7th Multidisciplinary International Conference on Scheduling : Theory and Applications (MISTA 2015)
25-28 August 2015, Prague, Czech Republic

MISTA 2015

Tropical optimization problems in project scheduling

Nikolai Krivulin

Abstract We consider a project that consists of activities operating in parallel under various temporal constraints, including start-to-start, start-to-finish and finish-to-start precedence relations, early-start, late-start and late-finish time boundaries, and due dates. Scheduling problems are formulated to find optimal schedules for the project with respect to different objective functions to be minimized, including the project makespan, the maximum deviation from the due dates, the maximum flow-time, and the maximum deviation of finish times. We represent the problems as optimization problems in terms of tropical mathematics, and then solve these problems by applying direct solution methods of tropical optimization. As a result, new direct solutions of the problems are obtained in a compact vector form, which is ready for further analysis and practical implementation.

1 Introduction

Tropical optimization problems, which are formulated and solved in the framework of tropical mathematics, find increasing use in various fields of operations research, including project scheduling. As an applied mathematical discipline concentrated on the theory and applications of idempotent semirings, tropical (idempotent) mathematics dates back to a few seminal papers by Pandit [30], Cuninghame-Green [8], Giffler [11], Hoffman [15], Vorob'ev [35], and Romanovskiĭ [31], including the papers [8] and [11] concerned with optimization problems drawn from machine scheduling.

In succeeding years, tropical optimization problems were investigated in a number of publications, in which scheduling issues frequently served to motivate and illustrate the study. Specifically, various scheduling problems are examined in terms of tropical optimization by Cuninghame-Green [7], U. Zimmermann [38], K. Zimmermann [36, 37], Bouquard et al. [4], Fiedler et al. [10], Butkovič and Tam [6], Butkovič [5], Houssin [16], and Aminu and Butkovič [2]. Many optimization problems are formulated in the tropical mathematics setting to minimize a linear or nonlinear function defined on vectors over an idempotent

Nikolai Krivulin
Faculty of Mathematics and Mechanics
Saint Petersburg State University
St. Petersburg, 198504, Russia
E-mail: nkk@math.spbu.ru

semifield (a semiring with multiplicative inverses), and may have constraints in the form of vector equalities and inequalities (see, e.g., an overview in [22]). For some problems, complete direct solutions are obtained in a closed form under fairly general assumptions. Other problems are solved algorithmically by means of iterative computational procedures.

In this paper, we examine several problems drawn from project scheduling, to provide new solutions on the basis of recent results in tropical optimization. For details on the models and methods of project scheduling, one can consult the monographs by Demeulemeester and Herroelen [9], Neumann et al. [29], T'kindt and Billaut [33], and Vanhoucke [34].

We consider scheduling problems, which are to find an optimal schedule for a project that involves a set of activities operating in parallel under various temporal constraints, including start-to-start, start-to-finish, finish-to-start, early-start, late-start, late-finish, and due-date constraints. As optimization criteria to minimize, we take the project makespan, the maximum deviation from due dates, the maximum flow-time, and the maximum deviation of finish times. The problems under study are known to have algorithmic solutions in the form of iterative computational procedures, and can also be solved as linear programming problems by an appropriate linear programming algorithm [9, 29, 33, 34].

We represent the scheduling problems as tropical optimization problems, which are then solved by applying direct solution methods developed by Krivulin in [20, 21, 23, 24, 27]. As a result, we offer new direct solutions to the scheduling problems considered, which, in contrast to the conventional algorithmic solutions, provide results in a compact explicit vector form, ready for further development and practical implementation. Specifically, the new solutions allow various constraints to be incorporated in a unified and constructive way. The calculation of the solutions involves simple matrix-vector computations according to explicit formulae, which forms a basis for efficient computational algorithms and software.

The rest of the paper is organized as follows. In Section 2, we present scheduling problems that motivate and illustrate the study, and formulate these problems by using the ordinary notation. Section 3 includes a brief overview of preliminary definitions and results of tropical mathematics to be used in the subsequent sections. In Section 4, we describe some tropical optimization problems together with their solutions. In Section 5, we first rewrite the scheduling problems as tropical optimization problems, and then solve them by applying the results from Section 4.

2 Project scheduling model and example problems

We start with the description of a project scheduling model and the formulation of example problems of optimal scheduling in a general form (see, e.g., [9, 29, 33, 34] for further details on the common terminology and notation used in the area).

Consider a project that consists of n activities operating in parallel under start-to-start, start-to-finish and finish-to-start precedence relations, due dates, and time boundaries in the form of early-start, late-start and late-finish constraints. To describe the temporal constraints and scheduling objectives, we use, for each activity $i = 1, \ldots, n$, the notation x_i and y_i to represent the unknown start and finish times of the activity, respectively.

2.1 Temporal constraints

We now examine constraints on the start and finish times of each activity $i = 1, \ldots, n$. First, we represent precedence relations, which link activity i with other activities. Let a_{ij} be the minimum time lag between the start of activity j and the finish of i. The time lag a_{ii} specifies the minimum duration of the activity (the duration provided that no other constraints are imposed). Note that a negative a_{ij} can be interpreted as the maximum time lag between the finish of j and the start of i. If there is no lag defined, we assume $a_{ij} = -\infty$.

The start-to-finish constraints take the form of the inequalities $a_{ij} + x_j \leq y_i$ holding for all $j = 1, \ldots, n$. The activity is assumed to finish immediately after all start-to-finish constraints are satisfied, and thus at least one of the inequalities holds as an equality. Then, these inequalities are equivalent to one equality

$$\max(a_{i1} + x_1, \ldots, a_{in} + x_n) = y_i.$$

Furthermore, we denote by b_{ij} the minimum time lag between the start of activity j and the start of i, and put $b_{ij} = -\infty$ if the lag is not indicated. The start-to-start constraints are given by the inequalities $b_{ij} + x_j \leq x_i$ for all j, which can be rewritten as one inequality

$$\max(b_{i1} + x_1, \ldots, b_{in} + x_n) \leq x_i.$$

Let the minimum time lag between the finish of activity j and the start of i be denoted by c_{ij}, with $c_{ij} = -\infty$ if undefined. The finish-to-start constraints are written as the inequalities $c_{ij} + y_j \leq x_i$ for all j, or as one inequality

$$\max(c_{i1} + y_1, \ldots, c_{in} + y_n) \leq x_i.$$

Finally, we introduce due dates and time boundary constraints. The due date indicates the time when the activity is expected to finish, and therefore, it is not actually a strict constraint. For activity i, we denote the due date by d_i.

