THE YOGA OF COMMUTATORS REVISTED, REVISITED

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Our results are mostly new already for the case of

- **General linear group** $\mathrm{GL}(n, R)$, $n \geq 3$.

But actually we have generalised many results — not all of them as yet! — to the following settings:

- **Chevalley groups** $G(\Phi, R)$, $\mathrm{rk}(\Phi) \geq 2$;
- **Bak's unitary groups** $\mathrm{GU}(2n, R, \Lambda)$.

Similar, but more complicated, due to different root lengths, non-trivial form parameters, involutions, etc.

1 Notation for $\mathrm{GL}(n, R)$

- R — and associative ring with 1;
- $M(n, R)$ — the full matrix ring of degree n over R;
- $\mathrm{GL}(n, R) = M(n, R)^*$ — general linear group of degree n over R,

$$\mathrm{GL}(n, R) = \{g \in M(n, R) \mid \exists h \in M(n, R),\ gh = e = hg\};$$

- e — identity matrix, whereas e_{ij}, $1 \leq i, j \leq n$, — standard matrix unit;
- $t_{ij}(c) = e + ce_{ij}$, $c \in R$, $1 \leq i \neq j \leq n$, — elementary transvection;
- $I \trianglelefteq R$ — a two-sided ideal of R;
- $\rho_I : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/I)$ — the reduction homomorphism modulo I.

• $\mathrm{GL}(n, I) = \mathrm{GL}(n, R, I) = \mathrm{Ker}(\rho_I)$ — the principal congruence subgroup of level I:

$$\mathrm{GL}(n, R, I) = \big\{ g = (g_{ij}) \in \mathrm{GL}(n, R) \mid g \equiv e \pmod{I} \big\};$$

• $E(n, I)$ — the [unrelative] elementary subgroup of level I:

$$E(n, I) = \big\langle t_{ij}(a),\ a \in I,\ 1 \leq i \neq j \leq n \big\rangle.$$

• $E(n, R)$ — the [absolute] elementary subgroup;

• $E(n, R, I)$ — the relative elementary subgroup of level I is the normal closure of $E(n, I)$ in the absolute elementary subgroup $E(n, R)$:

$$E(n, R, I) = E(n, I)^{E(n,R)}.$$

2 Early history of mixed commutator subgroups in $\mathrm{GL}(n,R)$

- Hyman Bass, 1964, started the study of

$$[\mathrm{GL}(n,R,I),\mathrm{GL}(n,R,J)], \qquad [\mathrm{GL}(n,R,I),E(n,R,J)],$$
$$[E(n,R,I),E(n,R,J)]$$

  - $\mathrm{sr}(R)$ — the stable rank of R, estimated in terms of *any* dimension.

Theorem. (Bass, 1964) Let $I,J\trianglelefteq R$ be two-sided ideals of R, $n\geq\max(\mathrm{sr}(A)+1,3)$, then

$$[\mathrm{GL}(n,R,I),E(n,R,J)]=[E(n,R,I),E(n,R,J)].$$

Theorem. (Mason—Stothers, 1974) Let $I,J\trianglelefteq R$ be two-sided ideals of R, $n\geq\max(\mathrm{sr}(A)+1,3)$, then

$$[\mathrm{GL}(n,R,I),\mathrm{GL}(n,R,J)]=[E(n,R,I),E(n,R,J)].$$

Question. Can one lift the dimension condition in the above results?

- R is **almost commutative** if it is **module-finite** over its centre $\mathrm{Cent}(R)$.

- R is **quasi-finite** over a commutative ring A if

$$R = \varinjlim R_i, \qquad A = \varinjlim A_i, \qquad R_i \text{ module finite over } A_i.$$

In the *absolute* case these are the **standard commutator formulae**.

