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On the Aizerman Problem: Coefficient Conditions for the Existence of Three- and Six-Period Cycles in a Second-Order Discrete-Time System

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Abstract—In this paper, I study an automatic control discrete-time system of the second order. The nonlinearity of this system satisfies the generalized Routh–Hurwitz condition. Systems of this type are widely used in solving modern applied problems of the theory of automatic control. This work is a continuation of the results of research presented in my paper "On the Problem of Aizerman: Coefficient Conditions for the Existence of a Four-Period Cycle in a Second-Order Discrete-Time System," in which systems with two-periodic nonlinearity lying in the Hurwitz angle were studied. In this paper, the conditions on the parameters under which a system with two-periodic nonlinearity can possess a family of nonisolated four-period cycles are indicated and a method for constructing such nonlinearity is proposed. In the current paper, we assume that the nonlinearity is three-periodic and lies in a Hurwitz angle. We study a system for all possible parameter values. We explicitly present the conditions for the parameters under which it is possible to construct a three-periodic nonlinearity in such a way that a system with specified nonlinearity is not globally asymptotically stable. We show that a family of three-period cycles and a family of six-period cycles can exist in the system with this nonlinearity. A method for constructing such nonlinearities is proposed. The cycles are nonisolated; any solution of the system with the initial data, which lies on a certain specified ray, is a periodic solution.

Keywords: second-order discrete-time system, Aizerman conjecture, absolute stability, periodic solution.

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1. INTRODUCTION

The problem of the absolute stability of systems of differential equations with sectoral nonlinearity formulated by A.I. Lurie and V.N. Postnikov is one of the main problems of control theory, and a great many Russian and foreign works are devoted to its solution. A review of results and a reference list on this topic may be found, e.g., in paper [1].

M.A. Aizerman [2] formulated the well-known hypothesis that a control system is completely stable in the class of nonlinear characteristics lying in a Hurwitz angle. V.A. Pliss [3, 4] disproved the Aizerman conjecture by constructing a counterexample for a system of three differential equations and suggested a method for determining periodic oscillations in such systems.

Although the Aizerman conjecture appeared to be false in the general case, the problem of determination of the classes of control systems where this hypothesis is borne out and the problem of searching for periodic oscillations in such systems are still topical. New methods for searching for periodic solutions in continuous-time systems may be found, e.g., in the works of G.A. Leonov and his students [5, 6].

In addition to the control theory for systems of ordinary differential equations, an analogous theory for discrete-time systems has been intensely developed over recent decades, which is due to the wide application of discrete models in solving many applied problems. For instance, in works [7–9], the issue of absolute stability of discrete systems with various types of nonlinearities is investigated, while the two counterexamples to the analog of the Aizerman conjecture are presented for second-order discrete systems in papers [10, 11]. In one of the constructed examples, the system has a three-period cycle, while it has a four-period cycle in the other one.

This work is a continuation of the research presented in my work [12], where we study the second-order discrete system with two-periodic nonlinearity satisfying the generalized Routh–Hurwitz conditions for all parameter values. In [12], I explicitly wrote the conditions on parameters under which this nonlinearity may be constructed in such a way that a family of four-period cycles exists in the system.

In the current work, I assume that the nonlinear characteristic of the system is three-periodic and lies in a Hurwitz angle and investigate the system for all possible parameter values. The results of the work are formulated in the form of two theorems. In the first theorem, I present the conditions on parameters under which the three-periodic nonlinearity may be constructed in such a way that the system with this nonlinearity has a family of three-period cycles. In the second theorem, I present the conditions for parameters under which nonlinearity may be constructed such that the system has a family of six-period cycles. I propose an approach to constructing these nonlinearities. The cycles existing in the system are nonisolated, and any solution of the system with the initial data that lies on a certain specified ray is a periodic solution.

2. PROBLEM FORMULATION. MAIN RESULT

Consider the second-order discrete-time system

$$\begin{cases} x_{j+1} = y_j, \\ y_{j+1} = -\alpha y_j - \beta x_j - \varphi(\sigma_j), \end{cases} \tag{1}$$

where $\sigma_j = ay_j + bx_j$, $j \in N$, and $a^2\beta - ab\alpha + b^2 \neq 0$. Assume that system (1) is asymptotically stable at $\varphi(\sigma) \equiv 0$, i.e., $|\alpha| - 1 < \beta < 1$. Assume for certainty that $a > 0$.