Let g_i and h_i be the earliest and latest possible times to start, and f_i be the latest possible time to finish. The early-start, late-start, and late-finish constraints provide strict boundaries for the start and finish times, given by

$$g_i \leq x_i \leq h_i, \qquad y_i \leq f_i.$$

2.2 Optimization criteria

To describe different scheduling objectives, we use several criteria that commonly arise in the development of optimal schedules in real-world problems. The criteria are written below in the form ready for further translation into terms of tropical mathematics.

We begin with the makespan, which is the interval between the earliest start time and the latest finish time of activities in a project. The makespan describes the total duration of the project and finds wide application as a natural objective function to be minimized in scheduling problems. With the notation introduced above, the makespan is given by

$$\max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} x_i = \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-x_i).$$

Another important criterion takes the form of the maximum absolute deviation of finish times of activities from given due dates for a project. The minimum value of this criterion corresponds to the minimal violation of due dates, which can be accomplished in the project. The maximum deviation from due dates is written as

$$\max_{1 \leq i \leq n} |y_i - d_i| = \max_{1 \leq i \leq n} \max(y_i - d_i, d_i - y_i).$$

The flow-time of an activity (also known as the system, throughput and turn-around time) is defined as the difference between its start and finish times, and can determine expenses related to undertaking the activity in the project. The flow-time of activity i is bounded from below by the value of a_{ii}, which is commonly assumed to be nonnegative. In general, the flow-time may be greater than this bound due to other temporal constraints.

In many real-world problems, the objective is formulated to minimize the maximum flow-time taken over all activities. The maximum flow-time is described by the expression

$$\max_{1\leq i\leq n}(y_i - x_i).$$

Finally, we consider the maximum deviation of completion times of all activities. The minimization of this criterion is equivalent to finding a schedule, where all activities have to finish simultaneously as much as possible. Such a problem can arise in just-in-time manufacturing, when certain delivery operations must be completed at once. The maximum deviation of completion times is given by

$$\max_{1\leq i\leq n} y_i - \min_{1\leq i\leq n} y_i = \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-y_i).$$

2.3 Examples of scheduling problems

We conclude with typical examples of scheduling problems, which are to serve to both motivate and illustrate the results in the rest of the paper. To formulate the problems, we use the notation and formulae introduced above to represent the unknown variables and given parameters, as well as to write the constraints and objectives for scheduling.

2.3.1 Minimization of maximum flow-time

First, we consider a problem of minimization of maximum flow-time under start-to-finish, start-to-start, finish-to-start and early-start temporal constraints. Given a_{ij}, b_{ij}, c_{ij} and g_i for all $i,j = 1,\dots,n$, the problem is to find the start and finish times x_i and y_i for each activity $i = 1,\dots,n$, that

$$\begin{aligned}
&\text{minimize} && \max_{1\leq i\leq n}(y_i - x_i),\\
&\text{subject to} && \max_{1\leq j\leq n}(a_{ij} + x_j) = y_i, \quad \max_{1\leq j\leq n}(b_{ij} + x_j) \leq x_i,\\
& && \max_{1\leq j\leq n}(c_{ij} + y_j) \leq x_i, \quad g_i \leq x_i, \qquad i = 1,\dots,n.
\end{aligned} \tag{1}$$

2.3.2 Minimization of maximum deviation from due dates

We now formulate a problem to minimize the maximum deviation from due dates under start-to-finish, start-to-start, finish-to-start and due dates constraints. Given parameters a_{ij}, b_{ij}, c_{ij} and d_i, the problem seeks to obtain the unknowns x_i and y_i that

$$\begin{aligned}
&\text{minimize} && \max_{1\leq i\leq n}\max(y_i - d_i, d_i - y_i),\\
&\text{subject to} && \max_{1\leq j\leq n}(a_{ij} + x_j) = y_i, \quad \max_{1\leq j\leq n}(b_{ij} + x_j) \leq x_i,\\
& && \max_{1\leq j\leq n}(c_{ij} + y_j) \leq x_i, \qquad i = 1,\dots,n.
\end{aligned} \tag{2}$$

2.3.3 Minimization of makespan

Suppose that we need to minimize the makespan under start-to-finish, early-start, late-start and late-finish constraints. Given parameters a_{ij}, g_i, h_i and f_i, we find x_i and y_i that solve the problem

$$\begin{aligned} \text{minimize} \quad & \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-x_i), \\ \text{subject to} \quad & \max_{1\leq j\leq n}(a_{ij}+x_j) = y_i, \quad g_i \leq x_i \leq h_i, \\ & y_i \leq f_i, \qquad i = 1,\ldots,n. \end{aligned} \tag{3}$$

2.3.4 Minimization of maximum deviation of finish times

Consider the problem of minimizing the maximum deviation of finish times under start-to-finish, start-to-start, finish-to-start, and late-finish constraints: given a_{ij}, b_{ij}, c_{ij} and f_i, find x_i and y_i, that

$$\begin{aligned} \text{minimize} \quad & \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-y_i), \\ \text{subject to} \quad & \max_{1\leq j\leq n}(a_{ij}+x_j) = y_i, \quad \max_{1\leq j\leq n}(b_{ij}+x_j) \leq x_i, \\ & \max_{1\leq j\leq n}(c_{ij}+y_j) \leq x_i, \quad y_i \leq f_i, \qquad i = 1,\ldots,n. \end{aligned} \tag{4}$$

Note that such problems can normally be solved using iterative computational algorithms (see [9, 29, 33, 34] for overviews of available solutions). In addition, these problems can be reformulated as linear programming problems to solve them by applying one of the computational methods of linear programming. Below, we provide new solutions to the problems, which are based on optimization methods in tropical mathematics, and offer results in a compact explicit vector form rather than in the form of a numerical algorithm.

