Algebraic Solutions to Scheduling Problems in Project Management

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Abstract: An approach to schedule development in project management is proposed based on models and methods of idempotent algebra. The approach offers a way to represent various types of precedence relationships among activities in projects as linear vector equations in terms of an idempotent semiring. As a result, many issues in project scheduling reduce to solving computational problems in the idempotent algebra setting, like linear equations and the eigenvalue problem. The solutions to the problems are given in a compact vector form that provides the basis for the development of efficient computation procedures and related software applications.

Key-Words: idempotent semiring, linear equation, eigenvalue and eigenvector, project scheduling, activity precedence relations, activity flow time

1 Introduction

The problem of scheduling a large-scale set of activities is a key issue in project management [1, 2]. There is a variety of project scheduling techniques developed to handle different aspects of the problem, ranging from the classical Critical Path Method and the Program Evaluation and Review Technique marked the beginning of the active research in the area in 1950s, to more recent methods of idempotent algebra (see, e.g., [3–8] and references therein).

We describe a new computational approach to project scheduling problems based on implementation and further development of models and methods of idempotent algebra in [8–10]. The approach offers a useful way to represent different types of precedence relationships among activities in a project as linear vector equations written in terms of an idempotent semiring. As a result, many issues in project scheduling reduce to solving computational problems in the idempotent algebra setting, like linear equations and the eigenvalue-eigenvector problem. We give solutions to the problems in a compact vector form that provides a basis for the development of efficient computation algorithms and related software applications.

The rest of the paper is as follows. We start with a brief introduction to idempotent algebra, that provides main definitions and notation, and then outlines basic results underlying subsequent applications. Furthermore, examples of actual problems in project scheduling are considered. We show how to formulate the problems in terms of idempotent algebra, and present related algebraic solutions. To illustrate the application of the results, numerical examples are given.

2 Definitions and Notation

We start with a brief introduction to idempotent algebra based on [8–10]. Further details on the topic can be found in [3, 5–7, 11–13].

2.1 Idempotent Semifield

Consider a set $\mathbb{X}$ that is equipped with two operations $\oplus$ and $\otimes$ called addition and multiplication, and that has neutral elements $\mathbb{0}$ and $\mathbb{1}$ called zero and identity. We suppose that $\langle\mathbb{X},\mathbb{0},\mathbb{1},\oplus,\otimes\rangle$ is a commutative semiring, where addition is idempotent and multiplication is invertible. Since the nonzero elements in $\mathbb{X}$ form a group under multiplication, this semiring is often referred to as the idempotent semifield.

The idempotent property is expressed by the equality $x \oplus x = x$ that is true for all $x \in \mathbb{X}$. Let $\mathbb{X}_{+} = \mathbb{X} \setminus \{\mathbb{0}\}$. For each $x \in \mathbb{X}_{+}$, there exists its inverse x^{-1} such that $x \otimes x^{-1} = \mathbb{1}$.

The power notation is defined in the ordinary way.

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For any $x \in \mathbb{X}_+$ and integer $p > 0$, we have $x^0 = \mathbb{1}$, $\mathbb{0}^p = \mathbb{0}$, $x^p = x^{p-1} \otimes x = x \otimes x^{p-1}$, $x^{-p} = (x^{-1})^p$.

In what follows, the multiplication sign $\otimes$ is omitted as is usual in conventional algebra. The power notation is thought of as defined in terms of idempotent algebra. However, when writing exponents, we routinely use ordinary arithmetic operations.

Since the addition is idempotent, it induces a partial order $\leq$ on $\mathbb{X}$ according to the rule: $x \leq y$ if and only if $x \oplus y = y$. With this definition, it is easy to verify that $x \leq x \oplus y$, $y \leq x \oplus y$, and that both addition and multiplication are isotonic.

The relation symbols are understood below in the sense of this partial order. Note that according to the order, we have $x \geq \mathbb{0}$ for any $x \in \mathbb{X}$.

As an example of the semirings under study, one can consider the idempotent semifield of real numbers

$$\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle.$$

The semiring has the neutral elements $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. For each $x \in \mathbb{R}$, there exists its inverse x^{-1}, which is equal to $-x$ in ordinary arithmetics. For any $x, y \in \mathbb{R}$, the power x^y is equivalent to the arithmetic product xy. The partial order coincides with the natural linear order on $\mathbb{R}$.