Theorem. (Suslin, 1976, ..., Vaserstein, 1981, Borewicz—Vavilov, 1982, ..., Bak, 1991) Let R be *quasi-finite*, $I \trianglelefteq R$ be a two-sided ideal of R, and $n \geq 3$. Then

$$[\mathrm{GL}(n, R), E(n, R, I)] = [E(n, R), \mathrm{GL}(n, R, I)] = E(n, R, I).$$

Question. Can one remove commutativity conditions here?

Answer. NO, Gerasimov, 1989.

Many further generalisations by a number of authors. However, the absolute case is not our concern here.

3 The work of Hong You, Stepanov, Hazrat, and ourselves

Question. Does this generalise to the *birelative* case?

Theorem. (Hong You, 1992, Vavilov—Stepanov, 2008–2010, Hazrat—Zhang, 2011) Let R be *quasi-finite*, $I, J \trianglelefteq R$ be two-sided ideals of R, and $n \geq 3$. Then

$$[\mathrm{GL}(n, R, I), E(n, R, J)] = [E(n, R, I), E(n, R, J)].$$

We had *three* completely different proofs in various situations:

- Decomposition of unipotents,
- Relative versions of localisation:

  yoga of conjugations and **yoga of commutators**,
- Level calculations.

Actually, the level of these commutators is the symmetrised product:

$$I \circ J = IJ + JI.$$

Namely, below we reproduce the key step in deriving the above from the absolute case.

Theorem. Let R be any associative ring and I, J be two-sided ideals of R, ad $n \geq 3$. Then

$$\begin{aligned} &E(n, R, IJ + JI) \leq \\ &\qquad [E(n, I), E(n, J)] \leq [E(n, R, I), E(n, R, J)] \leq \\ &\qquad [E(n, R, I), \mathrm{GL}(n, R, J)] \leq [\mathrm{GL}(n, R, I), \mathrm{GL}(n, R, J)] \leq \\ &\qquad\qquad \mathrm{GL}(n, R, IJ + JI) \end{aligned}$$

However, these results are *weaker* than the classical ones.
In the spirit of Bass, Mason, Stothers, and Suslin, one could ask:

Question 1. Can one replace $[\mathrm{GL}(n, R, I), E(n, R, J)]$ here by $[\mathrm{GL}(n, R, I), \mathrm{GL}(n, R, J)]$?

Answer. NO.

Question 2. Are the commutators *equal* to $E(n, R, I \circ J)$?

Answer. NO.

The reason is **non-stable K-theory** = **non-abelian K-theory**.

Even for quasi-finite rings the quotient

$$K_1(n, R, I) = \mathrm{GL}(n, R, I)/E(n, R, I)$$

may be **non-abelian** of [arbitrarily] high nilpotency class.

Such counter-examples — and much fancier ones! — were known from the work of Mason, Bak, and van der Kallen (1982–1991).

4 Generation of $[E(n,R,I),E(n,R,J)]$, 1st installment

In general, $E(n,R,I) \gneqq E(n,I)$.

• The unrelative elementary group $E(n,I)$ is generated by the **elementary transvections** $t_{ij}(a)$, $a \in I$, of level I.

• To generate the relative elementary group $E(n,R,I)$ one needs *at least* the **elementary conjugates**

$$z_{ij}(a,c) = {}^{t_{ji}(c)}t_{ij}(a) = t_{ji}(c)t_{ij}(a)t_{ji}(-c),$$

where $a \in I$, $c \in R$.

Theorem. (Vaserstein—Suslin, 1976) Let R be an associative ring, I be a two-sided ideal of R, and $n \geq 3$. Then the group $E(n,R,I)$ is generated by the elementary conjugates of level I:

$$E(n,R,I) = \big\langle z_{ij}(a,c),\ 1 \leq i \neq j \leq n,\ a \in I,\ c \in R \big\rangle.$$

Similar results were also established for:

- Chevalley groups — Hurley, Stein, Tits, Vaserstein,
- Bak's unitary groups — Bak—Vavilov.