3 Preliminary algebraic definitions and results

In this section, we give a brief overview of preliminary definitions, notation and results of tropical algebra to provide a formal basis for describing and solving tropical optimization problems as well as for applying them in project scheduling in the next sections. Both introductory and advanced material on tropical mathematics is provided in many publications, including [1,3,5,12–14,17,28,32] to name only a few. The overview below is mainly based on the presentation of results in [21,23,24,27], which provides a useful framework to obtain direct solutions to the problems under study in a compact vector form.

3.1 Idempotent semifield

We consider a system $(\mathbb{X},\oplus,\otimes,\mathbb{0},\mathbb{1})$, where $\mathbb{X}$ is a set closed under addition $\oplus$ and multiplication $\otimes$ with zero $\mathbb{0}$ and identity $\mathbb{1}$, such that $(\mathbb{X},\oplus,\mathbb{0})$ is a commutative idempotent monoid, $(\mathbb{X}\setminus\{\mathbb{0}\},\otimes,\mathbb{1})$ is an Abelian group, multiplication is distributive over addition, and $\mathbb{0}$ is absorbing for multiplication. This system is usually called the idempotent semifield.

Addition is idempotent, which means that $x\oplus x = x$ for each $x\in\mathbb{X}$. The idempotent addition induces on $\mathbb{X}$ a partial order such that $x\leq y$ if and only if $x\oplus y = y$. It follows directly from the definition that $x \leq x \oplus y$ and $y \leq x \oplus y$ for all $x, y \in \mathbb{X}$. Moreover, the inequality $x \oplus y \leq z$ appears to be equivalent to the two inequalities $x \leq z$ and $y \leq z$, and both addition and multiplication are monotone in each argument. Finally, we assume that the partial order is extendable to a total order, and thus consider $\mathbb{X}$ to be linearly ordered.

Multiplication is invertible to let each nonzero $x \in \mathbb{X}$ have the inverse x^{-1} such that $x \otimes x^{-1} = \mathbb{1}$. The inverse operation is antitone, which implies that, for all nonzero x and y, the inequality $x \leq y$ yields $x^{-1} \geq y^{-1}$ and vise versa.

The power notation with integer exponents is used to represent repeated multiplication defined as $x^0 = \mathbb{1}$, $x^p = x^{p-1} \otimes x$ and $x^{-p} = (x^{-1})^p$ for all nonzero x and positive integer p. The integer power is assumed to extend to rational exponents to make $\mathbb{X}$ algebraically closed (algebraically complete, radicable). In the algebraic expressions below, we omit the multiplication sign to save writing, and read the exponents only in the above sense.

A typical example of the idempotent semifield under consideration is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, where the addition $\oplus$ is defined as maximum and the multiplication $\otimes$ as ordinary addition, with the zero $\mathbb{0}$ given by $-\infty$ and the identity $\mathbb{1}$ by 0. Each number $x \in \mathbb{R}$ has the inverse x^{-1}, which is equal to the opposite number $-x$ in the conventional notation. For all $x, y \in \mathbb{R}$, the power x^y is well-defined and coincides, in ordinary arithmetic, with the product xy. The partial order induced by the idempotent addition corresponds to the standard linear order given on $\mathbb{R}$.

As another example, consider the semifield $\mathbb{R}_{\min,\times} = (\mathbb{R}_+ \cup \{+\infty\}, \min, \times, +\infty, 1)$, where $\mathbb{R}_+$ is the set of positive real numbers, $\oplus = \min$, $\otimes = \times$, $\mathbb{0} = +\infty$ and $\mathbb{1} = 1$. In this semifield, the notation of inverses and exponents has the standard interpretation. The partial order defined by addition extends to a linear order that is opposite to the standard order on $\mathbb{R}$.

3.2 Matrices and vectors

We now examine matrices and vectors over the idempotent semifield introduced above. The set of matrices that have m rows and n columns with entries from $\mathbb{X}$ is denoted $\mathbb{X}^{m \times n}$. A matrix, in which all entries are $\mathbb{0}$, is the zero matrix. A matrix is called row-regular (column-regular), if it has no row (column) consisting entirely of $\mathbb{0}$. Provided that a matrix is both row-regular and column-regular, it is called regular.

For any matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{X}^{m \times n}$ and $\boldsymbol{C} \in \mathbb{X}^{n \times l}$, and a scalar $x \in \mathbb{X}$, the matrix addition, matrix multiplication and scalar multiplication follow the standard rules with the scalar operations $\oplus$ and $\otimes$ in the place of the ordinary addition and multiplication, given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = \{\boldsymbol{A}\}_{ij} \oplus \{\boldsymbol{B}\}_{ij}, \qquad \{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k=1}^{n} \{\boldsymbol{A}\}_{ik} \{\boldsymbol{C}\}_{kj}, \qquad \{x\boldsymbol{A}\}_{ij} = x\{\boldsymbol{A}\}_{ij}.$$

The partial order associated with the idempotent addition and its properties extend to those on the set of matrices, where the relations are expanded entry-wise.

Consider any matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{m \times n}$. The transpose of $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^T \in \mathbb{X}^{n \times m}$.

The multiplicative conjugate transpose of $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^- = (a_{ij}^-)$, where $a_{ij}^- = a_{ji}^{-1}$ if $a_{ji} \neq \mathbb{0}$, and $a_{ij}^- = \mathbb{0}$ otherwise.

Consider square matrices of order n, which form the set $\mathbb{X}^{n \times n}$. A matrix which has the diagonal entries equal to $\mathbb{1}$ and all off-diagonal entries equal to $\mathbb{0}$ is the identity matrix $\boldsymbol{I}$. The power notation with nonnegative integer exponent serves to represent iterated products as $\boldsymbol{A}^0 = \boldsymbol{I}$ and $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A}$ for any square matrix $\boldsymbol{A}$ and positive integer p.

