Tropical algebra solution of a project scheduling problem

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Introduction

Time-constrained project scheduling problems constitute an integral part of project management. These problems are to find an optimal schedule for a project that consists of a set of activities operating in parallel under various temporal constraints, including start-start, start-finish, finish-start, release time, deadline, due-dates and other constraints. As optimization criteria to minimize, one can take the project makespan, the maximum deviation from due dates, the maximum flow-time, the maximum deviation of start or finish times [1, 2].

Many time-constrained scheduling problems can be formulated as linear, integer, or mixed-integer linear programs, graph and network optimization problems, and then solved using appropriate computational algorithms. This approach usually allows one to obtain a numerical solution of the problem, but cannot provide a complete analytical solution in an explicit form.

In this paper, we consider a project, in which activities are performed under temporal constraints in the form of start-start precedence relationships, release start and release end times. The scheduling problem of interest is to find the start times of activities to provide the minimum deviation of start times. Such an optimality criterion can arise when the schedule has to provide a common start time for all activities in the project.

We represent the problem in terms of tropical mathematics, which deals with the theory and applications of algebraic systems with idempotent operations [3, 4, 5]. To solve the project scheduling problem, we apply methods and results of tropical optimization [6, 7, 8], and then obtain a new complete solution, which provides the result in an explicit analytical form, ready for further analysis and numerical implementation.

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1. A time-constrained project scheduling problem

Consider a project that consists of n activities operating under start-start, release start and release end temporal constraints. For each activities $i = 1, \ldots, n$ we denote the start time by x_i. Let g_i and h_i be release start and release end times, which specify the earliest and latest allowed time for activity i to start. Let b_{ij} be the minimum allowed time lag between the start of activity i and the start of j.

Suppose that the optimal schedule has to minimize the maximum deviation of start times x_i over all activities i. The project scheduling problem is formulated as follows: given the parameters g_i, h_i and b_{ij}, find the start times x_i for all $i = 1, \ldots, n$ to solve the minimization problem

$$\begin{aligned} &\max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i) \to \min, \\ &\max_{1\leq j\leq n}(x_j + b_{ij}) \leq x_i, \\ &g_i \leq x_i \leq h_i, \quad i = 1, \ldots, n. \end{aligned} \tag{1}$$

2. Elements of tropical algebra

Let $\mathbb{X}$ be a set endowed with two associative and commutative operations: $\oplus$ (addition) and $\otimes$ (multiplication), and equipped with additive and multiplicative neutral elements: $\mathbb{0}$ (zero) and $\mathbb{1}$ (unit). Addition is idempotent, which yields $x \oplus x = x$ for each $x \in \mathbb{X}$. Multiplication distributes over addition and is invertible to provide each nonzero $x \in \mathbb{X}$ with its inverse x^{-1} such that $x^{-1} \otimes x = \mathbb{1}$. The algebraic structure $(\mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes)$ is normally called the idempotent semifield.

Let $\mathbb{X}^{m\times n}$ be the set of matrices consisting of m rows and n columns with elements from $\mathbb{X}$. Matrix addition and multiplication and multiplication by scalars are performed according to the usual rules with replacement of arithmetic addition and multiplication by the operations $\oplus$ and $\otimes$.

Consider the set $\mathbb{X}^{n\times n}$ of square matrices of order n. A matrix with $\mathbb{1}$ on the diagonal and $\mathbb{0}$ elsewhere is the identity matrix denoted $\boldsymbol{I}$. The power notation with nonnegative integer exponents serves to represent iterated products of matrices as follows: $\boldsymbol{A}^0 = \boldsymbol{I}$ and $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A}$ for any matrix $\boldsymbol{A}$ and integer $p > 0$.

The trace of a matrix $\boldsymbol{A} = (a_{ij})$ is calculated as $\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$.

Furthermore, we introduce the function

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n.$$

If $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, we define a matrix, which is usually called the Kleene star matrix

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}.$$

Let $\mathbb{X}^n$ denote the set of column vectors of dimension n. A vector containing all elements as $\mathbb{0}$ is the zero vector. A vector without zero components is called regular. A vector with all elements equal to $\mathbb{1}$ is denoted by $\mathbf{1} = (\mathbb{1}, \ldots, \mathbb{1})^T$.

For any nonzero vector $\boldsymbol{x} = (x_i) \in \mathbb{X}^n$, its multiplicative conjugate transpose is the row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i^- > \mathbb{0}$, and $x_i^- = \mathbb{0}$ otherwise.

An example of the idempotent semifield under consideration is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, in which the addition $\oplus$ is defined as maximum, and the multiplication $\otimes$ is as ordinary addition, with the zero $\mathbb{0}$ given by $-\infty$, and the identity $\mathbb{1}$ by 0. Each number $x \in \mathbb{R}$ has the inverse x^{-1} equal to the opposite number $-x$ in the conventional notation. For all $x, y \in \mathbb{R}$, the power x^y is well-defined and coincides with the arithmetic product xy.

In the algebraic expressions below, the multiplication sign $\otimes$ is omitted to save writing: $x \otimes y = xy$.

