Exceptional Uniform Polytopes of the E_6, E_7 and E_8 Symmetry Types

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Abstract. The goal of this project is to review the structure and geometry of the nine Gosset—Elte uniform polytopes in dimensions 6 through 8, of exceptional symmetry types E_6, E_7 and E_8. These polytopes were extensively studied by Coxeter, Conway, Sloane, Moody, Patera, McMullen, and many other remarkable mathematicians. We develop a new easier approach towards their combinatorial and geometric properties. In particular, we propose a new way to describe the faces of these polytopes, and their adjacencies, inscribed subpolytopes, compounds, independent subsets, foldings, and the like. Our main tools — weight diagrams, description of root subsystems and conjugacy classes of the Weyl group — are elementary and standard in the representation theory of algebraic groups. But we believe their specific use in the study of polytopes might be new, and considerably simplifies computations. As an illustration of our methods that seems to be new, we calculate the cycle indices for the actions of the Weyl groups on the faces of these polytopes. With our tools, this can be done by hand in the easier cases, such as the Schläfli and Hesse polytopes for E_6 and E_7. Nevertheless, the senior polytopes and the case of E_8 require the use of computers anyway, even after all possible simplifications. Since our actual new results are mostly of technical and/or computational nature, the talk itself will be mostly expository, explaining the background and the basic ideas of our approach, and presenting much easier proofs of the classical results.

Introduction

Marcel Berger [11], p. 39–40, attributes to René Thom the division of mathematical structures into

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• **rich** = rigid, that become progressively scarce in higher dimensions, orders, ranks, etc., and

• **poor** = soft that abound in higher sizes, and that eventually become impossible to classify.

One of the classical examples of this phenomenon are regular polytopes and their kin, such as semiregular and other strictly uniform polytopes. They abound in dimension 2, are still quite freakish in dimension 3, and then eventually crystallise to very few possible shapes that self-reproduce throughout all dimensions.

Essentially everything that is of earnest mathematical interest takes place in dimensions 3 through 8, and is closely related to quaternions, octonions[1], exceptional root systems H_3, D_4, F_4, H_4, D_5, E_6, E_7 and E_8, see, *in particular*, [2, 3, 12, 18, 21, 22, 24, 25, 26, 27, 28, 29, 30, 41, 42, 51, 55, 63, 64, 66, 67, 72, 73, 78, 79, 80, 82, 85, 88, 89, 93, 94, 106, 107], see also [8, 14, 10, 20, 31, 33, 37, 43, 46, 47, 48, 56, 57, 58, 59, 60, 68, 69, 76, 77] for applications and more general contexts[2]. We highly recommend the reference book by Peter McMullen and Egon Schulte [75], and especially the recent book by McMullen [74], which contain systematic bibliographies.

The second-named author became genuinely interested in these matters in the process of his work with Alexander Luzgarev on the explicit equations defining the exceptional Chevalley groups of types E_6, E_7 and E_8, see, in particular, [4, 61, 98, 101, 102, 103, 104], and references therein.

There, the polynomial equations themselves and/or the occurring monomials would correspond to the faces of the Schläfli, Hesse and Gosset polytopes, and their kin, with some weird coincidences and kinky symmetries.

Thus, for instance, the highest Weyl orbit of equations on the orbit of the highest weight vector consists of

• 27 Borel—Freudenthal equations defining the projective octave plane E_6/P_1 for (E_6, ϖ_1);

• 126 Freudenthal equations defining the 27-dimensional Freudenthal variety E_7/P_7 for (E_7, ϖ_7);

• 270 quadratic equations in the adjoint representation (E_6, ϖ_2);

• 756 quadratic equations in the adjoint representation (E_7, ϖ_1);

• 2160 quadratic equations in the adjoint representation (E_8, ϖ_8);

[1]Even professional mathematicians seldom realise that the fact that in dimension $n = 3$ the regular tetrahedron can be vertex embedded in a cube is just another expression of the existence of quaternions, and that the next dimension, where the same happens, is $n = 7$, see [23].

[2]There are also hundreds of papers of related interest in physics, crystallography, chemistry and biology journals. We browsed through a few dozens of those. The upside of it is that mostly they are highly repetitive. It comes with a penalty, though: nobody cares. Our overall impression can be summarised by the identity $9 \cdot 6 \cdot 8 = 192$ that we've found at the top of page 11207 in [52]. In another paper the number of roots in E_7 is listed as 128 instead of 126. That's about everything you wanted to know about those papers. They can be used as raw experimental material, which may contain amusing observations, but otherwise precarious.

and similar results for other orbits. In the two senior cases the explanation of these numbers in terms of the embeddings $\mathrm{A}_7 \subseteq \mathrm{E}_7$ and $\mathrm{D}_8 \subseteq \mathrm{E}_8$ were not immediate to us, and required separate explanation, see [99]. They are now, in terms of the faces of the corresponding Gosset—Elte polytopes!

This is why, after encountering the same exceptional symmetries in his recent research on higher symbols and presentations of the exceptional Chevalley groups in the framework of his "Basis" project, he felt committed, rather than merely involved.

The current talk is based on the Diploma project by the first-named author under the supervision of the second-named author, whose idea was to reconsider and recast the theory of exceptional uniform polytopes from scratch, with the tools standard in the structure theory and the representation theory of algebraic groups alone.

1. Semiregular polytopes

Here we very briefly describe the place of the Gosset—Elte polytopes in a general context.

1.1. Regular polytopes

For **regular** polytopes their symmetry group acts transitively on flags (a vertex, an edge containing this vertex, a 2-face containing this edge, etc.). Their are three series of **classical** regular polytopes that are present in all dimensions $n \geq 2$ and constructed by obvious generic procedures from the smaller-dimensional ones.

• Cones: **simplices**[3] $\alpha_n = \{3. \dots, 3\}$ with $n{+}1$ vertices = the weight polytopes of the vector representation $(\mathrm{A}_{n-1}, \varpi_1)$;

• Suspensions: **hyperoctahedra** $\beta_n = \{3, \dots, 3, 4\}$ with $2n$ vertices = orthoplexes = cross-polytopes = the weight polytopes of the vector representation (D_l, ϖ_1);

• Products: **hypercubes** $\gamma_n = \{4, 3, \dots, 3\}$ with 2^n vertices = the weight polytopes of the spin representation (B_l, ϖ_n).