For any matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{n\times n}$, the trace is given by

$$\operatorname{tr}\boldsymbol{A} = \bigoplus_{i=1}^{n} a_{ii}.$$

Every matrix that has only one row (column) is considered a row (column) vector. All vectors below are column vectors unless otherwise specified. The set of column vectors of order n is denoted $\mathbb{X}^n$. A vector with all elements equal to $\mathbb{0}$ is the zero vector. If a vector has no zero elements, it is called regular. The vector of all ones is $\boldsymbol{1} = (\mathbb{1},\dots,\mathbb{1})^T$.

Let $\boldsymbol{A} \in \mathbb{X}^{n\times n}$ be a row regular matrix and $\boldsymbol{x} \in \mathbb{X}^n$ be a regular vector. Then, the vector $\boldsymbol{A}\boldsymbol{x}$ is regular. If the matrix $\boldsymbol{A}$ is column regular, then the row vector $\boldsymbol{x}^T\boldsymbol{A}$ is also regular.

The multiplicative conjugate transpose of a nonzero column vector $\boldsymbol{x} = (x_i) \in \mathbb{X}^n$ is a row vector $\boldsymbol{x}^- = (x_i^-)$ with the entries $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$, and $x_i = \mathbb{0}$ otherwise.

It is not difficult to verify that the conjugate transposition has the following useful properties. First, the equality $\boldsymbol{x}^-\boldsymbol{x} = \mathbb{1}$ is valid for any nonzero vector $\boldsymbol{x}$.

Furthermore, suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are regular vectors of the same order. Then, it is easy to see that the element-wise inequality $\boldsymbol{x} \leq \boldsymbol{y}$ is equivalent to $\boldsymbol{x}^- \geq \boldsymbol{y}^-$. In addition, the matrix inequality $\boldsymbol{x}\boldsymbol{y}^- \geq (\boldsymbol{x}^-\boldsymbol{y})^{-1}\boldsymbol{I}$ holds, and becomes $\boldsymbol{x}\boldsymbol{x}^- \geq \boldsymbol{I}$ when $\boldsymbol{y} = \boldsymbol{x}$.

Finally, consider a square matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$. A scalar $\lambda \in \mathbb{X}$ is an eigenvalue of $\boldsymbol{A}$, if there exists a nonzero vector $\boldsymbol{x} \in \mathbb{X}^n$ to satisfy the equality $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. The maximum eigenvalue with respect to the order defined on $\mathbb{X}$ is called the spectral radius and given by

$$\lambda = \bigoplus_{k=1}^{n} \operatorname{tr}^{1/k}(\boldsymbol{A}^k).$$

3.3 Solution to linear inequalities

We conclude the overview of the preliminary definitions and results with the solution of linear inequalities to be used below in the analysis of tropical optimization problems.

Suppose that, given a matrix $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ and a regular vector $\boldsymbol{d} \in \mathbb{X}^m$, we need to find vectors $\boldsymbol{x} \in \mathbb{X}^n$ that satisfy the inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}. \tag{5}$$

A direct complete solution to the problem under various assumptions is obtained in [23, 27] in the following form.

Lemma 1 *For any column-regular matrix $\boldsymbol{A}$ and regular vector $\boldsymbol{d}$, all solutions to inequality* (5) *are given by*

$$\boldsymbol{x} \leq (\boldsymbol{d}^-\boldsymbol{A})^-.$$

Furthermore, we consider the problem: given a matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$, find regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ to solve the inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}. \tag{6}$$

To represent a solution to the problem for any matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$, we define a function that takes $\boldsymbol{A}$ to produce the scalar

$$\operatorname{Tr}(\boldsymbol{A}) = \bigoplus_{k=1}^{n} \operatorname{tr}\boldsymbol{A}^k,$$

and use the asterisk operator (the Kleene star), which maps $\boldsymbol{A}$ to the matrix

$$\boldsymbol{A}^* = \bigoplus_{k=0}^{n-1} \boldsymbol{A}^k.$$

The next result obtained in [24, 26, 27] by using various arguments offers a direct, complete solution to inequality (6).

Theorem 1 *For any matrix $\boldsymbol{A}$, the following statements hold:*

1. *If* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, *then all regular solutions to* (6) *are given by* $\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{u}$, *where* $\boldsymbol{u}$ *is any regular vector.*
2. *If* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, *then there is no regular solution.*

4 Tropical optimization problems

Tropical optimization problems present an area in tropical mathematics, which is of both theoretical interest and practical importance (see, e.g., [22] for an overview, and [5, 7, 38] for further details on particular problems). Many problems are formulated in the framework of tropical mathematics to minimize or maximize nonlinear functions defined on vectors over idempotent semifields and calculated using multiplicative conjugate transposition of vectors. These problems may have constraints given by linear inequalities and equalities.

There are problems that can be solved directly in a rather general setting. For other problems, only algorithmic solution are known in the form of an iterative computational scheme to produce a particular solution, if there is any, or signify that no solutions exist.

The purpose of this section is twofold: first, to offer representative examples to demonstrate a variety of optimization problems under study, and second, to provide an efficient basis for the solution of scheduling problems in the next section. We consider examples of both unconstrained and constrained optimization problems with different objective functions defined in the common setting in terms of a general idempotent semifield. For all problems, direct solutions are given in a compact vector form ready for further analysis and straightforward computations. For some problems, the solutions obtained are complete solutions.

We start with the following problem: given matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{g} \in \mathbb{X}^n$, find regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ that

$$\begin{aligned} &\text{minimize} && \boldsymbol{x}^-\boldsymbol{A}\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}. \end{aligned} \tag{7}$$

A direct complete solution to the problem is given in [21, 24] as follows.

Theorem 2 *Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$ and $\boldsymbol{B}$ a matrix with* $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$. *Then, the minimum value in problem* (7) *is equal to*

$$\theta = \lambda \oplus \bigoplus_{k=1}^{n-1} \bigoplus_{1\leq i_1+\cdots+i_k\leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k}),$$

and all regular solutions are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^*\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{g}.$$

Furthermore, suppose that, given a matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$ and vectors $\boldsymbol{f},\boldsymbol{g},\boldsymbol{h} \in \mathbb{X}^n$, we need to find regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ that solve the problem

$$\begin{aligned} &\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}. \end{aligned} \tag{8}$$

The following direct exact solution is proposed in [21].