We use this semiring as the basis for the development of algebraic solutions to project scheduling problems in subsequent sections.

2.2 Vector and Matrix Algebra

Vector and matrix operations are routinely introduced on the basis of the scalar operations. Consider a Cartesian product $\mathbb{X}^n$ with its elements represented as column vectors. For any two vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ from $\mathbb{X}^n$, and a scalar $x \in \mathbb{X}$, vector addition and scalar multiplication follow the rules

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_i = a_i \oplus b_i, \qquad \{x\boldsymbol{a}\}_i = x a_i.$$

A vector with all entries equal to zero is called the zero vector and denoted by $\mathbb{0}$.

A vector is regular if it has no zero elements.

As usual, a vector $\boldsymbol{b} \in \mathbb{X}^n$ is linearly dependent on vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{X}^n$ if there are scalars $x_1, \ldots, x_m \in \mathbb{X}$ such that $\boldsymbol{b} = x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_m\boldsymbol{a}_m$. In particular, $\boldsymbol{b}$ is collinear with $\boldsymbol{a}$ when $\boldsymbol{b} = x\boldsymbol{a}$.

For any regular column vector $\boldsymbol{x} = (x_i)$, we define a row vector $\boldsymbol{x}^- = (x_i^-)$ with entries $x_i^- = x_i^{-1}$.

We define the distance between any two regular vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with a metric

$$\rho(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{b}^-\boldsymbol{a} \oplus \boldsymbol{a}^-\boldsymbol{b}.$$

When $\boldsymbol{b} = \boldsymbol{a}$ we have $\rho(\boldsymbol{a}, \boldsymbol{b}) = \mathbb{1}$, where $\mathbb{1}$ is the minimum value the metric ρ can take.

Specifically, in $\mathbb{R}^n_{\max,+}$, we have $\mathbb{1} = 0$, whereas the metric takes the form $\rho(\boldsymbol{x}, \boldsymbol{y}) = \max_i |x_i - y_i|$, and thus coincides with the Chebyshev metric.

For conforming matrices $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ with entries in $\mathbb{X}$, matrix addition and multiplication together with multiplication by a scalar $c \in \mathbb{X}$ are performed in accordance with the formulas

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \qquad \{BC\}_{ij} = \bigoplus_k b_{ik} c_{kj},$$
$$\{cA\}_{ij} = c a_{ij}.$$

A matrix with all entries equal to zero is called the zero matrix and denoted by $\mathbb{0}$.

A matrix is regular if it has no zero rows.

Consider the set of square matrices $\mathbb{X}^{n \times n}$. A matrix is diagonal if its off-diagonal entries are zero. The matrix $I = \operatorname{diag}(\mathbb{1}, \ldots, \mathbb{1})$ is the identity matrix.

A matrix is reducible if it can be put in a block triangular form by simultaneous permutations of rows and columns. Otherwise, the matrix is irreducible.

For any matrix $A \neq \mathbb{0}$ and integer $p > 0$, we have $A^0 = I$, $A^p = A^{p-1}A = AA^{p-1}$.

The trace of a matrix $A = (a_{ij})$ is defined as

$$\operatorname{tr} A = a_{11} \oplus \cdots \oplus a_{nn}.$$

2.3 Linear Vector Equations

Suppose $A, C \in \mathbb{X}^{m \times n}$ and $\boldsymbol{b}, \boldsymbol{d} \in \mathbb{X}^m$ are given matrices and vectors. A general linear equation in the unknown vector $\boldsymbol{x} \in \mathbb{X}^n$ is written in the form

$$A\boldsymbol{x} \oplus \boldsymbol{b} = C\boldsymbol{x} \oplus \boldsymbol{d}.$$

Note that due to the lack of additive inverse, one cannot put the equation in the form where all terms involving the unknown $\boldsymbol{x}$ are brought to one side of the equation while those without $\boldsymbol{x}$ go to another side.

Many practical problems reduce to solution of the following particular cases of the general equation

$$A\boldsymbol{x} = \boldsymbol{d}, \qquad A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}.$$

By analogy with linear integral equations, the above two equations are respectively referred to as that of the first kind and that of the second kind. The second-kind equations $A\boldsymbol{x} = \boldsymbol{x}$ and $A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ are also known in the literature as homogeneous and nonhomogeneous Bellman equations.