In dimension 2 regularity is a very weak requirement and there are regular polygons $\{m\}$ with an arbitrary number of vertices m. Their Weyl groups are dihedral groups D_m, denoted in this context as $\mathrm{I}_2(m)$. The junior of these symmetry types have separate names

$$\mathrm{I}_2(2) = \mathrm{A}_1 + \mathrm{A}_1, \quad \mathrm{I}_2(3) = \mathrm{A}_2, \quad \mathrm{I}_2(4) = \mathrm{B}_2 = \mathrm{C}_2, \quad \mathrm{I}_2(5) = \mathrm{H}_2, \quad \mathrm{I}_2(6) = \mathrm{G}_2.$$

• As we all know, in dimension $n = 3$ there are two such **exceptional** regular polytopes, the 12-cell = **dodecahedron** $\{5, 3\}$ and its dual the 20-cell = **icosahedron** $\{3, 5\}$ of symmetry type H_3. Collectively, the 5 regular polyhedra of dimension $n = 3$ are called **Platonic solids**.

[3] As everybody knows, complices are made of simplices.

• Through 1850–1852 Ludwig Schläfli discovered *three* exceptional regular polytopes in dimension $n = 4$, the **24-cell** $\{3, 4, 3\}$, the **600-cell** $\{3, 3, 5\}$, and the **120-cell** $\{5, 3, 3\}$. For us they are the (self-dual) root polytope of type F_4 — or, what is the same, of type D_4; the root polytope of type H_4; and its dual. However, his *opus magnum* was not published until 1901, and was partially rediscovered by several mathematicians in the meantime.

That's about it. Starting with dimension $n \geq 5$ regular polytopes become exceedingly dull, they are all classical. This means that to get further fascinating examples one has to relax the transitivity condition.

1.2. Uniform polytopes

The regularity condition is too rigid, one has to slack it. One of the most exoteric generalisations is due to Coxeter.

• A polytope is called **uniform** if its symmetry group is **vertex transitive** and its **facets** (= the faces of codimension 1) are themselves *uniform*.

• A uniform polytope is called **semiregular** if its facets are *regular*. This is Gosset's definition. Elte would define semiregularity inductively and allow the facets themselves to be *semiregular*.

• A classification of semiregular polyhedra of dimension $n = 3$ was obtained by Johannes Kepler in 1596–1620. The 13 such semiregular polyhedra, different from Platonic solids, prisms and anti-prisms, are called **Archimedean solids**.

The two most interesting Archimedean solids in our context are the **cuboctahedron** $r\{4, 3\}$, and the **icosidodecahedron** $r\{5, 3\}$, which are the root polytopes of types A_3 and H_3, respectively.

• In 1900 Thorold Gosset published a list of 7 semiregular polytopes, 3 in dimension $n = 4$ and one in dimensions $n = 5, 6, 7, 8$ each, the four remarkable semiregular polytopes of symmetry types D_5, E_6, E_7 and E_8, the **Clebsch polytope** 1_{21}, the **Schläfli polytope** 2_{21}, the **Hesse polytope** 3_{21}, and the **Gosset polytope** 4_{21}.

• In 1912 Emanuel Elte rediscovered those, relaxed the notion of semiregularity, and constructed further exceptional polytopes of symmetry types E_6, E_7 and E_8, the **Elte polytopes** 1_{22}, 2_{31}, 1_{32}, 2_{41} in Coxeter notation.

In the next section we start discussing these polytopes in more details.

Comment 1. Without additional regularity assumptions on faces, vertex transitivity itself imposes essentially no restrictions on the symmetry type. Laszlo Babai [5] has proven that essentially any finite group, is the full symmetry group of some vertex regular polytope. The only exceptions are [most of] the abelian ones and [some of] those that have an abelian subgroup of index 2,

Comment 2. On the other hand, in dimension $n = 3$ regularity of faces alone without some kind of vertex transitivity or the like produces scores of warped polytopes with low symmetry, whose classification becomes an exacting problem of metric geometry. The classification of such solids, known as **Johnson solids**, is highly non-trivial, and the completeness of the Norman Johnson list of 92 such convex polytopes [50] was only established by Viktor Zalgaller [108] in 1967, compare [44] for a comprehensive historical account.

In dimensions $d \geq 5$, a similar classification up to isomorphism was completed by Gerd and Roswitha Blind before 1980. There, nothing unexpected occurs, just the regular and semiregular solids, pyramids and bipyramids. *Morally*, this tells us that in dimensions $n \geq 5$ there are no regular-faced polytopes, apart from the semi-regular ones.

But for $n = 4$ regular-faced polytopes thrive outlandishly. Namely, truncating the regular 600-cell by ≤ 24 icosahedral pyramids at non-adjacent vertices, Mathieu Dutour Sikirić and Wendy Myrvold in [34] produce as many as 314 248 344 isomorphism classes of polytopes whose facets are regular tetrahedra and icosahedra[4].

Comment 3. There are a number of constructions that allow to create new uniform polytopes of the same symmetry type from the old ones, most notably the **Wythoff constructions** proposed by Willem Wythoff in 1918. These constructions admit very natural interpretations from the viewpoint of representation theory, but we cannot discuss them here.

Comment 4. Another possible generalisation is to renounce convexity, allow self-intersections, hidden faces, etc. Such **stellated polytopes**, in particular, the **Kepler—Poinsot star polyhedra**, polytope **compounds**, and the like were studied at least since Luca Paciolis's *De Divina Proportione*, 1509.

Comment 5. Even while considering graphs and other face complexes of the exceptional semiregular polytopes purely combinatorially, one should bear in mind that they come from actual *geometric* polytopes in Euclidean spaces. *Abstract* semiregular polytopes are not nearly as symmetric. In fact, even in the case of polytopes all of whose facets are isomorphic, one can construct examples with arbitrarily large number of orbits on flags, or even on smaller dimensional faces, see [85].

