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MATHEMATICS

Two-Dimensional Homogeneous Cubic Systems: Classification and Normal Forms—VI

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Abstract—This paper is the sixth in a series of papers devoted to two-dimensional homogeneous cubic systems. It considers a case where a homogeneous vectorial polynomial in the right-hand part of the system does not have a common multiplier. A set of such systems is divided into classes of linear equivalence; in each of them, the simplest system is a third-order normal form which is separated on the basis of properly introduced principles. Such a form is defined by the matrix of its right-hand part coefficients, which is called the canonical form (CF). Each CF has its own arrangement of non-zero elements, their specific normalization and a canonical set of permissible values for the unnormalized elements, which relates the CF to the selected equivalence class. In addition to the classification, each CF is provided with: a) the conditions on the coefficients of the initial system, b) non-singular linear substitutions that reduce the right-hand side of the system under these conditions to the selected CF, c) obtained values of CF's unnormalized elements. The proposed classification was primarily created to obtain all possible structures of generalized normal forms for the systems with a CF in the unperturbed part. This paper presents another application of the resulting classification related to finding phase portraits in the Poincare circle for the CF.

Keywords: homogeneous cubic system, normal form, canonical form.

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1. INTRODUCTION

This study continues a series of papers [1–5] and all the designations introduced previously are used here. There are references to the proofs performed using the Maple software package and these are accessible in any of the following archives: https://github.com/Vladimir-Basov/DE, https://github.com/A-Cherm/DE.

This work completes the classification of the real systems (2.1) from [1, Sec. 2]

$$\dot{x}_1 = P_1(x_1, x_2), \quad \dot{x}_2 = P_2(x_1, x_2) \quad (P_i = a_i x_1^3 + b_i x_1^2 x_2 + c_i x_1 x_2^2 + d_i x_2^3 \not\equiv 0),$$

which are identified with the matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}. \tag{1.1}$$

We consider a case which was not investigated in previous studies, namely, $l = 0$, which means that the polynomials P_1 and P_2 does not have a common multiplier of nonzero degree.

Earlier, in [2] for the systems with $l = 3$, in [3, 4] for the systems with $l = 2$, and in [5] for the systems with $l = 1$, where l is the degree of the common multiplier of P_1 and P_2, the following results were obtained.

The set of systems (1.1) has been divided into the equivalence classes with respect to the linear nonsingular substitutions (2.2) from [1, Sec. 2] $x_1 = r_1 y_1 + s_1 y_2$, $x_2 = r_2 y_1 + s_2 y_2$, which are identified with the matrix

$$L = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}, \quad \delta = \det L \neq 0. \tag{1.2}$$

In each class, we have succeeded in separating the generator being the "simplest" system called a normal cubic form, with the matrix A called a canonical form; this form is designated $CF_i^{m,l}$ (see [2, Def. 1.10]), where m is the number of nonzero elements of A and l is the maximal degree of the real common multiplier.

The separation of canonical forms is based on the structural and normalization principles developed in [1, Secs. 1.1 and 1.2] to maximally simplify the reduction of perturbed systems with different CF in the unperturbed part to the generalized normal forms. The definition of the generalized normal forms and the constructive method for obtaining their various structures are presented in [1, Sec. 1.3].

In addition to the selection of arrangement of non-zero elements and the selection of normalized elements for each $CF_i^{m,l}$, the notion of canonical form presupposes the presence of the canonical set $cs_i^{m,l}$, introduced in ([2], Def. 1.9), that describes the set of values of the unnormalized elements of the canonical form at which it cannot be reduced to no one of the previous structural forms SF (see [2, Def. 1.3]) by using substitution (1.2). In [2–5] we have succeeded in writing out all $cs_i^{m,l}$ with $l = 1, 2, 3$.

In addition, for each canonical form obtained with $l \geq 1$, we have written out explicitly the conditions on the coefficients of the initial system and the substitution (1.2) that reduces this system to the selected form $CF_i^{m,l}$, as well as the concrete values of the parameters from $cs_i^{m,l}$.