Finally note that along with the equations, one can consider inequalities of the first and second kind, that have the form $A\boldsymbol{x} \leq \boldsymbol{d}$ and $A\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$.

3 Preliminary Results

Now we outline some basic results from [8–10] that underlie subsequent applications of idempotent algebra to project scheduling.

3.1 The First-Kind Equation and Inequality

Given a matrix $A \in \mathbb{X}^{m\times n}$ and a vector $\boldsymbol{d} \in \mathbb{X}^m$, the problem is to find all solutions $\boldsymbol{x} \in \mathbb{X}^n$ of the equation

$$A\boldsymbol{x} = \boldsymbol{d}, \tag{1}$$

and the inequality

$$A\boldsymbol{x} \leq \boldsymbol{d}. \tag{2}$$

A solution $\boldsymbol{x}_1$ to equation (1) is called maximum if $\boldsymbol{x}_1 \geq \boldsymbol{x}$ for all respective solutions $\boldsymbol{x}$ of (1).

We present a solution to equation (1) based on the analysis of the distance between vectors in $\mathbb{X}^m$. The solution involves the introduction of a new symbol

$$\Delta = (A(\boldsymbol{d}^{-}A)^{-})^{-}\boldsymbol{d}$$

to represent a residual quantity associated with (1).

We start with a result that gives the distance from the vector $\boldsymbol{d}$ to a set $\{A\boldsymbol{x}|\boldsymbol{x} \in \mathbb{X}^n\}$ that is the linear span of columns (range) for the matrix A.

Lemma 1. *Suppose $A \in \mathbb{X}^{m\times n}$ and $\boldsymbol{d} \in \mathbb{X}^m$ are regular matrix and vector. Then it holds that*

$$\min_{\boldsymbol{x}\in\mathbb{X}^n} \rho(A\boldsymbol{x}, \boldsymbol{d}) = \Delta^{1/2}$$

with the minimum attained at $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{d}^{-}A)^{-}$.

As a consequence, we get the following result.

Theorem 1. *Suppose $A \in \mathbb{X}^{m\times n}$ and $\boldsymbol{d} \in \mathbb{X}^m$ are regular matrix and vector. Then a solution of equation* (1) *exists if and only if $\Delta = \mathbb{1}$. If solvable, the equation has the maximum solution given by*

$$\boldsymbol{x} = (\boldsymbol{d}^{-}A)^{-}.$$

Suppose that $\Delta > \mathbb{1}$. It follows from Lemma 1 that in this case equation (1) has no solution. As an approximate solution one can get $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{d}^{-}A)^{-}$ which is the best in the sense of the metric ρ.

Finally, it is not difficult to obtain the next result.

Lemma 2. *For any matrix $A \in \mathbb{X}^{m\times n}$ and vector $\boldsymbol{d} \in \mathbb{X}^m$, the solution to inequality* (2) *is given by*

$$\boldsymbol{x} \leq (\boldsymbol{d}^{-}A)^{-}.$$

3.2 Second-Kind Equations and Inequalities

Suppose a matrix $A \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbb{X}^n$ are given, whereas $\boldsymbol{x} \in \mathbb{X}^n$ is an unknown vector. We examine the equation

$$A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}, \tag{3}$$

and the inequality

$$A\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}. \tag{4}$$

To solve equation (3) we propose an approach based on the use of a function $\mathrm{Tr}(A)$ that takes each square matrix A to a scalar according to the definition

$$\mathrm{Tr}(A) = \mathrm{tr}\, A \oplus \cdots \oplus \mathrm{tr}\, A^n.$$

The function is exploited to examine whether the equation has a unique solution, many solutions, or no solution, and so may play the role of the determinant.

The solution involves evaluation of matrices A^*, $A^\times$, and A^+. The matrices A^* and $A^\times$ are given by

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}, \quad A^\times = A \oplus \cdots \oplus A^n.$$

Let $\boldsymbol{a}_i^\times$ be column i in $A^\times$, and $a_{ii}^\times$ be its diagonal element, $i = 1, \ldots, n$. To construct the matrix A^+ we take the set of columns $\boldsymbol{a}_i^\times$ such that $a_{ii}^\times = \mathbb{1}$, and then reduce it by removing those columns that are linearly dependent on others. Finally, the columns in the reduced set are put together to form a matrix A^+.