1.3. The status of the classification of semiregular polytopes

Most laymen — initially including ourselves! — believe that the combinatorial structure of the Gosset polytopes was known to Gosset and Elte more than a century ago and that the classification of semiregular polytopes in *all* dimensions was completed by Coxeter not later than 1948.

Both claims are *outrageous* oversimplifications!

It is a fact that Coxeter made *absolutely amazing* discoveries, and was instrumental in reviving the interest in the subject. However, in what regards higher-dimensional uniform polytopes, the proofs in the books [26, 29] are far from being conclusive. Therein, neither the tables of the (non-convex) uniform polytopes themselves, nor the description of the faces of the exceptional polytopes and their adjacency, are consummate.

[4] John Stembridge observed: “It is a general principle that unique or canonical objects are easier to construct than those that require choices to be made”, [91].

In fact, we do not believe that even the completeness of the Gosset list has been rigorously validated before the 1990-ies. Even for $n = 4$ the first accepted proof was only published in 1988 by Petr Makarov[5] [63, 64]. Whereas the completeness of the Gosset list in dimensions $d \geq 5$ was only established by Gerd and Roswitha Blind [12] in 1991, as a spin-off of their classification of the regular-faced polytopes.

The above referred to *convex* polytopes. One can imagine the mess with the rest! Thus, in 2018 Peter McMullen [73] has found 6 new regular compounds in dimension 4. Five of them are vertex-embedded in the 120-cell and consisting, respectively, of

- 720 copies of the 4-simplex,
- 120 copies of the 4-simplex[6],
- 75 copies of the 4-hyperoctaheder,
- 75 copies of the 4-hypercube,
- 25 copies of the 24-cell.

• The last example is the *dual* of the previous example, consisting of 25 copies of the 24-cell vertex-embedded in the 600-cell.

We believe that the first definitive classification of such polytopes was obtained not by Coxeter in 1948, but rather 70+ years later by McMullen [74].

The same applies to the combinatorial structure of these polytopes. Gosset himself has not given a complete combinatorial description of the polytopes, just their *facets* and *some* incidence properties of the following type: a $(d-3)$-face of the d-dimensional polytope is contained in two $(d-1)$-hyperoctahedra and one $(d-1)$-simplex, etc.

The detailed proofs of such a description announced by Coxeter in 1940–1948 were never published before 1988–1992, by Coxeter himself, Conway, Sloane, Moody, and Patera, see [28, 22, 79], with some circumstantials being clarified long after that.

2. Roots, weights and symmetries

2.1. Basic notation

In all that concerns root systems, including the numbering of simple roots, we follow Bourbaki [15], see also [49]. In particular, Φ is a reduced irreducible root system of rank l, whereas $W = W(\Phi)$ is its Weyl group. For a root $\alpha \in \Phi$ we denote by $w_\alpha \in W$ the corresponding root reflection.

[5] Not to be confused with his father Vitaly Makarov, famous for his work on regular polytopes in Lobachevsky spaces.

[6] A different configuration from the one listed by Coxeter in [26].

The Weyl groups of senior exceptional types have the following orders

$$|W(\mathrm{E}_6)| = 51840 = 72 \cdot 6! = 2^7 \cdot 3^4 \cdot 5,$$
$$|W(\mathrm{E}_7)| = 2903040 = 72 \cdot 8! = 2^{10} \cdot 3^4 \cdot 5 \cdot 7,$$
$$|W(\mathrm{E}_8)| = 696729600 = 192 \cdot 10! = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

We fix an order on Φ, and let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the corresponding set of fundamental roots, Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Usually we denote the fundamental root reflection w_{α_i} simply by w_i. The Weyl group is generated already by the fundamental reflections, $W = \langle w_1, \ldots, w_l \rangle$.

Let $\overline{\Pi}$ be the extended fundamental system of Φ. It is obtained by appending to Π the root $\alpha_0 = -\delta$, where δ is the maximal root of Φ with respect to the fundamental system Π. Recall that in the Dynkin form the maximal roots of E_6, E_7 and E_8 are depicted as

$$\begin{smallmatrix} 12321 \\ 2 \end{smallmatrix}, \qquad \begin{smallmatrix} 234321 \\ 2 \end{smallmatrix}, \qquad \begin{smallmatrix} 2465432 \\ 3 \end{smallmatrix}.$$

For two root systems Δ and Σ we denote by $\Delta + \Sigma$ their orthogonal sum. In particular, $k\ lta = \Delta + \ldots + \Delta$ is the orthogonal sum of k isomorphic summands. It is sometimes convenient to consider also the empty root system $\mathrm{A}_0 = \emptyset$ of rank 0. Recall that $\mathrm{D}_1 = \mathrm{D}_0 = \emptyset$.

Here, we are only interested in the simply laced systems, in which case the roots are usually normalised so that $(\alpha, \alpha) = 2$.

Further, we denote by $Q(\Phi)$ the root lattice, generated by $\alpha_1, \ldots, a_l$, and by $P(\Phi)$ the [dual] weight lattice[7] generated by the fundamental weights $\varpi_1, \ldots, \varpi_l$. Recall that $(\varpi_i, \alpha_j) = \delta_{ij}$.

2.2. Hyperbolic realisation of E_l

Most of the non-trivial calculations with root systems pertain to the cases $\Phi = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. As in our previous works that relied on massive computations in root systems, such as [45, 96, 97, 98, 100, 101, 102, 103, 104] we use the *hyperbolic* realisation of these systems in the $(l+1)$-dimensional Minkowsky space [65]. This realisation is *considerably* more adapted to the large-scale calculations, than the usual realisations in Euclidean space.

Consider a real vector space $U = \Re^{l,1}$ of dimension $l+1$ endowed with a non-degenerate symmetric inner product $(\ ,\ ) : U \times U \to \mathbb{R}$ of signature $(l, 1)$. Fix an orthonormal base $e_0, e_1, \ldots, e_l$ such that $(e_0, e_0) = -1$ and $(e_i, e_i) = 1$ for all $1 \le i \le l$. We are primarily interested in the case $l = 8$.