In this study, analogous investigations have been carried out for the systems with $l = 0$.

The list of canonical forms with $l = 0$ and $m \leq 4$ is presented in Section 2; the canonical sets of parameters' values corresponding to them are separated. Unfortunately, it seems to be impossible to obtain $CF_i^{m,0}$ and $cs_i^{m,0}$ with $m \geq 5$ because of insurmountable technical difficulties. This does not allow us to perform a complete classification of homogeneous cubic systems, however, the absent part represents a minimal practical interest, since the main purpose of this classification related to normalization of the perturbed systems having a canonical form in the unperturbed part, in fact, cannot be fulfilled at $m \geq 5$, also because of insurmountable technical difficulties. Even at $m = 4$, we were able to obtain all structures of generalized normal forms only with one canonical form defined by the quadratic polynomial $\left(-x_1^2 - x_1x_2\right)/\left(x_1^2 + x_1x_2\right)$.

In Section 3, it is completely investigated under what conditions the initial system (1.1) can be reduced to any of the separated forms $CF_i^{m,0}$ with $m = 1, 2, 3$.

Another application of the classification obtained is presented in Section 4, namely: for one of the canonical forms, the topological classification in the Poincare circle is performed. For this purpose, the classification obtained is compared with another classification proposed by A. Cima and J. Llibre in [6], in which the set of systems (1.1) is divided into ten linearly nonequivalent classes with explicitly written generators, after that, for all representatives of each class at $l = 0$, all possible phase portraits in the Poincare circle are found.

2. SEPARATION OF CANONICAL FORMS AND CANONICAL SETS AT $l = 0$

Let us separate from the list 1.1 of [2] the structural forms $SF_i^{m,0}$ with $m = 2, 3, 4$ of system (1.1) (there are 32 such forms), normalize them according to the normalization principles, and find out which of the obtained normalized structural forms designated as NSF (see ([2, Def. 1.6]) are the canonical forms.

Statement 2.1. *Only* $NSF_4^{4,0} = \sigma\begin{pmatrix} u & v & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *and* $NSF_9^{4,0} = \sigma\begin{pmatrix} u & 0 & v & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *at all nonzero values of parameters* u, v *are reduced by means of substitutions* (1.2) *to some previous structural forms, according to structural principles.*

Proof. $NSF_4^{4,0}$ and $NSF_9^{4,0}$ are reduced by the substitution with $r_2 = 0$, $s_2 = \theta_* s_1$, in which $\theta_* : \theta^3 - \theta^2 + v\theta + u$, to $SF_2^{4,0}$. Examination shows that 30 remaining forms $NSF^{m,0}$ ($m = 2, 3, 4$) are $CF^{m,0}$.

Notation 2.1. Here and in what follows: (1) the phrase "is reduced to some $SF^{m,0}$" means that the indicated form or one of the forms previous to it is obtained; (2) the phrase "θ_*: polynomial of θ" means that θ_* is any real zero of the polynomial.

Let us write down all canonical forms, their canonical sets, and resultants R (see ([7; 1, Def. 2.1]), the form of $cs^{m,0}$ will be justified below in Statement 2.2.

