spectral asymptotic formula

$$\mathcal{N}(t, \mathbb{A}) \sim (2\pi)^{-1} \text{meas} \{x, \xi \in (0, 1) \times \mathbb{R} \mid |a(h(x))\mathbf{p}(\xi)^{-(m+h(x))}| > t^{-1}\}$$

is valid for operator \mathbb{A} . Asymptotics (22) is just concretization of Weyl's formula for our problem with the remainder estimate.

2. If $h_1(x) \equiv 0$, then formula (22) is valid with $O(t^{-r})$ remainder estimate for some $r > 0$.

Corollary 1 1. One can consider the operator in formula (20) as an operator acting in $L_2(0, 1)$ if we extend $u \in L_2(0, 1)$ by zero. Formula (22) remains valid in this case.

2. Since Fourier transform is the unitary operator in $L_2(\mathbb{R})$, formula (22) holds also for the operator

$$(\mathbb{A}f)(x) := (2\pi)^{-\frac{1}{2}} a(h(x)) \int_{-\infty}^{\infty} e^{ix\xi} (\mathbf{p}(\xi))^{-(m+h(x))} f(\xi) d\xi.$$

5.2 Estimates for singular values of integral operator

We need the following results in asymptotic perturbations theory.

Proposition 1 1. Let $\mathbb{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $\mathbb{B} : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = O(t^{p_1})$ and

$$\mathcal{N}(t, \mathbb{B}) = O(t^{p_2}) \text{ as } t \rightarrow \infty \text{ then } \mathcal{N}(t, \mathbb{B}\mathbb{A}) = O(t^p) \text{ with } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

2. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = O(t^{p_1})$ and $\mathcal{N}(t, \mathbb{B}) = O(t^{p_2})$ as $t \rightarrow \infty$ then $\mathcal{N}(t, \mathbb{A} + \mathbb{B}) = O(t^p)$.

3. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = t^p V(t)$ as $t \rightarrow \infty$ with a slowly varying function V , and $\mathcal{N}(t, \mathbb{B}) = O(t^q)$ for some $q < p$, then

$$\mathcal{N}(t, \mathbb{A} + \mathbb{B}) = t^p (V(t) + O(t^{-r})), \quad \text{where } r < \frac{p-q}{q+1}.$$

4. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be compact operators. Assume that

$$\mathcal{N}(t, \mathbb{A}) = t^p (\delta + \varkappa \log^{-a} t V(\log t)) \quad \text{as } t \rightarrow \infty$$

with $a > 0$ and a slowly varying function V , and $\mathcal{N}(t, \mathbb{B}) = O(t^p \log^{-b} t)$ for some $b > a$. Then

$$\mathcal{N}(t, \mathbb{A} + \mathbb{B}) = t^p (\delta + \varkappa \log^{-a} t V(\log t) + O(\log^{-r} t)),$$

where r is arbitrary exponent such that

$$r < \frac{b}{p+1}, \quad \text{if } \delta > 0; \quad r < \frac{ap+b}{p+1}, \quad \text{if } \delta = 0.$$

The assertions 1 and 2 are elementary consequences of the inequalities (17) and (19) in [7, Sec. 11.1], respectively. The statements 3 and 4 refine the estimate in [6, Lemma 3.1] and are contained in [13, Lemma 2.1].

Now we formulate the singular values estimates for integral operator in terms of the properties of its kernel. The results are given for d -dimensional domains, though we need only the case $d = 1$. In what follows W_2^λ stands for the standard Sobolev–Slobodetskii space, see, e.g., [27, Sec. 2.3.1].

Let Ω be a bounded domain in \mathbb{R}^d with Lipschitz boundary. We put

$$\mathfrak{R} : L_2(\mathbb{R}^d) \rightarrow L_2(\Omega), \quad (\mathfrak{R}f)(x) := a(x) \int_{\mathbb{R}^d} R(x, y) f(y) dy,$$

where $a \in L_\infty(\Omega)$.