[7] In the textbooks on lattices and sphere packings the root lattices $Q(\mathrm{E}_l)$ are usually denoted simply by E_6, E_7 and E_8, whereas the weight lattices $P(\mathrm{E}_l)$ are denoted by E_6^* and E_7^*. The lattice $P(\mathrm{E}_8)$ is unimodular and self-dual, $\mathrm{E}_8^* = \mathrm{E}_8$.

In this realisation, up to sign every element of $\Phi = \mathrm{E}_8$ has one of the following forms:

$$\begin{cases} e_i - e_j, \quad i > j, \\ e_0 + e_i + e_j + e_h, \\ 2e_0 + e_1 + \ldots + \widehat{e_i} + \ldots + \widehat{e_j} + \ldots + e_8, \\ 3e_0 + e_1 + \ldots + 2e_i + \ldots + e_8, \end{cases}$$

where indices $i, j, h = 1, \ldots, 8$ are pair-wise distinct, while the hat $\hat{}$ over an index signifies that this index should be omitted.

Fix the following fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_8\}$ in $\Phi = \mathrm{E}_8$:

$$\alpha_1 = e_2 - e_1, \quad \alpha_2 = e_0 + e_1 + e_2 + e_3, \quad \alpha_3 = e_3 - e_2, \quad \alpha_4 = e_4 - e_3,$$
$$\alpha_5 = e_5 - e_4, \quad \alpha_6 = e_6 - e_5, \quad \alpha_7 = e_7 - e_6, \quad \alpha_8 = e_8 - e_7.$$

The root system $\Phi = \mathrm{E}_7$ and its fundamental root system are obtained by dropping e_8 and α_8, thereby the last type of roots disappear. The root system $\Phi = \mathrm{E}_6$ is obtained by dropping both e_8 and e_7, or, respectively, both α_8 and α_7, thereby the third type of roots reduces to the single root $2e_0 + e_1 + \ldots + e_6$.

2.3. Subsystems of root systems

Classification of all subsystems of root systems, including the maximal ones, was obtained by Borel—de Siebenthal [13] and Dynkin [35], many further details are produced in [105, 17, 45, 100, 36, 83] Their construction can be described as follows.

• For every r, $1 \leq r \leq l$, we consider the subsystem Δ_r in Φ, generated by $\overline{\Pi} \setminus \{\alpha_r\}$, or, in other words, the smallest subsystem containing these roots. It is the smallest closed set of roots containing both these roots themselves and their opposites.

Subsystems Δ_r are the **maximal rank** subsystems, they have rank l. In general subsystems Δ_r are not necessarily irreducible. Moreover, they are only maximal, when the coefficient with which α_r occurs in α_0 is prime[8].

• Applying the above procedure to all irreducible components of the systems Δ_r and repeating this process until complete satisfaction, we obtain all subsystems $\Delta \subseteq \Phi$ of maximal rank.

• Now all subsystems $\Delta \subseteq \Phi$ can be obtained as follows. Let $\Sigma \subseteq \Phi$ be a subsystem of maximal rank, Ξ be its fundamental system. Then the subsystem of Δ generated by any subset $J \subseteq \Xi$ is a closed subsystem of Φ and all closed subsystems of Φ can be obtained this way.

For exceptional systems there are — up to conjugacy by an element of $W(\Phi)$ — the following number of *proper* subsystems: 20 for E_6, 46 for E_7 and 76 for E_8,

[8] Otherwise they either coincide with Φ, when the coefficient is 1, or are sub-maximal, when it is the product of two primes.

Two subsystems $\Delta, \Sigma \subseteq \Phi$ of the same type are *almost* always conjugate by an element of $W(\Phi)$. All possible exceptions are listed below[9].

• Subsystems D_2 and $2\mathrm{A}_1$, as also subsystems D_3 and A_3 are isomorphic, but not conjugate subsystems of D_l. Moreover, for $l \geq 5$ they are not conjugate not only by an element of $W(\mathrm{D}_l)$, but even by an external automorphism $\mathrm{Aut}(\mathrm{D}_l) = W(\mathrm{B}_l)$. In fact, for $l = 4$ they become conjugate by an element of $\mathrm{Aut}(\mathrm{D}_4) = W(\mathrm{F}_4)$, the phenomenon known as triality.

• When $k_1, \ldots, k_t$ are all odd, then D_l has two conjugacy classes of subsystems

$$\Delta = \mathrm{A}_{k_1} + \ldots + \mathrm{A}_{k+t}, \qquad l = (k_1 + 1) + \ldots + (k_t + 1),$$

which are conjugate by an external automorphism.

• In the root system E_7 there are two conjugate classes of subsystems of the following types:

$$\mathrm{A}_5 + \mathrm{A}_1, \quad \mathrm{A}_5, \quad \mathrm{A}_3 + 2\mathrm{A}_1, \quad \mathrm{A}_3 + \mathrm{A}_1, \quad 4\mathrm{A}_1, \quad 3\mathrm{A}_1.$$

In turn, in the root system E_8 there are two conjugate classes of subsystems of the following types:

$$\mathrm{A}_7, \quad \mathrm{A}_5 + \mathrm{A}_1, \quad 2\mathrm{A}_3, \quad \mathrm{A}_3 + 2\mathrm{A}_1, \quad 4\mathrm{A}_1.$$

Observe that in the case of E_7 these are exactly the pairs of subsystems that were not conjugate in $\mathrm{D}_6 + \mathrm{A}_1$, whereas in the case of E_8 these are exactly the pairs of subsystems that were not conjugate in D_8.

We denote by Δ' the subsystem of type Δ, that *is* contained in A_7 or, respectively, in A_8, and by Δ'' we denote the one that is not contained there. It should be noted that in [17] the notation Δ' and Δ'' has the opposite meaning!

2.4. Conjugacy classes of the Weyl group

Most of our actual computations depend on an explicit knowledge of the conjugacy classes of the Weyl groups. An *ad hoc* description of the conjugacy classes of the exceptional Weyl groups was given by Sutherland Frame [38, 39]. Roger Carter [16, 17] proposed a conceptual explanation. Roughly the situation can be described as follows.