Question. What about the generation of mixed commutators

$$[E(n, R, I), E(n, R, J)]?$$

- To generate the relative elementary commutator subgroups $[E(n, R, I), E(n, R, J)]$ one needs *at least* the **elementary commutators**

$$y_{ij}(a, b) = [t_{ij}(a), t_{ji}(b)],$$

where $a \in I$, $b \in J$.

The yoga of commutators and subsequent work of Stepanov and ourselves on the commutator width depended on:

Theorem. (Hazrat—Zhang, 2011, and HVZ, 2016) Let R be a quasi-finite ring, $n \geq 3$, and let I, J be two-sided ideals of R. Then the mixed commutator subgroup $\big[E(n,R,I), E(n,R,J)\big]$ is generated as a group by

- the elementary conjugates $z_{ij}(ab,c)$ or $z_{ij}(ba,c)$,
- the elementary commutators $y_{ij}(a,b)$,
- the HZ-generators $\big[t_{ij}(a), z_{ij}(b,c)\big]$,

where in all cases $a \in I$, $b \in J$, $c \in R$.

Corollary 1. Assume as above, then

$$[E(n,I), E(n,R,J)] = [E(n,R,I), E(n,J)] = [E(n,R,I), E(n,R,J)].$$

Corollary 2. Assume as above, then

$$[E(n,I), E(n,J)] \trianglelefteq E(n,R).$$

5 Generation of $[E(n,R,I),E(n,R,J)]$, 2nd installment

In October 2018, revising our work with Stepanov, in an attempt to answer a question by Raimund Preusser, I noticed that together with the above it implies:

Theorem. (Vavilov, 2018) Let I and J be two ideals of a *commutative* ring R, $n \geq 3$. Then

$$[E(n,I),\mathrm{GL}(n,J)] = [E(n,I),E(n,J)].$$

Theorem. (Vavilov, 2018) Let I and J be two ideals of a *commutative* ring R, $n \geq 3$. Then

$$[E(n,I),E(n,J)] = [E(n,R,I),E(n,R,J)]$$

Zuhong immediately asked:

Question. Does this mean that HZ-generators are superfluous?
Could one then give a direct proof?
Could one relax or remove commutativity condition?

The answers, November 2018—October 2019 are YES, YES, YES. This is what I reported in September—October 2019 at the conferences dedicated to the 60-th birthday of Ivan Panin, and the 70-the birthday of Alexander Generalov.

Theorem. (Vavilov—Zhang, 2018–2019) Let R be any *associative* ring with 1, let $n \geq 3$, and let I, J be two-sided ideals of R. Then the mixed commutator subgroup $[E(n, R, I), E(n, R, J)]$ is generated by:

- the elementary conjugates $z_{ij}(ab, c)$ and $z_{ij}(ba, c)$,
- the elementary commutators $y_{ij}(a, b)$,

where $1 \leq i \neq j \leq n$, $a \in I$, $b \in J$, $c \in R$.

Hold on, hold on, that's not the end of the story!

But first some corollaries of what we already have.

The unrelative group $E(n, I)$ is not normal in $E(n, R)$, but the commutator of two of these guys very much is! In fact:

Theorem. (Vavilov—Zhang, 2019) Let R be any *associative* ring with 1, let $n \geq 3$, and let I, J be two-sided ideals of R. Then one has

$$\big[E(n, I), E(n, J)\big] = \big[E(n, R, I), E(n, R, J)\big].$$

Theorem. (Vavilov—Zhang, 2019) Let R be a *quasi-finite* ring with 1, let $n \geq 3$, and let I, J be two-sided ideals of R. Then the following commutator formula holds

$$\big[E(n, R, I), \mathrm{GL}(n, R, J)\big] = \big[E(n, I), E(n, J)\big].$$

Theorem. (Vavilov—Zhang, 2019) Let I be an ideal of a *quasi-finite* ring R with 1, $n \geq 3$. Then $E(n, I)$ is normal in $\mathrm{GL}(n, I)$.

This last result is a broad generalisation of Mennicke and Nica.