Theorem 3 *Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$, and $\boldsymbol{h}$ be a regular vector such that $\boldsymbol{h}^{-}\boldsymbol{g} \leq \mathbb{1}$. Then the minimum value in problem* (8) *is equal to*

$$\theta = \lambda \oplus \bigoplus_{k=1}^{n-1} (\boldsymbol{h}^{-}\boldsymbol{A}^{k}\boldsymbol{g})^{1/k},$$

and all regular solutions are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{h}^{-}(\theta^{-1}\boldsymbol{A})^{*})^{-}.$$

Given a matrix $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ and a vector $\boldsymbol{d} \in \mathbb{X}^m$, consider the problem to find regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ that

$$\text{minimize} \quad \boldsymbol{d}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{d}. \tag{9}$$

A direct solution proposed to the problem in [18, 25, 27] is as follows.

Theorem 4 *Let $\boldsymbol{A}$ be a row-regular matrix and $\boldsymbol{d}$ be a regular vector. Then, the minimum value in problem* (9) *is equal to*

$$\Delta = ((\boldsymbol{A}(\boldsymbol{d}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{d})^{1/2},$$

and the maximum solution is given by

$$\boldsymbol{x} = \Delta(\boldsymbol{d}^{-}\boldsymbol{A})^{-}.$$

Finally, we present a solution to the problem: given matrices $\boldsymbol{A},\boldsymbol{B} \in \mathbb{X}^{m\times n}$ and vectors $\boldsymbol{p},\boldsymbol{q} \in \mathbb{X}^m$, find regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ to

$$\text{minimize} \quad \boldsymbol{q}^{-}\boldsymbol{B}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}. \tag{10}$$

The next statement offers a direct solution to the problem [19].

Theorem 5 *Let $\boldsymbol{A}$ be row-regular and $\boldsymbol{B}$ column-regular matrices, $\boldsymbol{p}$ be nonzero and $\boldsymbol{q}$ regular vectors. Then, the minimum value in problem* (10) *is equal to*

$$\Delta = (\boldsymbol{A}(\boldsymbol{q}^{-}\boldsymbol{B})^{-})^{-}\boldsymbol{p},$$

and attained at any vector

$$\boldsymbol{x} = \alpha(\boldsymbol{q}^{-}\boldsymbol{B})^{-}, \qquad \alpha > \mathbb{0}.$$

5 Application to project scheduling

We are now in a position to derive solutions to the scheduling problems formulated in the beginning of the paper. In this section, we first represent each problem in the framework of the idempotent semifield $\mathbb{R}_{\max,+}$ in both scalar and vector forms, and then solve this problem by reducing to an optimization problem of the previous section.

5.1 Minimization of maximum flow-time

Consider problem (1) and describe it in terms of the semifield $\mathbb{R}_{\max,+}$. By replacing the usual operations by those of $\mathbb{R}_{\max,+}$, we obtain the problem to find the unknowns x_i and y_i for all $i = 1,\dots,n$, which

$$\begin{aligned}
&\text{minimize} && \bigoplus_{i=1}^{n} x_i^{-1} y_i,\\
&\text{subject to} && \bigoplus_{j=1}^{n} a_{ij} x_j = y_i, \quad \bigoplus_{j=1}^{n} b_{ij} x_j \leq x_i, \quad \bigoplus_{j=1}^{n} c_{ij} y_j \leq x_i,\\
& && g_i \leq x_i, \qquad i = 1,\dots,n.
\end{aligned}$$

To put the problem in a vector form, we introduce the following matrix-vector notation:

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{C} = (c_{ij}), \qquad \boldsymbol{g} = (g_i), \qquad \boldsymbol{x} = (x_i).$$

With this notation, the problem is to find vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ that

$$\begin{aligned}
&\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{y},\\
&\text{subject to} && \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{y} \leq \boldsymbol{x},\\
& && \boldsymbol{g} \leq \boldsymbol{x}.
\end{aligned} \tag{11}$$

A direct complete solution of the problem is given as follows.

Theorem 6 *Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be matrices such that* $\mathrm{Tr}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A}) \leq \mathbb{1}$. *Then, the minimum value in problem* (11) *is equal to*

$$\theta = \lambda \oplus \bigoplus_{k=1}^{n-1} \bigoplus_{1 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{i_1} \cdots \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{i_k}),$$

and all regular solutions are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{A}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{g}.$$

Proof By substitution of the first equality constraint $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we eliminate the vector $\boldsymbol{y}$. Then, we combine all inequality constraints into one, to write the problem

$$\begin{aligned}
&\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x},\\
&\text{subject to} && (\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}.
\end{aligned}$$

Clearly, this problem has the form of that at (7), where $\boldsymbol{B}$ is replaced by $\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A}$. Thus, a direct application of Theorem 2 yields the desired result. $\square$

5.2 Minimization of maximum deviation from due dates

Consider problem (2), which, in terms of the semifield $\mathbb{R}_{\max,+}$, takes the form

$$\begin{aligned} &\text{minimize} && \bigoplus_{i=1}^{n}(d_i^{-1}y_i \oplus y_i^{-1}d_i), \\ &\text{subject to} && \bigoplus_{j=1}^{n} a_{ij}x_j = y_i, \quad \bigoplus_{j=1}^{n} b_{ij}x_j \leq x_i, \quad \bigoplus_{j=1}^{n} c_{ij}y_j \leq x_i, \qquad i = 1,\ldots,n. \end{aligned}$$

In addition to the previously introduced matrix-vector notation, we define the vector $\boldsymbol{d} = (d_i)$, and write the problem as

$$\begin{aligned} &\text{minimize} && \boldsymbol{d}^{-}\boldsymbol{y} \oplus \boldsymbol{y}^{-}\boldsymbol{d}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{y} \leq \boldsymbol{x}. \end{aligned} \tag{12}$$

The next result offers a solution to the problem.