The general solutions to both equation and inequality of the second kind in the case of irreducible matrices are given by the following results.

Theorem 2. *Let $\boldsymbol{x}$ be the solution of equation* (3) *with an irreducible matrix A. The next statements hold:*

1) *if $\mathrm{Tr}(A) < \mathbb{1}$, then $\boldsymbol{x} = A^*\boldsymbol{b}$;*
2) *if $\mathrm{Tr}(A) = \mathbb{1}$, then $\boldsymbol{x} = A^*\boldsymbol{b} \oplus A^+\boldsymbol{v}$ for any vector $\boldsymbol{v}$;*
3) *if $\mathrm{Tr}(A) > \mathbb{1}$, then $\boldsymbol{x} = \mathbb{0}$ provided that $\boldsymbol{b} = \mathbb{0}$, and there is no solution otherwise.*

Lemma 3. *Let $\boldsymbol{x}$ be the solution of inequality* (4) *with an irreducible matrix A. The next statements hold:*

1) *if $\mathrm{Tr}(A) \leq \mathbb{1}$, then $\boldsymbol{x} = A^*(\boldsymbol{b} \oplus \boldsymbol{v})$ for any vector $\boldsymbol{v}$;*
2) *if $\mathrm{Tr}(A) > \mathbb{1}$, then $\boldsymbol{x} = \mathbb{0}$ provided that $\boldsymbol{b} = \mathbb{0}$, and there is no solution otherwise.*

3.3 Eigenvalues and Eigenvectors

A scalar λ is an eigenvalue of a matrix $A \in \mathbb{X}^{n\times n}$ if there is a nonzero vector $\boldsymbol{x} \in \mathbb{X}^n$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Any vector $\boldsymbol{x} \neq \mathbb{0}$ that satisfies the above equality is an eigenvector of A, corresponding to λ.

If the matrix $A \in \mathbb{X}^{n\times n}$ is irreducible, then it has only one eigenvalue given by

$$\lambda = \operatorname{tr} A \oplus \cdots \oplus \operatorname{tr}^{1/n}(A^n). \tag{5}$$

The corresponding eigenvectors of A have no zero entries and take the form

$$\boldsymbol{x} = A_\lambda^+ \boldsymbol{v},$$

where $A_\lambda = \lambda^{-1}A$, and $\boldsymbol{v}$ is any nonzero vector.

We conclude with an extremal property of the eigenvalue and eigenvectors of irreducible matrices.

Lemma 4. *Suppose A is an irreducible matrix with an eigenvector λ. Then it holds that*

$$\min_{\boldsymbol{x}\in\mathbb{X}_+^n} \rho(A\boldsymbol{x}, \boldsymbol{x}) = \lambda \oplus \lambda^{-1}$$

with the minimum attained at any eigenvector of A.

4 Applications to Project Scheduling

In this section we show how to apply results presented above to solve scheduling problems under various constraints (for further details on the schedule development in project management see, e.g., [1, 2]).

As the underlying idempotent semiring, we use $\mathbb{R}_{\max,+}$ in all examples under discussion.

4.1 Start-to-Finish Precedence Constraints

Consider a project that involves n activities. Activity dependencies are assumed the form of Start-to-Finish relations that do not allow an activity to complete until some predefined time after initiation of other activities. The scheduling problem of interest consists in finding the latest initiation time for all activities subject to given constraints on their completion time.

For each activity $i = 1, \ldots, n$, denote by x_i its initiation time, and by y_i its completion time. Let d_i be a due date, and a_{ij} be a minimum possible time lag between initiation of activity $j = 1, \ldots, n$ and completion of i.

Given a_{ij} and d_i, the completion time of activity i must satisfy the relations

$$y_i = d_i, \qquad x_j + a_{ij} \leq y_i, \quad j = 1, \ldots, n.$$

When a_{ij} is not actually given for some j, it is assumed to be $\mathbb{0} = -\infty$.

The relations can be combined into one equation in the unknown variables $x_1, \ldots, x_n$,

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = d_i.$$

By replacing the ordinary operations with those in $\mathbb{R}_{\max,+}$ in all equations, we get

$$a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n = d_i, \quad i = 1, \ldots, n.$$

Now we introduce an $n \times n$ matrix $A = (a_{ij})$, and n-vectors $\boldsymbol{d} = (d_i)$ and $\boldsymbol{x} = (x_i)$.