7 Generation of $[E(n,R,I),E(n,R,J)]$, 3rd installment

In September 2019, as part of a joint work with Capdeboscq, Kuniavsky, and Plotkin, we started to seriously study the *arithmetic* case — below! In particular, the proofs in the works on the

- **Congruence subgroup problem**,
- **Bounded generation** by elementaries.

In October 2019 I noticed a congruence between $y_{ij}(a,b)$ modulo $E(n,R,I\circ J)$, Zuhong simplified and generalised it.

Theorem. (Vavilov—Zhang, 2019) Let R be any *associative* ring with 1, let $n\geq 3$, and let I,J be two-sided ideals of R. Then the mixed commutator subgroup $[E(n,R,I),E(n,R,J)]$ is generated by:

- the elementary conjugates $z_{ij}(ab,c)$ and $z_{ij}(ba,c)$,
- the elementary commutators $y_{ij}(a,b)$,

where $1\leq i\neq j\leq n$, $a\in I$, $b\in J$, $c\in R$. **Moreover**, for the second type of generators, it suffices to fix one pair of indices $(i,j)=(h,k)$.

The proof of the above result relies on:

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let I, J be two-sided ideals of R. Then $\big[E(n,I), E(n,J)\big]$ is central in $E(n,R)$ modulo $E(n,R,IJ+JI)$. In other words,

$$\big[\big[E(n,I), E(n,J)\big], E(n,R)\big] = E(n,R,IJ+JI).$$

In fact, modulo $E(n,R,IJ+JI) = E(n,R,I \circ J)$ the elementary commutators behave as **symbols** in classical algebraic K-theory, such as **Mennicke symbols**, or **Steinberg symbols**.

The proofs of the theorems below imitate Mennicke, Bass, Milnor, Serre, van der Kallen, Carter, Keller, Tavgen, Morris, Nica, ...

Hold on, hold on, that's not the end of the story! Now we have *ultimate* explanations!

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let I, J be two-sided ideals of R. Then for any $1 \leq i \neq j \leq n$, $a, a_1, a_2 \in I$, $b, b_1, b_2 \in J$ one has

$$\begin{aligned}
&y_{ij}(a_1 + a_2, b) \equiv y_{ij}(a_1, b) \cdot y_{ij}(a_2, b) \pmod{E(n, R, I \circ J)}\,,\\
&y_{ij}(a, b_1 + b_2) \equiv y_{ij}(a, b_1) \cdot y_{ij}(a, b_2) \pmod{E(n, R, I \circ J)}\,,\\
&y_{ij}(a, b)^{-1} \equiv y_{ij}(-a, b) \equiv y_{ij}(a, -b) \pmod{E(n, R, I \circ J)}\,,\\
&y_{ij}(ab_1, b_2) \equiv y_{ij}(a_1, a_2 b) \equiv e \pmod{E(n, R, I \circ J)}\,,\\
&y_{ij}(a_1 a_2, b) \equiv y_{ij}(a, b_1 b_2) \equiv e \pmod{E(n, R, I \circ J)}\,.
\end{aligned}$$

And, most importantly:

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let I, J be two-sided ideals of R. Then for any $1 \leq i \neq j \leq n$, any $1 \leq k \neq l \leq n$, and all $a \in I$, $b \in J$, $c \in R$, one has

$$y_{ij}(ac, b) \equiv y_{kl}(a, cb) \pmod{E(n, R, I \circ J)}\,.$$

9 Generation of $[E(n,R,I),E(n,R,J)]$, 4th installment

Morally, these computations go back to the verification of properties of Mennicke symbols in the works of Mennicke himself — **rolling over elementary commutators**, modulo $E(n,R,I\circ J)$.

Of course, Mennicke, Bass, Milnor, Serre,... stated this calculation in terms of *one* ideal. What we now need, are their *birelative* versions.

But this can be used in the opposite direction, to get rid of the elementary conjugates! This is what we noticed in March 2020.