Proposition 2 (see [7, §11.8]). *Let the function $R(\cdot, y) \in W_2^\lambda(\Omega)$ for a.e. $y \in \mathbb{R}^d$, and let*

$$M^2 := \int_{\mathbb{R}^d} \|R(\cdot, y)\|_{W_2^\lambda(\Omega)}^2 dy < \infty. \quad (23)$$

Then the following estimate holds:

$$\mathcal{N}(t, \mathfrak{R}) \leq C(\Omega) (M \|a\|_{L_\infty(\Omega)})^p t^p, \quad \frac{1}{p} = \frac{1}{2} + \frac{\lambda}{d}.$$

Now we take into account the decay of the kernel with respect to the second variable. Let

$$\tilde{\mathfrak{R}} : L_2(\mathbb{R}^d) \rightarrow L_2(\Omega), \quad (\tilde{\mathfrak{R}}f)(x) := \int_{\mathbb{R}^d} \tilde{R}(x, \xi) f(\xi) d\xi.$$

Lemma 3 *Let the function $\tilde{R}(\cdot, \xi) \in W_2^\lambda(\Omega)$ for a.e. $\xi \in \mathbb{R}^d$, and let*

$$\|\tilde{R}(\cdot, \xi)\|_{W_2^\lambda(\Omega)} \leq M(\ell, \lambda)(1 + |\xi|)^{-\ell} \quad \text{for some } \ell > \frac{d}{2}. \quad (24)$$

Assume also that $\tilde{R}(x, \cdot) \in C^\infty(\mathbb{R}^d)$, and

$$|\partial_\xi^\alpha \tilde{R}(x, \xi)| \leq C(\alpha)(1 + |\xi|)^{-(\ell + \rho|\alpha|)}, \quad x \in \Omega, \xi \in \mathbb{R}^d$$

for some $\rho > 0$ and any multi-index α .

Then

$$\mathcal{N}(t, \tilde{\mathfrak{R}}) \leq C(\ell, \lambda, p, \Omega)t^p \quad \text{for any } p > \frac{d}{\ell + \lambda}.$$

Proof. Since the Fourier transform $\mathfrak{F} : L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$ is unitary, the singular values of operators $\tilde{\mathfrak{R}}$ and $\mathfrak{R} = \tilde{\mathfrak{R}}\mathfrak{F}$ coincide.

We split \mathfrak{R} into two parts:

$$\mathfrak{R} = \mathfrak{R}_0 + \mathfrak{R}_1,$$

with the kernels

$$R_0(x, y) = \theta(|y|)R(x, y), \quad R_1(x, y) = (1 - \theta(|y|))R(x, y),$$

where the cut-off function θ is defined in (15), while

$$R(x, y) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \tilde{R}(x, \xi) e^{-iy\xi} d\xi, \quad x \in \Omega, y \in \mathbb{R}^d.$$

Integrating by parts with respect to ξ and using the identity

$$e^{-iy\xi} = (-\Delta_\xi)^N |y|^{-2N} e^{-iy\xi},$$

we obtain that the function $R_1(x, \cdot)$ belongs to the Schwartz class uniformly with respect to $x \in \Omega$. By [5, theorem 4.8], this gives $\mathcal{N}(t, \mathfrak{R}_1) = O(t^\varepsilon)$ for any $\varepsilon > 0$. So, part 2 of Proposition 1 shows that the estimate of $\mathcal{N}(t, \mathfrak{R})$ is governed by the estimate of $\mathcal{N}(t, \mathfrak{R}_0)$.

We choose some $0 < l < \ell - \frac{d}{2}$ and write \mathfrak{R}_0 in the form $\mathfrak{R}_0 = \mathfrak{R}_{0,0}\mathfrak{R}_{0,1}$, where the kernels of $\mathfrak{R}_{0,j}$, $j = 0, 1$, are

$$\begin{aligned} R_{0,0}(x, \xi) &= (2\pi)^{-\frac{d}{2}} \tilde{R}(x, \xi)(1 + |\xi|^2)^{\frac{l}{2}}, \quad x \in \Omega, \xi \in \mathbb{R}^d; \\ R_{0,1}(\xi, y) &= e^{-iy\xi}(1 + |\xi|^2)^{-\frac{l}{2}}\theta(|y|), \quad \xi, y \in \mathbb{R}^d. \end{aligned}$$

For operator $\mathfrak{R}_{0,0}$ we have the estimate $\mathcal{N}(t, \mathfrak{R}_{0,0}) \leq C(\Omega)t^{\frac{d}{2}}$ (this is a particular case of the Rozenblum–Lieb–Cwikel estimate, see, e.g., [5, theorem 6.5]). The estimate (24) ensures the inequality (23), and Proposition 2 yields the estimate $\mathcal{N}(t, \mathfrak{R}_{0,0}) \leq C(\ell, \lambda, l)t^{p_1}$, $\frac{1}{p_1} = \frac{1}{2} + \frac{\lambda}{d}$. By part 1 of Proposition 1, we have

$$\mathcal{N}(t, \mathfrak{R}_0) \leq C(\ell, \lambda, l, \Omega)t^p, \quad \frac{1}{p} = \frac{1}{2} + \frac{\lambda}{d} + \frac{l}{d} < \frac{\ell + \lambda}{d},$$

and the statement follows. \square

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