The scheduling problem under the Start-to-Finish constraints leads us to the derivation of the solution for the equation $A\boldsymbol{x} = \boldsymbol{d}$.

Consider the residual $\Delta = (A(\boldsymbol{d}^- A)^-)^- \boldsymbol{d}$ and suppose that $\Delta = \mathbb{1} = 0$. According to Theorem 1, the equation has a maximum solution $\boldsymbol{x} = (\boldsymbol{d}^- A)^-$.

As an example, consider a project with a constraint matrix and due date vector

$$A = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} 14 \\ 11 \\ 16 \\ 15 \end{pmatrix}.$$

First we calculate $\Delta = (A(\boldsymbol{d}^- A)^-)^- \boldsymbol{d} = 0$, and then get the solution

$$\boldsymbol{x} = (\boldsymbol{d}^- A)^- = (6, 4, 5, 3)^T.$$

4.2 Start-to-Start Precedence Constraints

Suppose there is a project consisting of n activities and operating under Start-to-Start precedence constraints that determine the minimum allowed time intervals between initiation of activities. The problem is to find the earliest initiation time for each activity that does not violate these constraints.

For each activity $i = 1, \ldots, n$, let b_i be an early possible initiation time, and let a_{ij} be a minimum possible time lag between initiation of activity $j = 1, \ldots, n$ and initiation of i. The initiation time x_i for activity i is subject to the relations

$$b_i \leq x_i, \qquad a_{ij} + x_j \leq x_i, \quad j = 1, \ldots, n,$$

where at least one must hold as an equality.

We can replace the relations with one equation

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}, b_i) = x_i.$$

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Representation in terms of $\mathbb{R}_{\max,+}$, gives the scalar equations

$$a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n \oplus b_i = x_i, \quad i = 1, \ldots, n.$$

With the notation $A = (a_{ij})$, $\boldsymbol{b} = (b_1, \ldots, b_n)^T$, $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ we arrive at a problem that is to solve the nonhomogeneous equation $A\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$.

Assume the matrix A to be irreducible. It follows from Theorem 2 that if $\mathrm{Tr}(A) \leq \mathbb{1} = 0$ then the equation has a solution given by $\boldsymbol{x} = A^*\boldsymbol{b} \oplus A^+\boldsymbol{v}$, where $\boldsymbol{v}$ is any vector of appropriate size.

Consider a project with Start-to-Start relations and examine two cases, with and without early initiation time constraints. Let us define a matrix

$$A = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix},$$

and two vectors

$$\boldsymbol{b}_1 = \mathbb{0}, \qquad \boldsymbol{b}_2 = (1, 1, 2, 1)^T.$$

Let us first calculate the initiation time of activities in the project when $\boldsymbol{b} = \boldsymbol{b}_1 = \mathbb{0}$ (that is, without early initiation time constraints imposed). Under this assumption, the equation takes the form $A\boldsymbol{x} = \boldsymbol{x}$.

The matrix A is irreducible and $\mathrm{Tr}(A) = 0$. Therefore, the equation has a solution.

Simple algebra gives

$$A^* = A^\times = \begin{pmatrix} 0 & -2 & 1 & -3 \\ 2 & 0 & 3 & -1 \\ -1 & -3 & 0 & -4 \\ 2 & 0 & 3 & 0 \end{pmatrix}.$$

Note that all diagonal entries in $A^\times$ are equal to $\mathbb{1} = 0$. However, considering that the first three columns are proportional, we take only one of them to form the matrix

$$A^+ = \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix}.$$

The solution to the equation is given by

$$\boldsymbol{x} = A^+\boldsymbol{v} = \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix} \boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^2_{\max,+}.$$

Consider the case when the equation takes the form $A\boldsymbol{x} \oplus \boldsymbol{b}_2 = \boldsymbol{x}$. Now we have

$$A^*\boldsymbol{b}_2 = (3, 5, 2, 5)^T,$$

and then get

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 5 \end{pmatrix} \oplus \begin{pmatrix} -2 & -3 \\ 0 & -1 \\ -3 & -4 \\ 0 & 0 \end{pmatrix} \boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^2_{\max,+}.$$