Theorem. (Vavilov—Zhang, 2020) Let I and J be two ideals of an *associative* ring R and let $n\geq 3$. Then the mixed commutator subgroup $[E(n,R,I),E(n,R,J)]$ is generated by

• the elementary commutators $[t_{ij}(a),t_{hk}(b)]$,

where $1\leq i\neq j\leq n$, $1\leq h\neq k\leq n$, $a\in I$ and $b\in J$.

No elementary conjugates at all!

10 Counter-examples

One may ask, whether double commutators $[E(n, I), E(n, J)]$ are themselves always equal to $E(n, R, I \circ J)$?

• It's not the case, when $I = J$, there are counter-examples even for such nice rings as Gaussian integers $\mathbb{Z}[i]$ — Mason—Stothers, 1974.

There are counter-examples even *at the stable level*, Geller—Weibel. These counter-examples work already for $\mathrm{GL}(3, R)$ or $\mathrm{GL}(4, R)$!

• Let $R = \mathbb{Q}[x, y]$, $I = xR + yR$. Then

$$z = \begin{pmatrix} 1 - xy & x^2 & 0 \\ -y^2 & 1 + xy & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & x & 1 \end{pmatrix} \right] \in [E(3, I), E(3, I)].$$

But Weibel proved that $K_1(R, I^2) = \mathrm{GL}(R, I^2)/E(R, I^2) \cong \mathbb{Q}$ and under this isomorphism the Mennicke symbol $\begin{bmatrix} x^2 \\ 1 - xy \end{bmatrix}$ goes to $2 \in \mathbb{Q}$. This means that $z \notin E(R, I^2)$ and thus $z \notin E(3, R, I^2)$.

- Similarly, let $R = \mathbb{Z}[x]$, $I = xR$. Then clearly

$$y_{21}(x, x) = \begin{pmatrix} 1 - x^2 & x^3 & 0 \\ -x^3 & 1 + x^2 + x^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in [E(3, I), E(3, I)].$$

But the Mennicke symbol $\begin{bmatrix} x^3 \\ 1 - x^2 \end{bmatrix}$ is non-trivial, so that $y_{21}(x, x) \notin E(R, I^2)$ and thus $y_{21}(x, x) \notin E(3, R, I^2)$.

However, for comaximal ideals this is indeed the case.

Theorem. (Vavilov—Zhang, 2019) Let R be any *associative* ring with 1, let $n \geq 3$, and let I and J be two-sided ideals of R. If I and J are comaximal, $I + J = R$, then

$$[E(n, I), E(n, J)] = E(n, R, I \circ J).$$

Before 2019, it was only known for *commutative* and then *quasi-finite* rings, Vavilov—Stepanov, 2008, 2010.

In particular, it explains and generalises the *absolute* case in the works of Bass, Suslin, Vaserstein, Borewicz—Vavilov,...

Another peculiar case, when this holds.

Theorem. (Mason—Stothers, 1974) Let I and J be two comaximal ideals of a Dedekind ring of arithmetic type $R = \mathcal{O}_S$, $I + J = R$ and $n \geq 3$. Then

$$\big[\,\mathrm{GL}(n, R, I), \mathrm{GL}(n, R, J)\big] = E(n, R, IJ).$$

If I and J are not comaximal, there are counter-examples for Gaussian integers $R = \mathbb{Z}[i]$ and Eisensteinian integers $R = \mathbb{Z}[\omega]$.

Amazingly, in the arithmetic case these are essentially the only such counter-examples!

Theorem. (Vavilov, 2019) Let I and J be two ideals of a Dedekind ring of arithmetic type $R = \mathcal{O}_S$. Assume that the multiplicative group R^* is infinite and that $n \geq 3$. Then

$$\big[\,\mathrm{GL}(n, R, I), \mathrm{GL}(n, R, J)\big] = E(n, R, IJ).$$

13 Triple congruence

The true calculation behind all works starting with Mennicke, Bass—Milnor—Serre, etc. is an identity of the Hall—Witt type. Essentially, the **three ideal lemma** — can be stated as such!