3. Representation and solution of project scheduling problem

Consider problem (1) and represent it in terms of the semifield $\mathbb{R}_{\max,+}$. The constraints in the problem take the form

$$\bigoplus_{j=1}^{n} b_{ij} x_j \leq x_i, \qquad g_i \leq x_i \leq h_i, \qquad i = 1, \ldots, n.$$

Next, we introduce the matrix-vector notation $\boldsymbol{B} = (b_{ij})$, $\boldsymbol{g} = (g_i)$ and $\boldsymbol{h} = (h_i)$ to represent the constraints in the vector form

$$\begin{aligned} &\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \\ &\boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}, \end{aligned}$$

and note that the inequalities $\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$ and $\boldsymbol{g} \leq \boldsymbol{x}$ are equivalent to the inequality $\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}$.

In terms of the semifield $\mathbb{R}_{\max,+}$, the objective function becomes

$$\bigoplus_{1 \leq i \leq n} x_i \bigoplus_{1 \leq j \leq n} x_j^{-1} = \boldsymbol{1}^{\mathrm{T}} \boldsymbol{x} \boldsymbol{x}^- \boldsymbol{1} = \boldsymbol{x}^- \boldsymbol{1} \boldsymbol{1}^{\mathrm{T}} \boldsymbol{x}.$$

By combining the objective function with constraints, we obtain the tropical optimization problem

$$\begin{aligned} &\boldsymbol{x}^- \boldsymbol{1} \boldsymbol{1}^{\mathrm{T}} \boldsymbol{x} \to \min, \\ &\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}, \\ &\boldsymbol{x} \leq \boldsymbol{h}. \end{aligned} \tag{2}$$

The following result offers a complete solution to the problem.

Theorem 1. *Let $\boldsymbol{B}$ be a matrix, $\boldsymbol{g}$ be a vector, and $\boldsymbol{h}$ be a regular vector such that* $\mathrm{Tr}(\boldsymbol{B}) \oplus \boldsymbol{h}^- \boldsymbol{B}^* \boldsymbol{g} \leq \mathbb{1}$. *Then the minimum in problem* (2) *is equal to*

$$\theta = \bigoplus_{i=0}^{n-1} \boldsymbol{1}^{\mathrm{T}} \boldsymbol{B}^i (\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{h}^-)(\boldsymbol{I} \oplus \boldsymbol{B})^{n-1-i} \boldsymbol{1}, \tag{3}$$

and all regular solutions are given by

$$\boldsymbol{x} = (\theta^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \quad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{h}^{-}(\theta^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} \oplus \boldsymbol{B})^{*})^{-}. \tag{4}$$

If $\mathrm{Tr}(\boldsymbol{B}) \oplus \boldsymbol{h}^{-}\boldsymbol{B}^{*}\boldsymbol{g} > \mathbb{1}$, *then there are no regular solutions.*

4. A numerical example

Let us examine a project that involves $n = 4$ activities under constraints given by the matrix and the vectors

$$\boldsymbol{B} = \begin{pmatrix} 0 & 2 & 3 & 1 \\ \mathbb{0} & 0 & 6 & 7 \\ \mathbb{0} & \mathbb{0} & 0 & 3 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \qquad \boldsymbol{h} = \begin{pmatrix} 20 \\ 15 \\ 10 \\ 10 \end{pmatrix}.$$

We start with the verification of existence conditions for regular solutions in Theorem 2. We obtain

$$\boldsymbol{B}^{*} = \begin{pmatrix} 0 & 2 & 8 & 11 \\ \mathbb{0} & 0 & 6 & 9 \\ \mathbb{0} & \mathbb{0} & 0 & 3 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}, \qquad \boldsymbol{h}^{-}\boldsymbol{B}^{*}\boldsymbol{g} = -2, \qquad \mathrm{Tr}\,\boldsymbol{B} = 0.$$

Since $\mathrm{Tr}\,\boldsymbol{B} \oplus \boldsymbol{h}^{-}\boldsymbol{B}^{*}\boldsymbol{g} = 0$, the problem has regular solutions. As the next step, we find the minimum value θ by application of (3). We have

$$\theta = 11.$$

To describe the solution set defined by (4), we obtain

$$(\theta^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} \oplus \boldsymbol{B})^{*} = \begin{pmatrix} 0 & 2 & 8 & 11 \\ -2 & 0 & 6 & 9 \\ -8 & -6 & 0 & 3 \\ -11 & -9 & -3 & 0 \end{pmatrix}, \qquad (\boldsymbol{h}^{-}(\theta^{-1}\mathbf{1}\mathbf{1}^{\mathrm{T}} \oplus \boldsymbol{B})^{*})^{-} = \begin{pmatrix} 17 \\ 15 \\ 9 \\ 6 \end{pmatrix}.$$

With (4), all solutions $\boldsymbol{x}$ to the problem are given by

$$\boldsymbol{x} = \begin{pmatrix} 0 & 2 & 8 & 11 \\ -2 & 0 & 6 & 9 \\ -8 & -6 & 0 & 3 \\ -11 & -9 & -3 & 0 \end{pmatrix}\boldsymbol{u}, \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \leq \boldsymbol{u} \leq \begin{pmatrix} 17 \\ 15 \\ 9 \\ 6 \end{pmatrix}.$$

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