4.3 Mixed Precedence Relations

Consider a project that has both Start-to-Finish and Start-to-Start constraints. Let A_1 be a given Start-to-Finish constraint matrix, $\boldsymbol{d}$ a vector of due dates, and $\boldsymbol{x}$ an unknown vector of activity latest initiation time. To meet the constraints, the vector $\boldsymbol{x}$ must satisfy the inequality

$$A_1\boldsymbol{x} \leq \boldsymbol{d}.$$

Furthermore, there are also Start-to-Start constraints defined by a constraint matrix A_2. This leads to the equation

$$A_2\boldsymbol{x} = \boldsymbol{x}.$$

Suppose the equation has a solution $\boldsymbol{x} = A_2^+\boldsymbol{v}$. Substitution into the inequality gives $A_1A_2^+\boldsymbol{v} \leq \boldsymbol{d}$.

Since the maximum solution to the last inequality is $\boldsymbol{v} = (\boldsymbol{d}^- A_1 A_2^+)^-$, the solution to the whole problem is written in the form $\boldsymbol{x} = A_2^+(\boldsymbol{d}^- A_1 A_2^+)^-$.

As an illustration, we evaluate the solution to the problem under the condition that

$$A_1 = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix},$$

and

$$\boldsymbol{d} = (13, 11, 15, 15)^T.$$

By using results of previous examples, we successively get

$$A_1A_2^+ = \begin{pmatrix} 10 & 9 \\ 8 & 8 \\ 12 & 11 \\ 12 & 12 \end{pmatrix}, \quad (\boldsymbol{d}^- A_1 A_2^+)^- = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Finally, we have

$$\boldsymbol{x} = A_2^+(\boldsymbol{d}^- A_1 A_2^+)^- = (1, 3, 0, 3)^T.$$

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4.4 Minimization of Maximum Flow Time

Assume that a project has n activities and operates under Start-to-Finish constraints. For each activity, consider the time interval between its initiation and completion, which is usually referred to as the flow time, the turnaround time or the processing time. The problem of interest now is to construct a schedule that minimizes the maximum flow time over all activities.

Let A be an irreducible constraint matrix, $\boldsymbol{x}$ a vector of initiation time, and $\boldsymbol{y} = A\boldsymbol{x}$ a vector of completion time for the project. The problem can be formulated as that of finding a vector $\boldsymbol{x}$ that provides

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \max(|y_1 - x_1|, \ldots, |y_n - x_n|).$$

In terms of $\mathbb{R}_{\max,+}$ the problem takes the form

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \rho(A\boldsymbol{x}, \boldsymbol{x})$$

and can be solved by the application of Lemma 4.

Let $\boldsymbol{d}$ be a given vector of activity due dates. Consider a problem of finding the latest initiation time for all activities so as to provide both the due date constraints in the form

$$A\boldsymbol{x} \le \boldsymbol{d}$$

and the condition of minimization of maximum flow time over all activities in the project.

By Lemma 4, the last condition is satisfied when $\boldsymbol{x}$ is an eigenvector of A. The eigenvectors take the form $\boldsymbol{x} = A_\lambda^+\boldsymbol{v}$, where $A_\lambda = \lambda^{-1}A$, λ is an eigenvalue of A, and $\boldsymbol{v}$ is any vector of appropriate size.

By combining the result with the due date constraints, we get the inequality $AA_\lambda^+\boldsymbol{v} \le \boldsymbol{d}$.

With the maximum solution to the inequality given by $\boldsymbol{v} = (\boldsymbol{d}^- AA_\lambda^+)^-$, we arrive at the solution to the whole problem in the form $\boldsymbol{x} = A_\lambda^+(\boldsymbol{d}^- AA_\lambda^+)^-$.

Let us evaluate the solution with the constraint matrix and due date vector defined as

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 2 & 3 & 5 \\ 3 & 2 & 3 \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} 9 \\ 8 \\ 9 \end{pmatrix}.$$

First we get $\lambda = 4$ with (5), and define the matrix

$$A_\lambda = \begin{pmatrix} -2 & 0 & 0 \\ -2 & -1 & 1 \\ -1 & -2 & -1 \end{pmatrix}.$$

Furthermore, we have the matrices

$$A_\lambda^* = A_\lambda^\times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad A_\lambda^+ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, we arrive at the solution

$$\boldsymbol{x} = A_\lambda^+(\boldsymbol{d}^- AA_\lambda^+)^- = (4, 4, 3)^T.$$

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