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let A, B, C be two-sided ideals of R. Then for any three distinct indices i, j, h such that $1 \leq i, j, h \leq n$, and all $a \in A$, $b \in B$, $c \in C$, one has

$$y_{ij}(ab, c) y_{jh}(ca, b) y_{hi}(bc, a) \equiv 1 \pmod{E(n, R, ABC + BCA + CAB)}\,.$$

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let A, B, C be two-sided ideals of R. Then

$$[E(n, AB), E(n, C)] \leq [E(n, BC), E(n, A)] \cdot [E(n, CA), E(n, B)].$$

The following result explains all examples in the works of Mason—Stothers, 1974, Mason 1974, 1981.

Theorem. (Vavilov—Zhang, 2019) Let I be an ideal of an associative ring R, $m \geq 1$. Then the generic lattice of elementary commutator subgroups

$$H(r) = [E(n, I^r), E(n, I^{m-r})] \geq E(n, R, I^m), \qquad 0 \leq r \leq m,$$

of level I^m is isomorphic to the lattice of divisors of m. In other words, generically,

$$[E(n, I^r), E(n, I^{m-r})] \leq [E(n, I^s), E(n, I^{m-s})] \iff \gcd(s, m) \mid \gcd(r, m).$$

Namely, for two ideals $I, J \trianglelefteq R$ we denote by $E(n, J, I)$ the smallest subgroup containing $E(n, I)$ and normalised by $E(n, J)$:

$$E(n, J, I) = E(n, I)^{E(n,J)} = [E(n, I), E(n, J)] \cdot E(n, I).$$

Clearly,

$$E(n, I) \leq E(n, J, I) \leq E(n, R, I).$$

Now, from our generation results for the double commutators, we can derive a very broad generalisation of Vaserstein—Suslin, Stein, Tits, ...

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with identity 1, $n \geq 3$, and let I and J be two-sided ideals of R. Then the partially relativised elementary subgroup $E(n, J, I)$ is generated by

- the elementary conjugates $z_{ij}(a, b)$,

for all $1 \leq i \neq j \leq n$, $a \in I$, $b \in J$.

Namely, let $H_1, \ldots, H_m \leq G$ be subgroups of G.

There are many ways to form higher commutators of these groups, depending on where we put the brackets.

Thus, for three subgroups $F, H, K \leq G$ one can form two triple commutator subgroups $[[F, H], K]$ and $[F, [H, K]]$, and they are **non-associative**!

In the sequel, we denote by $[\![H_1, H_2, \ldots, H_m]\!]$ *any* higher mixed commutator subgroup of $H_1, \ldots, H_m$, with an arbitrary placement of brackets.

Thus, for instance, $[\![F, H, K]\!]$ refers to any of the two arrangements above.

Actually, the primary attribute of such a bracket arrangement that plays major role in our results is its **cut point**.

Namely, every higher commutator subgroup $[\![H_1, H_2, \ldots, H_m]\!]$ can be uniquely written as a double commutator

$$[\![H_1, H_2, \ldots, H_m]\!] = \Big[[\![H_1, \ldots, H_s]\!], [\![H_{s+1}, \ldots, H_m]\!]\Big],$$

for some $s = 1, \ldots, m-1$.

This s is called the cut point of our multiple commutator.

For *non-commutative* rings there is another aspect that affects the final answer. Namely, in this case symmetrised product of ideals is not associative. For instance, for three ideals $A, B, C \trianglelefteq R$ one has

$$(A \circ B) \circ C = ABC + BAC + CAB + CBA,$$

whereas

$$A \circ (B \circ C) = ABC + ACB + BCA + CBA,$$

that in general do not coincide.