List 2.1. Thirty $CF_i^{m,0}$, $cs^{m,0}$, their R ($m = 2, 3, 4$; $\sigma, \kappa = \pm 1$; $R, u, v \neq 0$): $CF_{1,\kappa}^{2,0} = \sigma\begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_2$, $CF_{10,\kappa}^{2,0} = \sigma\begin{pmatrix} 0 & 0 & 0 & \kappa \\ 1 & 0 & 0 & 0 \end{pmatrix}_8$; $CF_1^{3,0} = \sigma\begin{pmatrix} u & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_4$; $CF_{2,\kappa}^{3,0} = \sigma\begin{pmatrix} \kappa & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_5$, $CF_4^{3,0} = \sigma\begin{pmatrix} u & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_6$, $CF_9^{3,0} = \sigma\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 0 & 1 & 0 \end{pmatrix}_7$, $CF_{15}^{3,0} = \sigma\begin{pmatrix} u & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}_8$, $CF_{20}^{3,0} = \sigma\begin{pmatrix} 1 & 0 & 0 & u \\ 1 & 0 & 0 & 0 \end{pmatrix}_9$, $CF_{23,\kappa}^{3,0} = \sigma\begin{pmatrix} 0 & u & 0 & \kappa \\ 1 & 0 & 0 & 0 \end{pmatrix}_{10}$, $CF_{24}^{3,0} = \sigma\begin{pmatrix} 0 & 0 & 1 & u \\ 1 & 0 & 0 & 0 \end{pmatrix}_{11}$, $CF_1^{4,0} = \sigma\begin{pmatrix} u & v & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}_6$; $CF_2^{4,0} = \sigma\begin{pmatrix} u & 1 & v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_7$, $CF_3^{4,0} = \sigma\begin{pmatrix} u & 0 & v & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}_7$, $CF_6^{4,0} = \sigma\begin{pmatrix} u & 0 & 0 & v \\ 0 & 0 & 1 & 1 \end{pmatrix}_8$, $CF_{8,\kappa}^{4,0} = \sigma\begin{pmatrix} u & 0 & v & 0 \\ 0 & 1 & 0 & \kappa \end{pmatrix}_8$, $CF_{10}^{4,0} = \sigma\begin{pmatrix} 1 & u & 0 & v \\ 0 & 0 & 1 & 0 \end{pmatrix}_9$, $CF_{13}^{4,0} = \sigma\begin{pmatrix} u & 0 & 0 & 1 \\ 0 & 1 & 0 & v \end{pmatrix}_9$, $CF_{16}^{4,0} = \sigma\begin{pmatrix} u & 0 & 1 & v \\ 0 & 0 & 1 & 0 \end{pmatrix}_{10}$, $CF_{17}^{4,0} = \sigma\begin{pmatrix} u & v & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{10}$, $CF_{21}^{4,0} = \sigma\begin{pmatrix} u & 0 & 0 & v \\ 1 & 0 & 0 & 1 \end{pmatrix}_{10}$, $CF_{22}^{4,0} = \sigma\begin{pmatrix} u & 0 & 0 & v \\ 0 & 1 & 1 & 0 \end{pmatrix}_{10}$, $CF_{25}^{4,0} = \sigma\begin{pmatrix} u & 0 & v & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{11}$, $CF_{26}^{4,0} = \sigma\begin{pmatrix} 1 & u & 0 & v \\ 1 & 0 & 0 & 0 \end{pmatrix}_{11}$, $CF_{28}^{4,0} = \sigma\begin{pmatrix} 0 & u & 0 & v \\ 1 & 0 & 0 & 1 \end{pmatrix}_{11}$, $CF_{31}^{4,0} = \sigma\begin{pmatrix} u & 0 & 1 & v \\ 1 & 0 & 0 & 0 \end{pmatrix}_{12}$, $CF_{32}^{4,0} = \sigma\begin{pmatrix} 0 & 0 & u & v \\ 1 & 0 & 0 & 1 \end{pmatrix}_{12}$, $CF_{34,\kappa}^{4,0} = \sigma\begin{pmatrix} 0 & u & 0 & v \\ 1 & 0 & \kappa & 0 \end{pmatrix}_{12}$, $CF_{35}^{4,0} = \sigma\begin{pmatrix} 0 & u & 1 & v \\ 1 & 0 & 0 & 0 \end{pmatrix}_{13}$, $CF_{36}^{4,0} = \sigma\begin{pmatrix} 0 & 0 & 1 & u \\ 1 & 0 & v & 0 \end{pmatrix}_{13}$, $CF_{37}^{4,0} = \sigma\begin{pmatrix} 0 & 0 & u & v \\ 1 & 1 & 0 & 0 \end{pmatrix}_{14}$;