Theorem 7 *Let $\boldsymbol{A}$ be a row-regular matrix, $\boldsymbol{B}$ and $\boldsymbol{C}$ matrices such that* $\mathrm{Tr}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A}) \leq \mathbb{1}$, *and $\boldsymbol{d}$ be a regular vector. Then, the minimum value in problem* (12) *is equal to*

$$\Delta = ((\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}(\boldsymbol{d}^{-}\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*})^{-})^{-}\boldsymbol{d})^{1/2},$$

and the maximum solution is given by

$$\boldsymbol{x} = \Delta(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})(\boldsymbol{d}^{-}\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*})^{-}, \qquad \boldsymbol{y} = \Delta\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})(\boldsymbol{d}^{-}\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*})^{-}.$$

Proof After substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we combine both inequality constraints into one inequality $(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})\boldsymbol{x} \leq \boldsymbol{x}$. Furthermore, application of Theorem 1 to the last inequality yields $\boldsymbol{x} = (\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}\boldsymbol{u}$, where $\boldsymbol{u}$ is any regular vector.

By substitution of this solution, we reduce problem (12) to the unconstrained problem

$$\text{minimize} \quad \boldsymbol{d}^{-}\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}\boldsymbol{u} \oplus (\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}\boldsymbol{u})^{-}\boldsymbol{d}.$$

This problem has the form of (9) with $\boldsymbol{A}$ replaced by $\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^{*}$. Therefore, we can apply Theorem 4 to obtain a solution in terms of the vector $\boldsymbol{u}$. Turning back to the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we complete the proof. □

5.3 Minimization of makespan

In the framework of the semifield $\mathbb{R}_{\max,+}$, problem (3) is rewritten as

$$\begin{aligned} &\text{minimize} && \bigoplus_{i=1}^{n} y_i \bigoplus_{j=1}^{n} x_j^{-1}, \\ &\text{subject to} && \bigoplus_{j=1}^{n} a_{ij}x_j = y_i, \quad g_i \leq x_i \leq h_i, \quad y_i \leq f_i, \qquad i = 1,\ldots,n. \end{aligned}$$

By adding the vector notation $\boldsymbol{f} = (f_i)$ and using $\mathbf{1}$ to indicate the vector of ones, we represent the problem in the form

$$\begin{aligned} &\text{minimize} && \mathbf{1}^{T}\boldsymbol{y}\boldsymbol{x}^{-}\mathbf{1}, \\ &\text{subject to} && \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}, \\ &&& \boldsymbol{y} \leq \boldsymbol{f}. \end{aligned} \tag{13}$$

Theorem 8 *Let $\boldsymbol{A}$ be a nonzero matrix, $\boldsymbol{h}$ and $\boldsymbol{f}$ be regular vectors satisfying the condition $(\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})\boldsymbol{g} \leq \mathbb{1}$. Then, the minimum makespan in problem* (13) *is equal to*

$$\theta = \boldsymbol{1}^T \boldsymbol{A}(\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{h}^{-})\boldsymbol{1},$$

and all regular solutions are given by

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})\boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{A}(\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})\boldsymbol{u},$$

where

$$\boldsymbol{g} \leq \boldsymbol{u} \leq ((\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})(\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}))^{-}.$$

Proof As before, we first substitute $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. An application of Lemma 1 to solve the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{f}$ yields $\boldsymbol{x} \leq (\boldsymbol{f}^{-}\boldsymbol{A})^{-}$. Then, we take the two upper boundaries $\boldsymbol{x} \leq \boldsymbol{h}$ and $\boldsymbol{x} \leq (\boldsymbol{f}^{-}\boldsymbol{A})^{-}$, and apply conjugate transposition to rewrite them as $\boldsymbol{x}^{-} \geq \boldsymbol{h}^{-}$ and $\boldsymbol{x}^{-} \geq \boldsymbol{f}^{-}\boldsymbol{A}$. By coupling both inequalities into one, and again taking the conjugate transposition, we obtain one upper bound $\boldsymbol{x} \leq (\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})^{-}$.

Finally, we represent the objective function $\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{1}$ in its equivalent form $\boldsymbol{x}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{x}$ to write the problem as

$$\begin{aligned} &\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{g} \leq \boldsymbol{x} \leq (\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})^{-}. \end{aligned} \tag{14}$$

The problem obtained is of the form of (8), where $\boldsymbol{A}$ is replaced by $\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}$ and $\boldsymbol{h}$ by $(\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})^{-}$. To apply Theorem 3, we first calculate

$$(\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})^k = (\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{k-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}, \qquad \mathrm{tr}(\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})^k = (\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^k, \qquad k = 1, \ldots, n;$$

from which it directly follows that the spectral radius of the matrix $\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}$ is equal to

$$\lambda = \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} > \mathbb{0}.$$

Furthermore, we consider the minimum, which is given by

$$\theta = \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} \oplus (\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1}) \bigoplus_{k=1}^{n-1} ((\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{-1}\boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g})^{1/k}.$$

First, suppose that $\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} \leq \boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g}$. Since the inequality $(\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{-1}\boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g} \geq \mathbb{1}$ holds, we have

$$((\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{-1}\boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g})^{1/k} \leq (\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{-1}\boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g},$$

and hence, $\theta = \boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g}$. On the other hand, if $\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} > \boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g}$, then we immediately find that $\theta = \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1}$. By combining both results, we finally obtain

$$\theta = \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} \oplus \boldsymbol{h}^{-}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g} = \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1} \oplus \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{g}\boldsymbol{h}^{-}\boldsymbol{1} = \boldsymbol{1}^T\boldsymbol{A}(\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{h}^{-})\boldsymbol{1}.$$

To describe the solution set according to Theorem 3, we examine the matrix

$$(\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})^* = \bigoplus_{k=0}^{n-1}(\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})^k = \boldsymbol{I} \oplus \theta^{-1} \bigoplus_{k=1}^{n-1}(\theta^{-1}\boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1})^{k-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}.$$

Considering that $\theta \geq \boldsymbol{1}^T\boldsymbol{A}\boldsymbol{1}$, we obtain $(\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})^* = \boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}$. Substitution into the solution provided by Theorem 3 yields

