COMPUTERS AS NOVEL MATHEMATICAL REALITY

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Doron Zeilberger (1993):
"The computer has already started doing to mathematics what the telescope and microscope did to astronomy and biology."

I cannot agree more! In Mathematics today we are in the same position as the XVII researchers in Natural Sciences.

Usually, one mentions two aspects:

- **Computer proofs**, computer aided proofs, formalised proofs, etc.

- Use of **computers in applications**.

But it's terribly much bigger!!!

In 2013–2014 Harald Helfgott has completely solved odd Gioldbach — IN THE ORIGINAL XVIII CENTURY ABSOLUTE FORM, not just in any of the later *asymptotic* forms.

That's what he says:

"The present work would most likely not have been possible without free and publicly available software: `PARI`, `Maxima`, `Gnuplot`, `VNODE-LP`, `PROFIL/BIAS`, `SAGE`, and, of course, LaTeX, `Emacs`, the `gcc` compiler and `GNU/Linux` in general. Some exploratory work was done in `SAGE` and `Mathematica`. Rigorous calculations used either D. Platt's interval-arithmetic package (based in part on `Crlibm`) or the `PROFIL/BIAS` interval arithmetic package underlying `VNODELP`."

I would like to stress three other aspects:

- **Mathematical experiment and exploration**,

- **Mathematical computation** – analysis of the small and intermediate cases,

- **Digital assistance**.

I believe, it started for real (there were attempts before that) in 1952, when Raphael Robinson has discovered new Mersenne primes, after a gap of 36 years.

For me personally, in 1990, when I've first seen `Mathematica` at the ICM-1990 in Kyoto.

1 Digital assistance in Mathematics

In 2011 the classics of experimental Mathematics David Bailey and Jonathan Borwein speak of **digital assistance** with the following scope:

• General purpose **Computer Algebra Systems** such as `Maple`, `Mathematica` — "or indeed `Matlab` and their open-source analogs"

Personally, I would add some others, in particular, `Axiom` and now, `Sage`.

• **Specialised packages** such as `CPLEX`, `PARI`, `SnapPea`, `Cinderella` or `MAGMA` — here one could add much more, according to the taste.

Again, I cannot agree more only that in my case it would be `GAP`, `CAYLEY`, `Lie`, `Chevie`, `Singular`, `CoCoA`, `Fermat`, `Macaulay`, ..., what not.

• **Programming Languages** such as `C`, `C++` and `Fortran-2000` — again, one could add their favourites.

• **Mathematical Applications** on the Web: `Sloane's Encyclopedia of Integer Sequences`, `Inverse Symbolic Calculator`, `Fractal Explorer`, `Jeff Weeks's Topological Games`, or even `Euclid` and `Java`.

• **Mathematical databases** and other **Information Resources** such as `Google`, `MathSciNet`, `arXiv`, `Wikipedia`, `MathWorld`, `MacTutor`, `Amazon`, `AmazonKindle`, and many others, that are seldom considered as part of computer-assisted mathematics — yet they are!

This is not limited to mathematics, of course, but one could mention also web-sites of individual mathematicians, seminars, departments, instituutes, mathematical societies, libraries, journals, publishers etc., etc.

This has completely changed access to information.

But I think there are two further types of products they *do not mention*, that have changed Mathematics as much — or at least the way that mathematical texts look — over the last 30 years:

• **Typesetting Systems**, text editors, symbol encoding systems, etc., such as TeX, and its dialects, extensions, but also `Unicode` or `Emacs`.

• Languages and **Systems of Computer Graphics**, graphic editors, etc. — such as `PostScript`, etc.

Even more amazingly, they do not mention:

• **Theorem Proving Systems**, proof asystants, etc., such as `Theorem`, `Coq`, `Isabelle`, `Mizar`, `HOL light`

For Bailey and Borwein, it is not part of **Experimental Mathematics**.

For me, it is not part of Mathematics *at all* — it is a respected and valuable activity, but not every respected and valuable activity is Mathematics.

The insistence on formalised proofs will kill Mathematics, as we know it, I say more on that aspect in the “Metaphor of Proof”.

2 Novel reality: How have computers changed our perception of Mathematics?

I mention some *obvious* aspects, there are more.

• **Contact with reality** – we can directly observe much larger chunks of mathematical reality, than ever before.

Jaffe and Quinn (as early as 1993):
"Mathematics may have even better experimental access to mathematical reality than the laboratory sciences have to physical reality".