$tcs_{1,\kappa}^{2,0}$, $R = \kappa$; $tcs_{10,\kappa}^{2,0}$, $R = -\kappa$; $tcs_1^{3,0}$, $R = u^3$; $cs_{2,\kappa}^{3,0} = \{u \neq 3/2 \text{ at } \kappa = -1\}$, $R = 1$; $tcs_4^{3,0}$, $R = u^3$; $tcs_9^{3,0}$, $R = -u$; $tcs_{15}^{3,0}$, $R = u$; $tcs_{20,}^{3,0}$, $R = -u^3$; $tcs_{23,\kappa}^{3,0}$, $R = -\kappa$; $tcs_{24}^{3,0}$, $R = -u^3$; $cs_1^{4,0} = \{v \neq u;\ (u, v) \neq (1/9, 1), (1/3, 1), (-1/9, -1)\}$, $R = u^2(u - v)$;

$cs_2^{4,0} = \{v \neq (3u)^{-1}, 1 + 2(9u)^{-1}, (12u + 1 \pm (1 - 8u)^{1/2})(8u)^{-1}\}$, $R = u^3$;

$cs_3^{4,0} = \{v \neq -u, 3/2 \pm (-2u)^{-1/2}\}$, $R = u^2(u + v)$;

$cs_6^{4,0} = \{v \neq u, (9u + 2)u^{-2}/27\}$, $R = u^2(u - v)$;

$cs_{8,\kappa}^{4,0} = \{v \neq u\kappa;\ (\kappa, u, v) \neq (1, -1/3, -3);\ v \neq \kappa(3u - 2)(2u - 1)^{-1} \text{ at } \kappa(1 - 2u) > 0\}$, $R = u\kappa(u\kappa - v)^2$;

$cs_{10}^{4,0} = \{u \neq 1;\ v \neq (2u - 1)^2(2 - u)/27, u^2(3 - 2u)/27;\ (u, v) \neq (2/3, 4/729)\}$, $R = -v$;

$cs_{13}^{4,0} = \{v \neq -u^{-2/3};\ u \neq 2/3 \text{ at } 3((\theta_*^2 - 4v)/2\theta_*)^2 + 12v \leq 0,\ \theta_* : \theta^3 + 3v\theta - 3\}$, $R = u(u^2v^3 + 1)$;

$cs_{16}^{4,0} = \{(u, v) \neq (1/3, -2/3);\ v \neq (1 - 9u \pm (1 - 3u)(1 - 12u)^{1/2})u^{-2}/27\}$, $R = -u^2v$;

$cs_{17}^{4,0} = \{(u, v) \neq (1/3, -6^{1/3});\ v \neq 2^{1/3}(2 - 3u)u^{-1/3}/2\}$, $R = u$;

$cs_{21}^{4,0} = \{v \neq u;\ v \neq u^{-2} \text{ at } u > 0\}$, $R = (u - v)^3$;

$cs_{22}^{4,0} = \{(u, v) \neq (1/3, 1/6);\ v \neq u, (2u - 1)(3u - 2)^{-3}, -16u^2(u - 1)^{-1}(3u + 1)^{-3}\}$, $R = uv(v - u)$;

$cs_{25}^{4,0} = \{(u, v) \neq (1/3, 3^{2/3}), (5/9, 3^{1/3}), ((\theta_*^3 + 1)\theta_*^{-3}/2, (\theta_*^3 - 3)(2\theta_*)^{-1}),\ \theta_* : \theta^9 - \theta^6 + 15\theta^3 + 9\}$, $R = u$;

$cs_{26}^{4,0} = \{(u, v) \neq (9/8, -27/32), (-18, 216)\}$, $R = -v^3$;

$cs_{28}^{4,0} = \{v \neq -u;\ (u, v) \neq (3(2 \pm \sqrt{2})^{2/3}/2, \mp 2^{-1/2}(2 \pm \sqrt{2})^{1/3}), (3 \times 2^{-1/3}(\pm\sqrt{5} - 1)^{1/3}, (28 \pm 12\sqrt{5})^{1/3}/2), (3^{2/3}, -3^{1/3})\}$, $R = -u^3 - v^3$;