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A})\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq ((\boldsymbol{h}^{-} \oplus \boldsymbol{f}^{-}\boldsymbol{A})(\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{A}))^{-}.$$

Finally, we represent the vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, which completes the proof. $\square$

5.4 Minimization of maximum deviation of finish times

After rewriting problem (4) in terms of $\mathbb{R}_{\max,+}$, the problem becomes

$$\begin{aligned} \text{minimize} \quad & \bigoplus_{i=1}^{n} y_i \bigoplus_{j=1}^{n} y_j^{-1}, \\ \text{subject to} \quad & \bigoplus_{j=1}^{n} a_{ij} x_j = y_i, \quad \bigoplus_{j=1}^{n} b_{ij} x_j \leq x_i, \\ & \bigoplus_{j=1}^{n} c_{ij} y_j \leq x_i, \quad y_i \leq f_i, \qquad i = 1, \dots, n. \end{aligned}$$

Switching to matrix-vector notation puts the problem in the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T \boldsymbol{y} \boldsymbol{y}^- \mathbf{1}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{y} \leq \boldsymbol{x}, \\ & \boldsymbol{y} \leq \boldsymbol{f}. \end{aligned} \tag{15}$$

The following result offers a solution to the problem.

Theorem 9 *Let $\boldsymbol{A}$ be row-regular and $\boldsymbol{B}$ column-regular matrices, $\boldsymbol{p}$ be nonzero and $\boldsymbol{q}$ regular vectors. Then, the minimum value in problem* (15) *is equal to*

$$\Delta = (\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-)^- \mathbf{1},$$

and attained if

$$\boldsymbol{x} = \alpha(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-, \qquad \boldsymbol{y} = \alpha \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-,$$

where

$$\alpha \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-)^{-1}.$$

Proof After substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we combine the first two inequalities in the constraints into one inequality $(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})\boldsymbol{x} \leq \boldsymbol{x}$. This inequality is then solved by using Theorem 1 to obtain $\boldsymbol{x} = (\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* \boldsymbol{u}$, where $\boldsymbol{u}$ is a regular vector.

Furthermore, we write the last inequality in the constraints as $\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* \boldsymbol{u} \leq \boldsymbol{f}$, and apply Lemma 1 to find $\boldsymbol{u} \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-$. The problem takes the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* \boldsymbol{u} (\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* \boldsymbol{u})^- \mathbf{1}, \\ \text{subject to} \quad & \boldsymbol{u} \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-. \end{aligned}$$

First, we remove the constraints and solve the obtained unconstrained problem. By applying Theorem 5, where both matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are replaced by $\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*$, and both vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ by $\mathbf{1}$, we find the minimum

$$\Delta = (\boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-)^- \mathbf{1},$$

which is attained at the vector

$$\boldsymbol{u} = \alpha(\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-, \qquad \alpha > \mathbb{0}.$$

To find the values of α, which satisfy the constraint $\boldsymbol{u} \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-$, we solve the inequality

$$\alpha(\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^- \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-.$$

By applying Lemma 1 with α as the unknown, we have

$$\alpha \leq (\boldsymbol{f}^- \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^* (\mathbf{1}^T \boldsymbol{A}(\boldsymbol{B} \oplus \boldsymbol{C}\boldsymbol{A})^*)^-)^{-1}.$$

It remains to turn back to vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ to complete the proof. □

Acknowledgements The author is very grateful to three referees for their careful reading of a previous draft of this paper, and for their constructive suggestions, which have been incorporated in the final version.

References

1. Akian, M., Bapat, R., Gaubert, S.: Max-plus algebra. In: L. Hogben (ed.) Handbook of Linear Algebra, Discrete Mathematics and Its Applications, pp. 25-1–25-17. Taylor and Francis, Boca Raton, FL (2007). DOI 10.1201/9781420010572.ch25
2. Aminu, A., Butkovič, P.: Non-linear programs with max-linear constraints: A heuristic approach. IMA J. Manag. Math. **23**(1), 41–66 (2012). DOI 10.1093/imaman/dpq020
3. Baccelli, F.L., Cohen, G., Olsder, G.J., Quadrat, J.P.: Synchronization and Linearity: An Algebra for Discrete Event Systems. Wiley Series in Probability and Statistics. Wiley, Chichester (1993)
4. Bouquard, J.L., Lenté, C., Billaut, J.C.: Application of an optimization problem in max-plus algebra to scheduling problems. Discrete Appl. Math. **154**(15), 2064–2079 (2006). DOI 10.1016/j.dam.2005.04.011
5. Butkovič, P.: Max-linear Systems: Theory and Algorithms. Springer Monographs in Mathematics. Springer, London (2010). DOI 10.1007/978-1-84996-299-5
6. Butkovič, P., Tam, K.P.: On some properties of the image set of a max-linear mapping. In: G.L. Litvinov, S.N. Sergeev (eds.) Tropical and Idempotent Mathematics, *Contemp. Math.*, vol. 495, pp. 115–126. American Mathematical Society (2009). DOI 10.1090/conm/495/09694
7. Cuninghame-Green, R.: Minimax Algebra, *Lecture Notes in Economics and Mathematical Systems*, vol. 166. Springer, Berlin (1979)
8. Cuninghame-Green, R.A.: Describing industrial processes with interference and approximating their steady-state behaviour. Oper. Res. Quart. **13**(1), 95–100 (1962). DOI 10.2307/3007584
9. Demeulemeester, E.L., Herroelen, W.S.: Project Scheduling: A Research Handbook. International Series in Operations Research and Management Science. Kluwer Acad. Publ., New York (2002)
10. Fiedler, M., Nedoma, J., Ramík, J., Rohn, J., Zimmermann, K.: Linear Optimization Problems with Inexact Data. Springer, Berlin (2006). DOI 10.1007/0-387-32698-7
11. Giffler, B.: Scheduling general production systems using schedule algebra. Naval Res. Logist. Quart. **10**(1), 237–255 (1963). DOI 10.1002/nav.3800100119
12. Golan, J.S.: Semirings and Affine Equations Over Them: Theory and Applications, *Mathematics and Its Applications*, vol. 556. Kluwer Acad. Publ., Dordrecht (2003)
13. Gondran, M., Minoux, M.: Graphs, Dioids and Semirings: New Models and Algorithms, *Operations Research / Computer Science Interfaces*, vol. 41. Springer, New York (2008). DOI 10.1007/978-0-387-75450-5
14. Heidergott, B., Olsder, G.J., van der Woude, J.: Max-plus at Work: Modeling and Analysis of Synchronized Systems. Princeton Series in Applied Mathematics. Princeton Univ. Press, Princeton, NJ (2006)
15. Hoffman, A.J.: On abstract dual linear programs. Naval Res. Logist. Quart. **10**(1), 369–373 (1963). DOI 10.1002/nav.3800100131
16. Houssin, L.: Cyclic jobshop problem and (max,plus) algebra. In: S. Bittanti, A. Cenedese, S. Zampieri (eds.) Proceedings of the 18th IFAC World Congress, 2011, *World Congress*, vol. 18, pp. 2717–2721. IFAC (2011). DOI 10.3182/20110828-6-IT-1002.03095
17. Kolokoltsov, V.N., Maslov, V.P.: Idempotent Analysis and Its Applications, *Mathematics and Its Applications*, vol. 401. Kluwer Acad. Publ., Dordrecht (1997)
18. Krivulin, N.: A solution of a tropical linear vector equation. In: S. Yenuri (ed.) Advances in Computer Science, *Recent Advances in Computer Engineering Series*, vol. 5, pp. 244–249. WSEAS Press (2012)