Denote by Ω a set of $S = \text{const}$ such that system (1) with the linear function $\varphi(\sigma_j) = S(ay_j + bx_j)$ is asymptotically stable:

$$\Omega = \{S : |\alpha + aS| - 1 < \beta + bS < 1\}.$$

Assume in system (1) that

$$\varphi(\sigma_j) = S_j\sigma_j = S_j(ay_j + bx_j), \tag{2}$$

where $S_j \in \Omega$ and $j \in N$. The nonlinearity so defined satisfies the generalized Routh–Hurwitz conditions.

If $4\beta \leq \alpha^2$, then we set $b_\pm = \frac{a}{2}(\alpha \pm \sqrt{\alpha^2 - 4\beta})$.

Let $b^* \in \left[-a, -\frac{a(1-\beta)}{2+\alpha}\right)$ be a root of the polynomial

$$(a-b)^2(a+b) + (b(1+\alpha) - a\beta)(b(1-\alpha) + a\beta)^2$$

as a function of b at fixed parameters a, α, and β (a value b^* exists and is unique in this interval that will be shown below).

To simplify the formulation of Theorem 1 for $0 < \alpha < 1$, we introduce the denotations

$$b_1 = \min\left(b^*, -\frac{a(1-\beta)}{\alpha}\right), \quad B_1 = \max\left(b^*, -\frac{a(1-\beta)}{\alpha}\right),$$

$$b_2 = \min\left(\frac{a(1+\beta)}{\alpha}, \frac{a(1-\beta)}{1-\alpha}\right), \quad B_2 = \max\left(\frac{a(1+\beta)}{\alpha}, \frac{a(1-\beta)}{1-\alpha}\right).$$

Note that $b_1 = b^*$ if $\beta \geq 1 - \alpha$ and $b_1 = -\frac{a(1-\beta)}{\alpha}$ in the opposite case; $b_2 = \frac{a(1+\beta)}{\alpha}$ if $\beta \leq 2\alpha - 1$ and $b_2 = \frac{a(1-\beta)}{1-\alpha}$ in the opposite case.

Theorem 1. *If one of the conditions*

1) $\alpha > 1$, $4\beta > \alpha^2$, $b \in \left(-\frac{a(1-\beta)}{\alpha-1}, b^*\right) \cup \left(-\frac{a(1-\beta)}{\alpha}, \frac{a(1+\beta)}{\alpha}\right)$,

2) $\alpha > 1$, $4\beta \leq \alpha^2$, $b \in \left(-\frac{a(1-\beta)}{\alpha-1}, b^*\right) \cup \left(-\frac{a(1-\beta)}{\alpha}, b_-\right) \cup \left(b_+, \frac{a(1+\beta)}{\alpha}\right)$,

3) $\alpha = 1$, $4\beta > 1$, $b \in (-\infty, b^*) \cup (-a(1-\beta), a(1+\beta))$,

4) $\alpha = 1$, $4\beta \le 1$, $b \in (-\infty, b^*) \cup (-a(1-\beta), b_-) \cup (b_+, a(1+\beta))$,

5) $0 < \alpha < 1$, $4\beta > \alpha^2$, $b \in (-\infty, b_1) \cup (B_1, b_2) \cup (B_2, +\infty)$,

6) $0 < \alpha < 1$, $4\beta \le \alpha^2$, $b \in (-\infty, b_1) \cup (B_1, b_-) \cup (b_+, b_2) \cup (B_2, +\infty)$,

7) $\alpha \le 0$, $4\beta > \alpha^2$, $b \in \left(b^*, \frac{a(1-\beta)}{1-\alpha}\right)$,

8) $\alpha \le 0$, $4\beta \le \alpha^2$, $b \in (b^*, b_-) \cup \left(b_+, \frac{a(1-\beta)}{1-\alpha}\right)$,*on the system parameters is met, then a three-periodic sequence* $\{S_j\}_{j=1}^{\infty} \subset \Omega$ *exists such that system* (1) *with the nonlinearity of form* (2) *has three-period cycles. The solution of the system with the initial data* $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ *is a three-period cycle if* $x_1 \in R$, $x_1 \ne 0$, *and* $y_1 = \frac{1-(\alpha+aS_2)(\beta+bS_1)}{(\alpha+aS_1)(\alpha+aS_2)-(\beta+bS_2)}x_1$.