• Most — but by no means all! — of the conjugacy classes of the Weyl group $W(\Phi)$ are represented by the class $C(\Delta$ of Coxeter elements

$$\mathrm{cox}_\Delta = w_{\beta_1} \ldots w_{\beta_r},$$

where $\beta_1, \ldots, \beta_r$ are the fundamental roots of a subsystem $\Delta \leq \Phi$.

• There are precisely two cases when for two *non-conjugate* subsystems $\Delta, \Sigma \subseteq \Phi$ their Coxeter classes $C(\Delta)$ and $C(\Sigma)$ coincide in $W(\Phi)$. The only one of interest for us[10] is $C(\mathrm{D}_5 + \mathrm{A}_3) = C(\mathrm{A}_7 + \mathrm{A}_1)$ in E_8.

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[9]For multiply laced systems one should also distinguish long root embeddings and short root embedding.

[10]The other one, $C(\mathrm{B}_2 + 2\mathrm{A}_1) = C(\mathrm{A}_3 + \mathrm{A}_1)$ occurs in a multiply laced root system F_4.

• However, the converse is far from being true. Other conjugacy classes come from **Carter graphs**, which are essentially bipartite Dynkin diagrams with cycles. The vertices of a Carter diagram C are partitioned into two sets $C = C_1 \sqcup C_2$, both consisting of pairwise orthogonal roots, and the remaining conjugacy classes are represented as products of two involutions $w = w_1 w_2$, where w_i, $i = 1, 2$, is the product of all root reflections w_α, $\alpha \in C_i$.

There is one new *indecomposable* Carter graph with cycles for D_4 and D_5 each; 2 for D_6 and E_6 each, 5 for E_8 and 10 for E_8.

Overall, with Carter graphs coming from smaller ranks and their sums with small rank Coxeter elements, this gives 4 non-Coxeter classes for E_6; 13 such classes for E_7; and 36 such classes for E_8, which by manual computations already produces some strain.

• A further remarkable conceptual advance was made by Rafael Stekolshchik [90], who has modified Carter's list by observing that Carter graphs with long cycles are equivalent to Carter graphs containing only cycles of length 4. In other words, the graphs with cycles of length 6 in the original Carter's list (in our case one for D_6, one for E_7 and two for E_8), can be reduced to other forms, more suitable for actual computations.

• Yet another remarkable conceptual advance was *not* made by Eugenii Dynkin and Andrei Minchenko [36]. They introduced a *marvelous* combinatorial tool, **enhanced Dynkin diagrams**, to explain the inclusions among root subsystems, but failed to notice their connection with the description of the conjugacy classes of the Weyl group[11].

In fact, all Carter diagrams, both in the original form and Stekolshchik form, can be readily accounted for by the enhanced Dynkin diagrams, which are as follows.

• The 8 vertex graph consisting of three squares with common edge, for E_6.

• The 11 vertex graph, consisting of the 4 vertices and the 6 edge midpoints of a tetrahedron + its centre joined to the vertices, for E_7.

• The 4×4 net on a torus[12], for E_8.

In this form, the contents of this and previous subsections should be in Bourbaki, Chapter $6\frac{1}{2}$, but it isn't!

2.5. Weight diagrams

Our *major* tool are weight diagrams, which are a standard tool in the representation theory of Lie algebras and algebraic groups, see [86, 95, 96, 97, 98] for the details and many further references. There are two usual ways to render exceptional polytopes as 2D pictures.

[11] The same graphs also occur in a completely different context, as graphs with certain extremal properties for their eigenvalues, in [70, 71].

[12] Exceptional behaviour of this net was observed by Vladimir Kornyak [53]. The second author immediately observed the connection with triality and F_4, but failed to notice the relation to E_8!

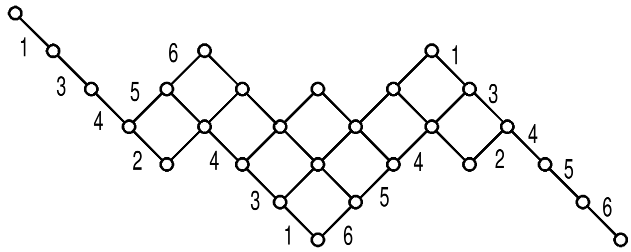

FIGURE 1. Weight diagram (E_6, ϖ_1)

• The usual "publicity photo" of E_8, as reproduced in hundreds of places [26, 6, 7, 14, 40, 54, 62, 92], are beautiful, but completely unsuitable for actual computations. The orthogonal projections to smaller dimensions, usually, 2D, 3D or 4D, which try to keep vertices distinct and faithfully depict all edges become a *complete mess*. Already for the root polytope of type E_8 with 240 vertices there are as many as 6720 edges, which makes the corresponding picture completely unfit for human calculations.

• The McMullen projections [74] are *terribly* much handier, but they give a very schematic picture, where some of the vertices represent actual vertices, whereas some other represent higher dimensional faces, sometimes the whole facet! As a result, you should be able to use several of those in conjunction, in the same calculation. To visualise the whole symmetry of a multidimensional object with these pictures is possible, as Peter McMullen himself amply and brilliantly illustrates, but it requires some *serious* mental exercise.

We chose the middle way. The pictures we use to visualise the polytopes are a blend of **Schreier graphs** depicting the cosets of the Weyl group modulo a parabolic subgroup, or **weight diagrams** common in representation theory of algebraic groups and related fields. Both are much more schematic and shorthand than the usual Coxeter like projections, and at the same time much more faithful and informative than McMullen diagrams. One such picture serves as a genuine shorthand reproduction of the whole multidimensional object. With *moderate* practice, all properties of this multidimensional object can be read off from such a picture purely combinatorially.

Roughly, the difference is as follows. All polytopes we consider can be scaled so that all of their vertices are *integral weights* = lattice point of $P(\Phi)$. We depict all vertices of the polytope, but [as a first approximation] only draw the edges that correspond to the *fundamental* roots, marking them accordingly.