Of course, I understand *experimental mathematics* and *theoretical mathematics* in tht strictly opposite sense.

For me a FORMALISED PROOF is *raw experimanetal material*, not any different from any other piece of calculation that we do not understand:

MATHEMATICS IS SOMETHING THAT WE UNDERSTAND.

• **Balance of ideas and computations**. Starting with the XIX century, there was a constant strife:

von Leibniz: REPLACE IDEAS WITH COMPUTATIONS.

Dirichlet: REPLACE COMPUTATIONS WITH IDEAS.

Observe how the style of Mathematics changed in the respect between 1896 and 1926 — the HILBERT REVOLUTION IN MATHEMATICS.

For me, large parts of both the XX century "pure" mathematics and "applied" mathematics and "computer science" do not make sense because they tilt this balance *far too much*:

René Thom: "Ce qui limite le vrai, ce n'est pas le faux, c'est l'insignifiant".

Now, for the first time in history, since the XVIII century, we enjoy the possibility to MERGE IDEAS AND COMPUTATIONS.

We should try to make the best use of this happy state of things in our research.

And even more so in teaching mathematics!!!

• **Finite and infinite** — the main philosophical lesson is that finite things can be so large that they become much less manageable, than infinite.

THERE IS NO DIFFERENCE BETWEEN INFINITE AND FINITE IN PRACTICAL TERMS.

David van Dantzig asked (1955): "Is $10 \uparrow\uparrow 3 = 10^{10^{10}}$ a finite number?"

Now we know it very much is.

Donlad Knuth (1976): "Is $10 \uparrow\uparrow\uparrow\uparrow 3$ a finite number?"

If the border has moved a bit since then, not much:

"Advances in our ability to compute are bringing us substantially closer to ultimate limitations."

Infinite in finite:

* Universal library,
* Ramsey type theorems,
* Goodstein sequences,
* Hercules and Hydra
* Kruskal-type tree theorems
* Busy beaver

etc., etc. I'll show some examples in Number Theory.

• **Feasible and unfeasible** — *large* finite can be as inaccessible as infinite:

Difference in size $10 \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow 3$ and $10 \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow 3$ far exceeds our imagination.

Continuing Knuth:

“Finite numbers can be really enormous, and the known universe is very small. Therefore, the distinction between 1nite and in1nite is not as relevant as the distinction between realistic and unrealistic”

• **Intermediate size phenomenon**:

■ General proof for large orders, dimensions, degrees,. . .

■ Direct verification for small ones.

■ But what to do with the intermediate ones?

Olivier Ramaré:

"Nous constatons dès lors qu'il reste une zone extrêmement étendue, typiquement entre 10^{10} et 10^{100000} où les moyens de calcul standarts ne suffisent plus et où les méthodes analytiques asymptotiques sont encore inopérantes."

Traditionally, classical mathematics was extremely effective in the opposite cases:

■ **Discrete Mathematics** — explicitly solvable models.

■ **Continuous Mathematics** — large, effecxtively infinite, number of objects.

■ But what about applications to biology, psycology, linguistics, analysis of historical, social and economic phenomena?

Both: LACK OF ADEQUATE MODELS, and

Grey zone: TOO LARGE FOR DIRECT ANALYSIS, TOO SMALL FOR ASYMPTOTICS.

• **Deterministic and random** — in individual human terms, THERE IS NO DIFFERENCE.

■ **Stochastic algorithms** — much faster than deterministic, absolute (not probabilistic!) answers.

■ **Probabilistic results** — any pair of elements generates the group, no such pair is known.

■ **Intrinsic indeterminism** — due to the lack of stability, etc.

■ But, first of all, lack of reliable *a priori* mechanisms to distinguish deterministic from random —

Like **Artcic Circle Theorem**, etc. — TRULY RANDOM IS HIGHLY REGULAR, what *seems* random, has to be specially constructed.

• **Derivable vs observable** — there is no difference between analytic and synthetic knowledge.

Stephen Wolfram (2002):

"But most of what's powerful out there in the computational universe is rife with computational irreducibility—so the only real way to see what it does is just to run it and watch what happens"

"Even when the underlying rules for a system are extremely simple, the behavior of the system as a whole can be essentially arbitrarily rich and complex"

There are many more aspects:

- Reconnect to the History of Mathematics
- Algorithmic thinking
- Theoretical and Experimental Mathematics

But probably I do not have time for that, so I'll pass to examples.

THANK YOU!