Let $\hat{b} \in \left(\frac{a(1-\beta)}{2-\alpha}, a\right]$ be a root of the polynomial

$$(b(1-\alpha)+a\beta)(b(1+\alpha)-a\beta)^2-(a-b)(a+b)^2$$

as the function b at fixed a, α, and β (value $\hat{b}$ exists and is unique in this interval).

To simplify the formulation of Theorem 2 for $-1 < \alpha < 0$, we introduce the denotations

$$b_3 = \min\left(-\frac{a(1-\beta)}{1+\alpha}, \frac{a(1+\beta)}{\alpha}\right), \quad B_3 = \max\left(-\frac{a(1-\beta)}{1+\alpha}, \frac{a(1+\beta)}{\alpha}\right),$$

$$b_4 = \min\left(-\frac{a(1-\beta)}{\alpha}, \hat{b}\right), \quad B_4 = \max\left(-\frac{a(1-\beta)}{\alpha}, \hat{b}\right).$$

Note that $b_3 = -\frac{a(1-\beta)}{1+\alpha}$ if $\beta \le -1-2\alpha$ and $b_3 = \frac{a(1+\beta)}{\alpha}$ in the opposite case; $b_4 = \hat{b}$ if $\beta \le 1+\alpha$ and $b_4 = -\frac{a(1-\beta)}{\alpha}$ in the opposite case.

Theorem 2. *If one of the conditions*

1) $\alpha < -1$, $4\beta > \alpha^2$, $b \in \left(\frac{a(1+\beta)}{\alpha}, \hat{b}\right) \cup \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{1+\alpha}\right)$,

2) $\alpha < -1$, $4\beta \le \alpha^2$, $b \in \left(\frac{a(1+\beta)}{\alpha}, b_-\right) \cup (b_+, \hat{b}) \cup \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{1+\alpha}\right)$,

3) $\alpha = -1$, $4\beta > 1$, $b \in (-a(1+\beta), \hat{b}) \cup (a(1-\beta), +\infty)$,

4) $\alpha = -1$, $4\beta \le 1$, $b \in (-a(1+\beta), b_-) \cup (b_+, \hat{b}) \cup (a(1-\beta), +\infty)$,

5) $-1 < \alpha < 0$, $4\beta > \alpha^2$, $b \in (-\infty, b_3) \cup (B_3, b_4) \cup (B_4, +\infty)$,

6) $-1 < \alpha < 0$, $4\beta \le \alpha^2$, $b \in (-\infty, b_3) \cup (B_3, b_-) \cup (b_+, b_4) \cup (B_4, +\infty)$,

7) $\alpha \ge 0$, $4\beta > \alpha^2$, $b \in \left(-\frac{a(1-\beta)}{1+\alpha}, \hat{b}\right)$,

8) $\alpha \ge 0$, $4\beta \le \alpha^2$, $b \in \left(-\frac{a(1-\beta)}{1+\alpha}, b_-\right) \cup (b_+, \hat{b})$,*on the system parameters is met, then a three-periodic sequence* $\{S_j\}_{j=1}^{\infty} \subset \Omega$ *exists such that system* (1) *with the nonlinearity of form* (2) *has six-period cycles. The solution of the system with the initial data* $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ *is a six-period cycle if* $x_1 \in R$, $x_1 \ne 0$, *and* $y_1 = -\frac{1+(\alpha+aS_2)(\beta+bS_1)}{(\alpha+aS_1)(\alpha+aS_2)-(\beta+bS_2)}x_1$.