The corresponding **weight graph** is obtained when you draw the edges corresponding to all positive roots, instead of drawing just the ones that correspond to the fundamental ones. The missing edges can be easily restored as those *paths* in these graphs, for which the multiplicities of marks coincide with the coefficients in the linear expansion of a given root with respect to the fundamental ones.

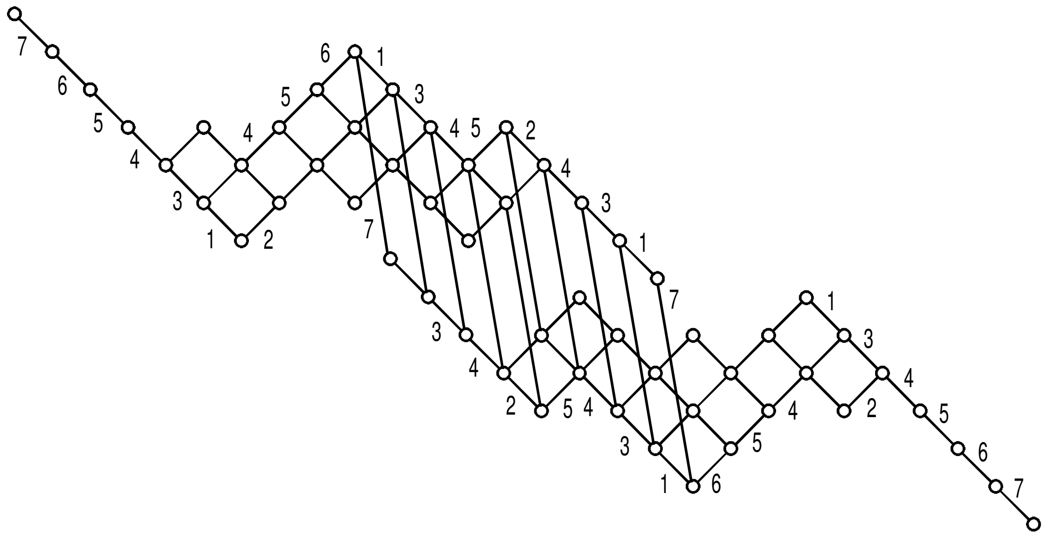

FIGURE 2. Weight diagram (E_7, ϖ_7)

Such similar pictures can be interpreted in a number of ways. Purely combinatorially, for uniform polytopes these pictures are Schreier graphs of the coset spaces $W(\Phi)/W(\Delta)$, where for type (Φ, ϖ_i) the subsystem Δ is the sub-maximal rank subsystem of Φ generated by $\Pi \setminus \{\alpha_i\}$.

What makes these pictures useful is that in the interesting cases they are intimately related to the actual *geometry* of weights and constitute part of what is known as weight diagrams, or crystal graphs in representation theory.

For the three *microweight polytopes* — the Clebsch, the Schläfli and the Hesse — all of their vertices are *extremal* weights of the corresponding representation. Moreover, the action of a root reflection consists in subtracting/adding the corresponding root. Thus, in these cases we get a *genuine* picture that fully captures all properties of the corresponding. These are precisely the (E_6, ϖ_1) and (E_7, ϖ_7), reproduced in dozens of texts, including [95, 86, 96, 97, 101, 103].

For the three *root polytopes* — of which only the one corresponding to E_8 is semiregular — the usual way to draw the weight diagrams as crystal graphs. Below we reproduce *half* of the picture of (E_8, ϖ_8), the diagrams of the adjoint representations (E_6, ϖ_2) and (E_7, ϖ_7) are part of it, and are reproduced in [86, 97]. The pictures of the senior polytopes such as (E_8, ϖ_1) or (E_8, ϖ_2) are too large anyway, to be used for manual computations.