19. Krivulin, N.: Explicit solution of a tropical optimization problem with application to project scheduling. In: D. Biolek, H. Walter, I. Utu, C. von Lucken (eds.) Mathematical Methods and Optimization Techniques in Engineering, pp. 39–45. WSEAS Press (2013)
20. Krivulin, N.: Complete solution of a constrained tropical optimization problem with application to location analysis. In: P. Höfner, P. Jipsen, W. Kahl, M.E. Müller (eds.) Relational and Algebraic Methods in Computer Science, *Lecture Notes in Computer Science*, vol. 8428, pp. 362–378. Springer, Cham (2014). DOI 110.1007/978-3-319-06251-8_22
21. Krivulin, N.: A constrained tropical optimization problem: Complete solution and application example. In: G.L. Litvinov, S.N. Sergeev (eds.) Tropical and Idempotent Mathematics and Applications, *Contemp. Math.*, vol. 616, pp. 163–177. AMS, Providence, RI (2014). DOI 10.1090/conm/616/12308
22. Krivulin, N.: Tropical optimization problems. In: L.A. Petrosyan, J.V. Romanovsky, D.W.K. Yeung (eds.) Advances in Economics and Optimization: Collected Scientific Studies Dedicated to the Memory of L. V. Kantorovich, Economic Issues, Problems and Perspectives, pp. 195–214. Nova Science Publ., New York (2014)
23. Krivulin, N.: Extremal properties of tropical eigenvalues and solutions to tropical optimization problems. Linear Algebra Appl. **468**, 211–232 (2015). DOI 10.1016/j.laa.2014.06.044
24. Krivulin, N.: A multidimensional tropical optimization problem with nonlinear objective function and linear constraints. Optimization **64**(5), 1107–1129 (2015). DOI 10.1080/02331934.2013.840624
25. Krivulin, N.K.: On solution of linear vector equations in idempotent algebra. In: M.K. Chirkov (ed.) Mathematical Models. Theory and Applications. Issue 5, pp. 105–113. Saint Petersburg University, St. Petersburg (2004). (in Russian)
26. Krivulin, N.K.: Solution of generalized linear vector equations in idempotent algebra. Vestnik St. Petersburg Univ. Math. **39**(1), 16–26 (2006)
27. Krivulin, N.K.: Methods of Idempotent Algebra for Problems in Modeling and Analysis of Complex Systems. Saint Petersburg Univ. Press, St. Petersburg (2009). (in Russian)
28. Litvinov, G.: Maslov dequantization, idempotent and tropical mathematics: A brief introduction. J. Math. Sci. (NY) **140**(3), 426–444 (2007). DOI 10.1007/s10958-007-0450-5
29. Neumann, K., Schwindt, C., Zimmermann, J.: Project Scheduling with Time Windows and Scarce Resources, 2 edn. Springer, Berlin (2003). DOI 10.1007/978-3-540-24800-2
30. Pandit, S.N.N.: A new matrix calculus. J. SIAM **9**(4), 632–639 (1961). DOI 10.1137/0109052
31. Romanovskiĭ, I.V.: Asymptotic behavior of dynamic programming processes with a continuous set of states. Soviet Math. Dokl. **5**(6), 1684–1687 (1964)
32. Speyer, D., Sturmfels, B.: Tropical mathematics. Math. Mag. **82**(3), 163–173 (2009)
33. T'kindt, V., Billaut, J.C.: Multicriteria Scheduling: Theory, Models and Algorithms. Springer, Berlin (2006)
34. Vanhoucke, M.: Project Management with Dynamic Scheduling. Springer, Berlin (2013). DOI 10.1007/978-3-642-40438-2
35. Vorob'ev, N.N.: The extremal matrix algebra. Soviet Math. Dokl. **4**(5), 1220–1223 (1963)
36. Zimmermann, K.: Some optimization problems with extremal operations. In: B. Korte, K. Ritter (eds.) Mathematical Programming at Oberwolfach II, *Mathematical Programming Studies*, vol. 22, pp. 237–251. Springer, Berlin (1984). DOI 10.1007/BFb0121020
37. Zimmermann, K.: Disjunctive optimization, max-separable problems and extremal algebras. Theoret. Comput. Sci. **293**(1), 45–54 (2003). DOI 10.1016/S0304-3975(02)00231-1
38. Zimmermann, U.: Linear and Combinatorial Optimization in Ordered Algebraic Structures, *Annals of Discrete Mathematics*, vol. 10. Elsevier, Amsterdam (1981)