3. PROOF OF THEOREMS

We write system (1) with nonlinearity (2) in the form

$$\begin{pmatrix} x_{j+1} \\ y_{j+1} \end{pmatrix} = P_j \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \tag{3}$$

where $P_j = \begin{pmatrix} 0 & 1 \\ -\beta_j & -\alpha_j \end{pmatrix}$,

$$\alpha_j = \alpha + aS_j, \quad \beta_j = \beta + bS_j, \tag{4}$$

$S_j \in \Omega, j \in N$.

For each fixed set of parameters a, b, α, and β of system (3), manifold Ω is the interval ($S_{\min}$, $S_{\max}$), where $S_{\min}$ and $S_{\max}$ depend on the parameters and are determined by the inequality

$$|\alpha_j| - 1 < \beta_j < 1. \tag{5}$$

$S_{\min}$ and $S_{\max}$ are written in explicit form in work [12].

Let the sequence $\{S_j\}_{j=1}^{\infty}$ be three-periodic (consequently, matrix P_j is also three-periodic). Together with system (3), consider the linear system

$$\begin{pmatrix} x_{j+1} \\ y_{j+1} \end{pmatrix} = P_{j+2}P_{j+1}P_j \begin{pmatrix} x_j \\ y_j \end{pmatrix} \tag{6}$$

with constant matrix

$$P_{j+2}P_{j+1}P_j = P_3P_2P_1 = \begin{pmatrix} \alpha_2\beta_1 & \alpha_1\alpha_2 - \beta_2 \\ \beta_1\beta_2 - \alpha_2\alpha_3\beta_1 & \alpha_1\beta_3 + \alpha_3\beta_2 - \alpha_1\alpha_2\alpha_3 \end{pmatrix},$$

$j = 3m - 2$, $m \in N$.

System (6) is asymptotically stable if and only if

$$|\mathrm{Sp}(P_3P_2P_1)| - 1 < \mathrm{Det}(P_3P_2P_1) < 1;$$

i.e., if the following inequalities hold

$$-1 - \beta_1\beta_2\beta_3 < -\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 < 1 + \beta_1\beta_2\beta_3, \tag{7}$$

$$\beta_1\beta_2\beta_3 < 1. \tag{8}$$

Note that inequality (8) is always true due to condition (5).

If inequality (7) is not valid, then system (6) is not asymptotically stable.

Proposition 1. *Let the equality*

$$-\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 = 1 + \beta_1\beta_2\beta_3 \tag{9}$$

be valid. Then, the solutions of system (3) *with the initial conditions* $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $j = 1$–3, *are three-period cycles if* $x_1 \in R$, $x_1 \neq 0$, *and* $y_1 = \dfrac{1 - \alpha_2\beta_1}{\alpha_1\alpha_2 - \beta_2} x_1$.

Proof of Proposition 1. Points $\begin{pmatrix} x_j \\ y_j \end{pmatrix}$, $j = 1$–3, determine a nontrivial three-periodic solution of system (3) if

$$\begin{cases} x_2 = y_1, \\ y_2 = -\alpha_1 y_1 - \beta_1 x_1, \end{cases} \quad \begin{cases} x_3 = y_2, \\ y_3 = -\alpha_2 y_2 - \beta_2 x_2, \end{cases} \quad \begin{cases} x_1 = y_3, \\ y_1 = -\alpha_3 y_3 - \beta_3 x_3. \end{cases} \tag{10}$$

If inequality (9) is valid, then the determinant of linear system (10) is equal to zero and the system has nontrivial solutions: $x_1 \neq 0$, $y_1 = \dfrac{1 - \alpha_2\beta_1}{\alpha_1\alpha_2 - \beta_2} x_1$, $x_2 = y_1$, $y_2 = -\dfrac{\alpha_1 - \beta_1\beta_2}{\alpha_1\alpha_2 - \beta_2} x_1$, $x_3 = y_2$, and $y_3 = x_1$, which make up three-periodic solutions of system (3) ($x_{j+3} = x_j$, $y_{j+3} = y_j$, $j \in N$). The proposition is proved.

The next proposition is proved similarly.