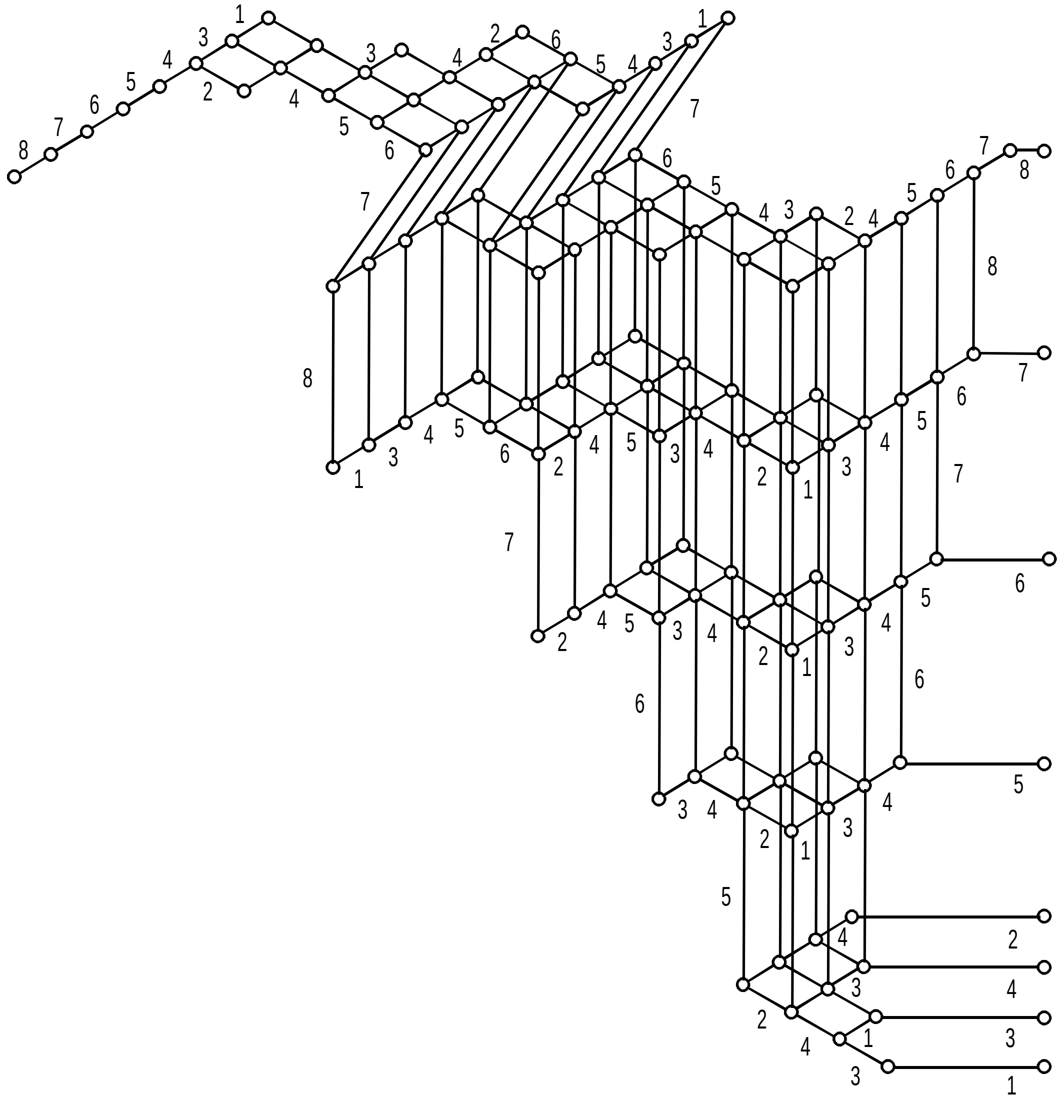

FIGURE 3. Weight diagram (E_8, ϖ_8)

3. Yet another look at the structure of the Gosset—Elte polytopes

In the talk we will show how to reconstruct all usual properties of the exceptional uniform polytopes from the weight diagrams, tables of roots, root subsystems, and conjugacy classes.

3.1. The treasury of exceptional polytopes

In the first place, we are interested with the 8 (or 9, depending on how you count them) classical Gosset—Elte polytopes in dimensions 6, 7 and 8. But they are closely related also to some further related Voronoi and Delaunay polytopes of exceptional lattices, further polytopes related to the Weyl orbits on weights, etc. Here is the beginning of the list.

Dimension 6:

- 2_{21} with 27 vertices — Schläfli polytope of type (E_6, ϖ_1) = Delaunay polytope for $Q(\mathrm{E}_6)$. Or, dually, 2_{12} of type (E_6, ϖ_6).

- 1_{22} with 72 vertices — the root polytope for E_6 of type (E_6, ϖ_2) = the contact polytope for $Q(\mathrm{E}_6)$.

There are many further *extremely* interesting related examples, which are not themselves on the Gosset—Elte list, but closely related to those, like, for instance:

- Diplo-Schläfli polytope with 54 vertices = the convex hull of two dual Schläfli polytopes 2_{21} and 2_{12} = Voronoi polytope for $Q(\mathrm{E}_6)$.

- Voronoi polytope for $P(\mathrm{E}_6)$ with 720 vertices.

Dimension 7:

- 3_{21} with 56 vertices = Hesse polytope of type (E_7, ϖ_7) = contact polytope for $P(\mathrm{E}_7)$.

- 2_{31} with 126 vertices = the root polytope for E_7 of type (E_7, ϖ_1) = the contact polytope for $Q(\mathrm{E}_7)$.

- 1_{32} with 576 vertices = Voronoi polytope of $P(\mathrm{E}_7)$ of type (E_7, ϖ_2).

Dimension 8

- 4_{21} with 240 vertices = Gosset polytope of type (E_8, ϖ_8) = the contact polytope for $Q(\mathrm{E}_8)$.

- 2_{41} with 2160 vertices = the deep hole polytope for $Q(\mathrm{E}_8)$ of type (E_8, ϖ_1).

- 1_{42} with 17280 vertices = the shallow hole polytope for $Q(\mathrm{E}_8)$ of type (E_8, ϖ_2).

The terminology for E_8 is borrowed from the book by Conway and Sloane. The 240 roots of E_8 are the lattice points of norm 2. A **hole** is a point of $\Re^8$, whose distance to $Q(\mathrm{E}_8)$ is a local maximum. The 2 160 **deep holes** near the origin are *halves* of the lattice points of norm 4. The 17540 lattice points of norm 8 fall into two orbits under the action of $W(\mathrm{E}_8)$, or which 240 are twice the roots, and 17280 are 3 times the **shallow holes** near the origin.

That's not the complete list even of the most interesting uniform exceptional polytopes, but that gives you some idea.

3.2. Structure of exceptional polytopes

We start with repeating with our methods all results on the structure, number and adjacency of faces of the above polytopes. With our tools, such a description becomes immediate.

For instance, look at Figure 1. Since the polytope is uniform, the highest weight ϖ_1 = the left-most node of the diagram, is incident to faces of all types, which thus correspond to parabolic root subsystems containing α_1.

Since from ϖ_1 there are unique descending paths of lengths 1, 2 and 3, and the roots susbystems they generate have types A_1, A_2 and A_3, respectively. This means that there are only one type of the faces of dimension 2,3, and that they are triangles and tetrahedra.

Their number can be easily computed as well. Since the Weyl group $W(\mathrm{E}_6)$ acts transitively on roots, the number of edges equals $6\cdot|\mathrm{E}_6^+|$. Alternatively, $W(\mathrm{E}_6)$ acts transitively on vertices, and there are 16 vertices at distance 1 from a given one in the weight graph. Thus, the number of edges equals $27\cdot 16/2 = 36\cdot 6 = 216$.

Obviously, in dimension 4 something funny happens. Namely, there are two *different* ways to embed $\mathrm{A}_3 = \langle\alpha_1, \alpha_3, \alpha_4$ intoA_4. One is to proceed with α_2, and this cannot be further embedded into A_5, and another one is to proceed with α_5, which can then be embedded into A_5 by further adjoining α_6. Both ways produce faces of type A_4, which are 4-simplices, but they form two distinct orbits.

Finally, there are two types of facets. There are 5-simplices α_5, 72 of them, that correspond to the roots of E_6, and there are 5-hyperoctahedra β_5, that correspond to the 27 pairs of non-comparable weights.