$cs_{31}^{4,0} = \{(u, v) \neq (-2^{1/3}/6, -2^{1/3}/3), (2^{1/3}/3, 2^{1/3}/3)\}$, $R = -v^3$;

$cs_{32}^{4,0} = \{(u, v) \neq (3, -1)\}$, $R = u^3 - v^3$;

$cs_{34,\kappa}^{4,0} = \{v \neq u\kappa;\ (u, v) \neq (-\kappa, 1/3), ((\sqrt{5} + 3\kappa)/2, -(3 + \sqrt{5}\kappa)/18), (2\kappa + \sqrt{3}, -(2 + \sqrt{3}\kappa)/3);\ (v, \kappa) \neq (u/9, 1);\ (u, v, \kappa) \neq (1, -1/9, -1);\ v \neq -(2u - \kappa)(u - 2\kappa)/9 \text{ at } u > \kappa/2;\ u \neq \kappa \text{ at } v > 0;\ (u, \kappa) \neq (-1, 1) \text{ at } v \geq -1/9;\ (u, \kappa) \neq (1, -1) \text{ at } v \in (-\infty, -1) \cup (0, +\infty)\}$, $R = -v(u\kappa - v)^2$;

$cs_{35}^{4,0} = \{(u, v) \neq (2 \times (3/5)^{1/3}, 2 \times 75^{-1/3}/3), (3^{1/6}(\sqrt{3} \pm 1)/2, 3^{-1/6}(\pm 3 - \sqrt{3})/6); v \neq (3u)^{-1}\}, R = -v^3;$

$cs_{36}^{4,0} = \{v \neq -u^2; (u, v) \neq ((2/81)^{1/3}, -(3/2)^{1/3}), (-2^{-5/3}, 2^{-4/3}), ((8/3 \pm 2\sqrt{2})^{1/3}/3, (3 \pm 3\sqrt{2})^{1/3}), (2^{2/3}, -2^{1/3}), (-2\theta_*^4, (1 - 3\theta_*^3)\theta_*^{-1}), \theta_* : 54\theta^9 - 18\theta^6 + 1\}, R = -u(u^2 + v);$

$cs_{37}^{4,0} = \{v \neq u; (u, v) \neq (1/9, -1/81); v \neq u^2$ at $u > 0; v \neq -u^2, u(4u \mp (3u - 1)(-u)^{1/2})(9u + 1)^{-1}$ at $u < 0\}, R = v^2(u - v).$

Here, the designation *tcs* (trivial cs) means that this set does not have limitations on parameters.

Statement 2.2. *Only at the indicated values of parameters, the forms* $NSF_i^{m,0}$ *presented below are reduced to previous forms according to the structural principles*:

1) $NSF_{2,\kappa}^{3,0}$ *at* $u = 3/2$, $\kappa = -1$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = \sqrt{2}s_1$ *to* $SF_1^{3,0}$;

2) $NSF_1^{4,0}$: a) *at* $u = 1/9$, $v = 1$ *is reduced by the substitution with* $r_1 = -3r_2$, $s_1 = 0$ *to* $SF_1^{3,0}$; b) *at* $u = 1/3$, $v = 1$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -s_1$ *to* $SF_9^{3,0}$; c) *at* $u = -1/9$, $v = -1$ *is reduced by the substitution with* $r_1 = 3r_2$, $s_1 = -3s_2$ *to* $SF_{23}^{3,0}$;

3) $NSF_2^{4,0}$: a) *at* $v = (12u + 1 \pm (1 - 8u)^{1/2})(8u)^{-1}$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = (-1 \pm (1 - 8u)^{1/2})s_1/2$ *to* $SF_1^{3,0}$; b) *at* $v = 1 + 2(9u)^{-1}$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -3us_1$ *to* $SF_2^{3,0}$; c) *at* $v = (3u)^{-1}$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -3us_1$ *to* $SF_4^{3,0}$;

4) $NSF_3^{4,0}$ *at* $v = 3/2 \pm (-2u)^{-1/2}$ *is reduced by the substitution with* $r_1 = \pm(-2u)^{-1/2}r_2$, $s_2 = 0$ *to* $SF_1^{4,0}$;