Proposition 2. *Let the equality*

$$-\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 = -1 - \beta_1\beta_2\beta_3 \quad (11)$$

be valid. Then, the solutions of system (3) *with the initial conditions* $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ *are six-period cycles if* $x_1 \in R$, $x_1 \neq 0$, *and* $y_1 = -\dfrac{1+\alpha_2\beta_1}{\alpha_1\alpha_2 - \beta_2}x_1$.

If condition (11) is met, points $\begin{pmatrix} x_j \\ y_j \end{pmatrix}$, $j = 1$–6, where $x_1 \neq 0$, $y_1 = -\dfrac{1+\alpha_2\beta_1}{\alpha_1\alpha_2 - \beta_2}x_1$, $x_2 = y_1$, $y_2 = \dfrac{\alpha_1 + \beta_1\beta_2}{\alpha_1\alpha_2 - \beta_2}x_1$, $x_3 = y_2$, and $y_3 = -x_1$, $x_4 = -x_1$, $y_4 = -y_1$, $x_5 = -x_2$, $y_5 = -y_2$, $x_6 = -x_3$, and $y_6 = -y_3$ determine a six-periodic solution of system (3) ($x_{j+6} = x_j$, $y_{j+6} = y_j$, $j \in N$).

If none of conditions (7), (9), and (11) is met, then system (3) has solutions unbounded at $j \to +\infty$.

Proof of Theorem 1. We find the conditions on the system parameters under which the sequence $\{S_j\}_{j=1}^{\infty} \subset \Omega$ mentioned in the formulation of the theorem exists.

It follows from (4) that $\beta_j = \dfrac{b\alpha_j + c}{a}$, where $c = a\beta - b\alpha$.

Equality (9) is true if $F(\alpha_1, \alpha_2, \alpha_3) = 0$, where

$$F(\alpha_1,\alpha_2,\alpha_3) = 1 + \alpha_1\alpha_2\alpha_3 + \frac{(b\alpha_1+c)}{a}\frac{(b\alpha_2+c)}{a}\frac{(b\alpha_3+c)}{a} - \alpha_1\frac{(b\alpha_3+c)}{a} - \alpha_2\frac{(b\alpha_1+c)}{a} - \alpha_3\frac{(b\alpha_2+c)}{a}.$$

We set $\alpha_{\max} = \alpha + aS_{\max}$, $\alpha_{\min} = \alpha + aS_{\min}$, $F(\alpha_{\max}, \alpha_{\min}, \alpha_{\min}) = F_1$, and $F(\alpha_{\min}, \alpha_{\max}, \alpha_{\max}) = F_2$.

We determine the parameter values under which F_1 and F_2 have opposite signs. If $F_1F_2 < 0$, then, due to continuity of function F, there exists a value $t \in (0, 1)$ such that $F(\alpha_1, \alpha_2, \alpha_3) = 0$ at

$$\alpha_1 = t\alpha_{\min} + (1-t)\alpha_{\max}, \quad \alpha_2 = \alpha_3 = (1-t)\alpha_{\min} + t\alpha_{\max}. \quad (12)$$

1. If $\alpha > 0$, $b \geq \dfrac{a(1+\beta)}{\alpha}$, then $\alpha_{\max} = \dfrac{a-c}{b}$, $\alpha_{\min} = \dfrac{a+c}{a-b}$,

$$F_1 = \frac{1}{b(a-b)^2}((a-b)^2(a+b) + (b-c)(b+c)^2),$$

$$F_2 = \frac{1}{b^2(b-a)}(c^2 - a^2)(a-b-c).$$

It is not hard to show that $F_1 > 0$ for such parameter values. $F_2 < 0$ if $\alpha < 1$, $\beta < 2\alpha - 1$, $b \in \left(\dfrac{a(1-\beta)}{1-\alpha}, +\infty\right)$ or $\alpha < 1$, $\beta \geq 2\alpha - 1$, $b \in \left(\dfrac{a(1+\beta)}{\alpha}, +\infty\right)$.