To account for this, in the sequel we write $(\!| I_1 \circ \cdots \circ I_m |\!)$ to denote the symmetrised product of $I_1, \ldots, I_m$ with an arbitrary placement of parenthesis. Thus, for instance, $(\!| A \circ B \circ C |\!)$ may refer either to $(A \circ B) \circ C$, or to $A \circ (B \circ C)$, depending. In the sequel the initial bracketing of higher commutators will be reflected in the parenthesizing of the corresponding multiple symmetrised products.

Double commutators of elementary subgroups are not elementary subgroups themselves. But higher commutators are double.

Theorem. (Vavilov—Zhang, 2019) Let R be any *associative* ring with 1, let $n \geq 4$, and let $I_i \trianglelefteq R$, $i = 1, \ldots, m$, be two-sided ideals of R. Consider an arbitrary arrangment of brackets $[\![\ldots]\!]$ with the cut point s. Then one has

$$[\![E(n, I_1), E(n, I_2), \ldots, E(n, I_m)]\!] = \\ \Big[E\big(n, (\!|I_1 \circ \ldots \circ I_s|\!)\big), E\big(n, (\!|I_{s+1} \circ \ldots \circ I_m|\!)\big)\Big],$$

where the bracketing of symmetrised products on the right hand side coincides with the bracketing of the commutators on the left hand side.

Corollary. Assume as above. Then one has

$$[\![E(n, I_1), E(n, I_2), \ldots, E(n, I_m)]\!] = \\ [\![E(n, R, I_1), E(n, R, I_2), \ldots, E(n, R, I_m)]\!].$$

Easily follows by induction from the cases of *triple* and *quadruple* commutators.

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 3$, and let A, B, C be two-sided ideals of R. Then

$$\big[\big[E(n, A), E(n, B)\big], E(n, C)\big] = \big[E(n, A \circ B), E(n, C)\big].$$

Theorem. (Vavilov—Zhang, 2019) Let R be an *associative* ring with 1, $n \geq 4$, and let A, B, C, D be two-sided ideals of R. Then

$$\big[\big[E(n, A), E(n, B)\big], \big[E(n, C), E(n, D)\big]\big] = \big[E(n, A{\circ}B), E(n, C{\circ}D)\big].$$

Problem. Weaken the condition here to $n \geq 3$.

18 Possible Application 5: standard and general multiple commutator formulae

Problem. Can one replace in the above multiple commutator formulae some/all elementary groups by the corresponding congruence subgroups?

That'd be a vast *simultaneous* generalisation of the **standard commutator formulae** above and of the **nilpotent filtrations** for K_1 and the like by Bak, Hazrat, Vavilov, Basu, etc.

Standard multiple commutator formula

Theorem. (Hazrat—Zhang, 2013) Let A be a *quasi-finite* algebra with 1 over a commutative ring R, let $n \geq 3$, and further let $I_i \trianglelefteq A$, $i = 1, \ldots, m$, be two-sided ideals of A. Then one has

$$[E(n, A, I_1), \mathrm{GL}(n, A, I_2), \ldots, \mathrm{GL}(n, A, I_m)] = [E(n, A, I_1), E(n, A, I_2), \ldots, E(n, A, I_m)].$$

General multiple commutator formula

Theorem. (Stepanov 2016, Hazrat—Stepanov—Vavilov—Zhang, 2012–2016) Let A be a *quasi-finite* algebra with 1 over a commutative ring R of finite Bass—Serre dimension $\delta(R)$, let $n \geq 3$, and further let $I_i \trianglelefteq R$, $i = 1, \ldots, m$, be two-sided ideals of A. Assume that $m \geq \max(\delta(R) + 3 - n, 1)$. Then

$$[\mathrm{GL}(n, A, I_1), \mathrm{GL}(n, A, I_2), \ldots, \mathrm{GL}(n, A, I_m)] = [E(n, A, I_1), E(n, A, I_2), \ldots, E(n, A, I_m)].$$

Published only in the *commutative* case, with GL replaced by SL, in the setting of *algebraic* groups, by Stepanov, via his **universal localisation**.

The proof in the general case, via the **multirelative localisation-completion** is still not published, due to immense technical difficulties in handling the generators.