This accounts for the distinction between two types of 4-dimensional faces. Indeed, α_5 has 6 facets, which gives $72\cdot 6 = 432$, whereas β_5 has 32 facets, which gives $27\cdot 32 = 864$. This means that 432 of the 4-faces are common faces of an α_5 and a β_5, whereas the 216 remaining ones are shared by two β_5. Clearly, they form two distinct Weyl orbits.

Of course, the case of (E_6, ϖ_1) is by far the simplest one. Nevertheless, for all other cases the types of faces, their incidence numbers, etc. can be easi8ly recuperated within half an hour by such similar means, perhaps with some little help of the tables of root subsystems, orders of the Weyl group, and the like. For that one even does not need the whole Schreier graph $W(\Phi)/W(\Delta)$/the whole the weight diagram (Φ, ϖ_i), just the neighbourhood of the highest weight.

3.3. Colourings of the exceptional polytopes

Coxeter discovered that instead of duality one should rather consider **triality**. The exceptional polytopes come in triples, the facets of each one of them corresponding to the vertices of the other two. As we already know, in dimension 6 the facets of 2_{21} correspond to the 27 vertices of 2_{12} *and* to the 72 vertices of 1_{22}.

Dually — or should one say *trially* in this case? — the facets of the root polytope of E_6 are all of them Clebsch polytopes = 5-demicubes, but they come in two denominations, the *positive half spin* and the *negative half spin*, $54 = 27 + 27$, the positive ones corresponding to the vertices of (E_6, ϖ_1) and the negative ones — to the vertices of the **dual** polytope (E_6, ϖ_6).

The 5-demicube has 16 facets α_4 and 10 facets β_4 and any two adjacent 5-demicubes of the same parity intersect in α_4, whereas two adjacent demicubes of different parities intersect in β_4. In particular, (E_6, ϖ_2) has two types of α_4 faces: the positive and the negative ones.

An easy calculation using Figure 1 and arguments of the above type shows that the cycle index of the action on the *vertices* of (E_6, ϖ_1) is

$$Z_{27}[x_1, \ldots x_{12}] = \frac{1}{51840}\Big(x_1^{27} + 36x_1^{15}x_2^6 + 270x_1^7x_2^{10} + 240x_1^9x_3^6 + 585x_1^3x_2^{12} + \\ 1440x_1^3x_2^3x_3^4x_6 + 1620x_1^5x_2x_4^5 + 2160x_1^3x_2^3x_3^4x_6 + 560x_3^9 + 3780x_1x_2^3x_4^5 + \\ 5184x_1^2x_5^5 + 1440x_1^3x_2^3x_6^3 + 540x_1^3x_4^6 + 1440x_3^5x_6^2 + 5184x_2x_5^3x_{10} + \\ 6480x_3x_6^4 + 6480x_1x_2x_8^3 + 4320x_1x_4^2x_6x_{12} + 4320x_3x_{12}^2 + 5760x_9^3\Big)$$

Observe the presence of all rotation axes of orders 5, 8, 10 and 12, which are already possible for crystals of dimensions 4 and 5[13], as also the appearance of a rotation axis of order 9, that first occurs in dimension 6.

However, since all elements of $W(\mathrm{E}_6)$ are *real* (= conjugate to their inverses), the action on (E_6, ϖ_6) is exactly the same. This means that to calculate the cycle index of the action of $W(\mathrm{E}_6)$ on the facets of (E_6, ϖ_2), one only has to replace the variables in the above formula by their squares. In particular, there are

$$350\,661\,193\,456$$

essentially different colourings of the facets of root polytope of type E_6 in 2 colours;

$$1\,121\,791\,681\,317\,791\,814\,588$$

such colourings in 3 colours, etc.

It's nothing special that such things can be easily done nowadays. What seems to be a bit special, is that this calculation was essentially done by us *manually* within a couple of evenings[14].

By hand, we have performed similar computations also for the Hesse polytope 3_{21} with 56 vertices, which by triality gives the colourings of the 56 facets of type 2_{21} of the root polytope 2_{31}. But we decided that to calculate the cycle index on the 576 simplicial facets would be a bit too much of a good thing.

Of course, performing similar calculations by hand for the 1_{23} — not to say for 2_{14} and 124 — would require much more leisure, and should be rather relegated to a computer. For the polytope 4_{12} it was indeed implemented by David Madore,

[13]The root polytope of type E_6 folds to icosidodecahedron in dimension 3, which inherits part of these symmetries, but that's not crystallographic.

[14]Samuel Wagstaff: "Multiply 2071723 × 5363222357 by hand. Feel the joy."

and can be found at his home page [62]. Computer realisations for other small cases are straightforward[15].

4. Whither from here?

Currently, we proceed to describing further structural properties of the exceptional polytopes, various generalisations, and applications of the above, such as, for instance:

- Classification of vertex embeddings in exceptional polytopes [1].
- Classification of d-codes and maximal independent sets [32].
- Explicit description of foldings $\mathrm{E}_6 \longrightarrow \mathrm{H}_3$, $\mathrm{E}_8 \longrightarrow \mathrm{H}_4$, and the like, [52].
- Classification of exceptional compounds. In particular, it seems quite plausible that the new 4D compounds discovered by Peter McMullen [73, 74] all come from the folding $\mathrm{E}_8 \longrightarrow \mathrm{H}_4$.
- Exceptional virtual polytopes [84].

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[15] Well, because we calculate in the smallest representations of the Weyl groups $W(\mathrm{E}_l)$, of dimensions 6, 7 and 8, respectively. If one is interested in the explicit construction of the irreducible constituents of the corresponding permutation representation considered as a linear representation, it becomes a computational challenge on a completely different scale. To give some idea, John Stembridge [91] has computed *explicit* matrices of all irreducible complex representations of the exceptional Weyl groups. Just the construction of the matrix for the fundamental reflection w_8 in the largest irreducible complex representation R_{7168} of $W(\mathrm{E}_8)$ of dimension 7168 involved solution of 14597 [quadratic] Coxeter relations in 593 variables + vanishing of 2 matrix entries + one clone equation to distinguish it from another representation of $W(\mathrm{E}_8)$ having the same restriction to $W(\mathrm{E}_7)$. Too many, for a general purpose CAS.

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