2. If $\alpha > 0$, $\dfrac{a(1-\beta)}{2-\alpha} \leq b < \dfrac{a(1+\beta)}{\alpha}$, then $\alpha_{\max} = \dfrac{a-c}{b}$, $\alpha_{\min} = -\dfrac{a+c}{a+b}$,

$$F_1 = \frac{1}{b(a+b)^2}(a+2b-c)((a+b)(a+c) + (b-c)^2), \quad (13)$$

$$F_2 = \frac{1}{b^2(a+b)}(a+c)(b+c-a)(a+2b-c). \quad (14)$$

In this case, $F_1 > 0$, because $b > 0$, $a + 2b - c > 0$, and multiplier

$$M_1(a,b,\alpha,\beta) = (a+b)(a+c) + (b-c)^2 = b^2(\alpha^2+\alpha+1) + ab(1-2\alpha\beta-\alpha-\beta) + a^2(\beta^2+\beta+1)$$

is positive for all parameter values (quadratic trinomial M_1 as a function of b at fixed parameters a, α, and β has a positive dominating coefficient and negative discriminant).

$F_2 < 0$ if $\alpha \geq 1$ or $\alpha < 1$, $\beta < 2\alpha - 1$, $b \in \left[\frac{a(1-\beta)}{2-\alpha}, \frac{a(1+\beta)}{\alpha}\right)$ or $\alpha < 1$, $\beta \geq 2\alpha - 1$, $b \in \left[\frac{a(1-\beta)}{2-\alpha}, \frac{a(1-\beta)}{1-\alpha}\right)$.

3. If $\alpha > 0$, $-\frac{a(1-\beta)}{2+\alpha} \leq b < \frac{a(1-\beta)}{2-\alpha}$, then $\alpha_{\max} = \frac{a+c}{a-b}$, $\alpha_{\min} = -\frac{a+c}{a+b}$,

$$F_1 = \frac{2}{(a-b)(a+b)^2}(a+c)((a+b)(a+c)+(b-c)^2), \tag{15}$$

$$F_2 = -\frac{4}{(a-b)^2(a+b)}(a+c)(ac+b^2). \tag{16}$$

In this case, $F_1 > 0$. $F_2 < 0$ if $4\beta > \alpha^2$, $b \in \left[-\frac{a(1-\beta)}{2+\alpha}, \frac{a(1-\beta)}{2-\alpha}\right)$ or $4\beta \leq \alpha^2$, $b \in \left[-\frac{a(1-\beta)}{2+\alpha}, b_-\right) \cup \left(b_+, \frac{a(1-\beta)}{2-\alpha}\right)$.

4. If $\alpha > 0$, $b < -\frac{a(1-\beta)}{2+\alpha}$, then $\alpha_{\max} = \frac{a+c}{a-b}$, $\alpha_{\min} = \frac{a-c}{b}$,

$$F_1 = \frac{1}{b^2(a-b)}(c^2-a^2)(b+c-a), \tag{17}$$

and $F_1 > 0$ if $\alpha \leq 1$, $b \in \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{2+\alpha}\right)$ or $\alpha > 1$, $b \in \left(-\infty, -\frac{a(1-\beta)}{\alpha-1}\right) \cup \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{2+\alpha}\right)$.

$$F_2 = \frac{1}{b(a-b)^2}((a-b)^2(a+b)+(b-c)(b+c)^2). \tag{18}$$

Polynomial

$$\begin{aligned} M_2(a,b,\alpha,\beta) &= (a-b)^2(a+b)+(b-c)(b+c)^2 = b^3(\alpha^3-\alpha^2-\alpha+2) \\ &+ ab^2(-3\alpha^2\beta+2\alpha\beta+\beta-1)+a^2b(3\alpha\beta^2-\beta^2-1)+a^3(1-\beta^3) \end{aligned} \tag{19}$$

in the right-hand side of Eq. (18) as a function of b at fixed parameters a, α, and β changes sign in the interval $\left(-a, -\frac{a(1-\beta)}{2+\alpha}\right)$ if $\beta \neq 1-\alpha$:

$$M_2\left(a, -\frac{a(1-\beta)}{2+\alpha}, \alpha, \beta\right) = \frac{8a^3}{(2+\alpha)^3}(1-\beta)(1+\alpha+\beta)^2 > 0, \tag{20}$$

$$M_2(a,-a,\alpha,\beta) = -a^3(1+\alpha+\beta)(1-\alpha-\beta)^2 < 0. \tag{21}$$

Consequently, there exists a root b^* of polynomial (19) that lies in the interval $\left[-a, -\frac{a(1-\beta)}{2+\alpha}\right)$. If $\beta = 1-\alpha$, then $b^* = -a$.