Now, eventually, we could simplify and finalise that!

For Chevalley groups assume

(*) In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of 2 elements and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2R + 2cR$.

Everything is commutative, but the additional complication are *two* root lengths.

Theorem. (Vavilov—Zhang, 2019) Let Φ be a reduced irreducible root system, $\mathrm{rk}(\Phi) \geq 2$. Further, let R be a commutative ring, and $I, J \trianglelefteq R$ be two ideals of R. For all $a \in I$, $b \in J$, $c \in R$, one has:

- $y_\alpha(a,b) \equiv y_\beta(a,b) \pmod{E(\Phi, R, IJ)}$,

for any roots $\alpha, \beta \in \Phi$ of the same length.

- $y_\alpha(a,b) \equiv y_\beta(a,b)^p \pmod{E(\Phi, R, IJ)}$,

if the root $\alpha \in \Phi$ is short, whereas the root $\beta \in \Phi$ is long, while $p = 2$ for $\Phi = \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4$, and $p = 3$ for $\Phi = \mathrm{G}_2$.

Thus, the short root type elementary commutators are expressed in terms of the long root type ones.

Theorem. (Vavilov—Zhang, 2019) Assume as above. Then for all $a \in I$, $b \in J$, $c \in R$, one has:

- $y_\alpha(ac, b) \equiv y_\alpha(a, cb) \pmod{E(\Phi, R, IJ)}$,

where either $\Phi \neq \mathrm{C}_l$, or α is short.

In the exceptional case when $\Phi = \mathrm{C}_l$ and α is long only the following weaker congruences hold:

- $y_\alpha(ac^2, b) \equiv y_\alpha(a, c^2b) \pmod{E(\Phi, R, IJ)}$,
- $y_\alpha(ac, b)^2 \equiv y_\alpha(a, cb)^2 \pmod{E(\Phi, R, IJ)}$.

In the case of Bak's unitary groups the same complication + form parameter, meaning that we need both one long *and* one short root.

Otherwise, everything similar, but technically more complicated, especially the case C_2 + some changes in the statements, like triple congruences become longer, etc.

20 Pending Application 6: subnormal subgroups

• Subgroups normalised by $E(n, R, J)$.

• Subnormal subgroups of $\mathrm{GL}(n, R)$ — and other groups! — with the best possible bounds

A joint project currenty under way, by Raimund Preusser, Zuhong Zhang, and myself — **reverse decomposition of unipotents**.

21 What's on?

• Three beautiful new ideas to fully solve the 50 years old problem.

Theorem. (van der Kallen, 1978, Lavrenov 2016, Sinchuk, 2017, Lavrenov—Sinchuk 2018, Voronetsky, 2020, Lavrenov—Sinchuk—Voronetsky, 2020) Let Φ be a reduced irreducible root system, $\mathrm{rk}(\Phi) \geq 3$, and let R be a commutative ring.

Then $K_2(\Phi, R)$ is central in the Steinberg group $\mathrm{St}(\Phi, R)$.

For A_2, C_2, G_2 there are counter-examples, Wendt, 2016.

• In 2020 we have constructed generalisation of **Steinberg symbols, Dennis—Stein symbols, Keune symbols, Kolster symbols.**

In terms of these **general symbols** we can generalise many results of the classical algebraic K-theory, and in particular the **Steinberg theorem** from fields and semi-local rings to Dedekind rings of arithmetic type.

• In 2020 we have constructed a whole new hierarchies of the higher **stability conditions** — vast generalisations of the usual Bass stable rank, unitary stable rank, absolute stable rank, and the like, and stated new stability theorems, *simultaneously* generalising all

◦ **usual stability theorems** (Bass—Vaserstein, Dennis—Vaserstein, Suslin—Tulenbaev, etc.), and

◦ **early stability theorems** (Bass—Milnor–Serre, Suslin, Vaserstein, van der Kallen, Kolster, ...)

But that's for the next talks!!!

THANK YOU!