We show that b^* is a unique root of M_2 in the interval $\left(-\infty, -\frac{a(1-\beta)}{2+\alpha}\right)$. Dominating term M_2 is larger than zero for $\alpha > 0$, and discriminant Δ_{M_2} of polynomial M_2 changes its sign at considered parameter values.

If $\Delta_{M_2} < 0$, then b^* is a unique real root of M_2. If $\Delta_{M_2} \geq 0$, then we denote by k_1, k_2 ($k_1 > k_2$) the points of local extremums of function M_2. In this case, it is not hard to show that $k_1 > -\frac{a(1-\beta)}{2+\alpha}$; therefore, the roots of the polynomial M_2 not equal to b^* lie outside of the interval $\left(-\infty, -\frac{a(1-\beta)}{2+\alpha}\right)$.

Thus, $F_2 > 0$ if $b \in (-\infty, b^*)$ and $F_2 < 0$ if $b \in \left(b^*, -\frac{a(1-\beta)}{2+\alpha}\right)$.

To compare b^* with $b = -\frac{a(1-\beta)}{\alpha}$ and with $b = -\frac{a(1-\beta)}{\alpha - 1}$ for $\alpha > 1$, we note that $M_2\left(a, -\frac{a(1-\beta)}{\alpha}, \alpha, \beta\right) > 0$ if $\beta > 1 - \alpha$ and, in addition, $M_2\left(a, -\frac{a(1-\beta)}{\alpha-1}, \alpha, \beta\right) < 0$ for $\alpha > 1$.

Consequently, F_1 and F_2 have opposite signs if $\alpha \le 1$, $\beta \ge 1 - \alpha, b \in (-\infty, b^*) \cup \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{2+\alpha}\right)$ or $\alpha \le 1$, $\beta < 1 - \alpha, b \in \left(-\infty, -\frac{a(1-\beta)}{\alpha}\right) \cup \left(b^*, -\frac{a(1-\beta)}{2+\alpha}\right)$ or $\alpha > 1$, $b \in \left(-\frac{a(1-\beta)}{\alpha-1}, b^*\right) \cup \left(-\frac{a(1-\beta)}{\alpha}, -\frac{a(1-\beta)}{2+\alpha}\right)$.

5. If $\alpha \le 0$, $b \ge \frac{a(1-\beta)}{2-\alpha}$, then $\alpha_{\max} = \frac{a-c}{b}$, $\alpha_{\min} = -\frac{a+c}{a+b}$. F_1 and F_2 are given by formulas (13) and (14). It is easy to show that $F_1 > 0$ here. In addition, $F_2 < 0$ if $b \in \left[\frac{a(1-\beta)}{2-\alpha}, \frac{a(1-\beta)}{1-\alpha}\right)$.

6. If $\alpha \le 0$, $-\frac{a(1-\beta)}{2+\alpha} \le b < \frac{a(1-\beta)}{2-\alpha}$, then $\alpha_{\max} = \frac{a+c}{a-b}$, $\alpha_{\min} = -\frac{a+c}{a+b}$. F_1 is determined by formula (15) and F_2 is determined by formula (16). In this case, $F_1 > 0$. $F_2 < 0$ if $4\beta > \alpha^2$, $b \in \left[-\frac{a(1-\beta)}{2+\alpha}, \frac{a(1-\beta)}{2-\alpha}\right)$ or $4\beta \le \alpha^2$, $b \in \left[-\frac{a(1-\beta)}{2+\alpha}, b_-\right) \cup \left(b_+, \frac{a(1-\beta)}{2-\alpha}\right)$.

7. If $\alpha = 0$, $b < -\frac{a(1-\beta)}{2}$ or $\alpha < 0$, $\frac{a(1+\beta)}{\alpha} \le b < -\frac{a(1-\beta)}{2+\alpha}$, then $\alpha_{\max} = \frac{a+c}{a-b}$, $\alpha_{\min} = \frac{a-c}{b}$, and F_1 and F_2 are determined by formulas (17) and (18). Here, $F_1 > 0$ if $\alpha = 0$ or $\alpha < 0$ and $b > \frac{a(1+\beta)}{\alpha}$.

Polynomial M_2 which is a multiplier in F_2 as a function of b at fixed a, α, β has root b^* in the interval $\left(-a, -\frac{a(1-\beta)}{2+\alpha}\right)$ according to (20) and (21).

It is easy to show (similar to the proof in clause 4) that b^* is a unique root of M_2 in the considered interval and $F_2 < 0$ for $b \in \left(b^*, -\frac{a(1-\beta)}{2+\alpha}\right)$.

8. If $\alpha < 0$ and $b < \frac{a(1+\beta)}{\alpha}$, then $\alpha_{\max} = -\frac{a+c}{a+b}$ and $\alpha_{\min} = \frac{a-c}{b}$, and in this case $F_1 > 0$ and $F_2 > 0$.

By combining all parameter values under which $F_1F_2 < 0$, we obtain the desired result. The theorem is proved.

Note 1. From (9) it follows that the three-periodic sequence $\{S_j\}_{j=1}^{\infty} \subset \Omega$ mentioned in the formulation of Theorem 1 is constructed as follows: S_1 belongs to the interval $(S_{\min}, S_{\max})$ and satisfies the equation

$$mS_1^3 - (2mS_{sum} + n)S_1^2 + (mS_{sum}^2 - r)S_1 + (nS_{sum}^2 + 2rS_{sum} + l) = 0,$$

where $S_{sum} = S_{\min} + S_{\max}$, $m = a^3 + b^3$, $n = a^2\alpha + b^2\beta - ab$, $r = a\alpha^2 + b\beta^2 - a\beta - b\alpha$, $l = \alpha^3 + \beta^3 - 3\alpha\beta + 1$, and $S_2 = S_3 = S_{sum} - S_1$.

Theorem 2 is proved similarly to the proof of Theorem 1.

Equation (11) is valid if $G(\alpha_1, \alpha_2, \alpha_3) = 0$, where

$$G(\alpha_1, \alpha_2, \alpha_3) = 1 - \alpha_1\alpha_2\alpha_3 + \frac{(b\alpha_1 + c)}{a}\frac{(b\alpha_2 + c)}{a}\frac{(b\alpha_3 + c)}{a} + \alpha_1\frac{(b\alpha_3 + c)}{a} + \alpha_2\frac{(b\alpha_1 + c)}{a} + \alpha_3\frac{(b\alpha_2 + c)}{a}.$$

As in the proof of Theorem 1, we determine the parameter values under which $G(\alpha_{\max}, \alpha_{\min}, \alpha_{\min})$ and $G(\alpha_{\min}, \alpha_{\max}, \alpha_{\max})$ have opposite signs. Under such parameter values, due to continuity of function G, there exists a value $t \in (0, 1)$ such that $G(\alpha_1, \alpha_2, \alpha_3) = 0$ if α_j ($j = 1$–3) are determined by Eqs. (12).

Note 2. From Eq. (11) it follows that the three-periodic sequence $\{S_j\}_{j=1}^{\infty} \subset \Omega$ mentioned in the formulation of Theorem 2 is constructed as follows: S_1 belongs to the interval $(S_{\min}, S_{\max})$ and satisfies the equation

$$\tilde{m}S_1^3 - (2\tilde{m}S_{sum} - \tilde{n})S_1^2 + (\tilde{m}S_{sum}^2 + \tilde{r})S_1 - (\tilde{n}S_{sum}^2 + 2\tilde{r}S_{sum} + \tilde{l}) = 0,$$

where $\tilde{m} = b^3 - a^3$, $\tilde{n} = a^2\alpha - b^2\beta - ab$, $\tilde{r} = a\alpha^2 - b\beta^2 - a\beta - b\alpha$, $\tilde{l} = \alpha^3 - \beta^3 - 3\alpha\beta - 1$, and $S_2 = S_3 = S_{sum} - S_1$.

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