5) $NSF_6^{4,0}$ *at* $v = (9u + 2)u^{-2}/27$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = 3us_1$ *to* $SF_1^{4,0}$;

6) $NSF_{8,\kappa}^{4,0}$: a) *at* $\kappa = 1$, $u = -1/3$, $v = -3$ *is reduced by the substitution with* $r_1 = -\sqrt{3}r_2$, $s_1 = \sqrt{3}s_2$ *to* $SF_{10}^{2,0}$; b) *at* $v = \kappa(3u - 2)(2u - 1)^{-1}$, $\kappa(1 - 2u) > 0$ *is reduced by the substitution with* $r_2 = (\kappa(1 - 2u))^{1/2}r_1$, $s_2 = -(\kappa(1 - 2u))^{1/2}s_1$ *to* $SF_1^{4,0}$;

7) $NSF_{10}^{4,0}$: a) *at* $v = (2u - 1)^2(2 - u)/27$ *is reduced by the substitution with* $r_1 = (1 - 2u)r_2/3$, $s_2 = 0$ *to* $SF_1^{4,0}$; b) *at* $v = u^2(3 - 2u)/27$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -3u^{-1}s_1$ *to* $SF_3^{4,0}$; c) *at* $u = 2/3$, $v = 4/729$ *is reduced by the substitution with* $r_2 = -9r_1$, $s_2 = 9s_1/2$ *to* $SF_3^{4,0}$; d) *at* $u = 1$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -3s_1$ *to* $SF_6^{4,0}$;

8) $NSF_{13}^{4,0}$ *at* $u = 2/3$, $3((\theta_*^2 - 4v)/2\theta_*)^2 + 12v \leq 0$, $\theta_*: \theta^3 + 3v\theta - 3$ *is reduced by the substitution with* $r_1 = -(\theta_*^2 - 4v)/4\theta_* + (-3((\theta_*^2 - 4v)/2\theta_*)^2 - 12v)^{1/2}/2$, $s_1 = \theta_* s_2$ *to* $SF_1^{4,0}$;

9) $NSF_{16}^{4,0}$: a) *at* $u = 1/3$, $v = -2/3$ *is reduced by the substitution with* $r_1 = 2r_2$, $s_1 = -s_2$ *to* $SF_{15}^{3,0}$; b) *at* $u \leq 1/12$, $v = (1 - 9u \pm (1 - 3u)(1 - 12u)^{1/2})u^{-2}/27$ *is reduced by the substitution with* $r_1 = (1 \pm (1 - 12u)^{1/2})(6u)^{-1}r_2$, $s_2 = 0$ *to* $SF_1^{4,0}$;

10) $NSF_{17}^{4,0}$: a) *at* $u = 1/3$, $v = -6^{1/3}$ *is reduced by the substitution with* $r_1 = -(60 + 36\sqrt{3})^{1/3}r_2/2$, $s_1 = ((9\sqrt{3} - 15)/2)^{1/3}s_2$ to $SF_8^{4,0}$; b) *at* $v = 2^{1/3}(2 - 3u)u^{-1/3}/2$ *is reduced by the substitution with* $r_2 = (u/2)^{1/3}r_1$, $s_2 = -(4u)^{1/3}s_1$ *to* $SF_{16}^{4,0}$;

11) $NSF_{21}^{4,0}$ *at* $u > 0$, $v = u^{-2}$ *is reduced by the substitution with* $r_2 = -u^{1/2}r_1$, $s_2 = u^{1/2}s_2$ *to* $SF_8^{4,0}$;

12) $NSF_{22}^{4,0}$: a) *at* $u = 1/3$, $v = 1/6$ *is reduced by the substitution with* $r_2 = 0$, $s_2 = -2s_1$ *to* $SF_8^{4,0}$; b) *at* $v = (2u - 1)(3u - 2)^{-3}$ *is reduced by the substitution with* $r_2 = (3u - 2)r_1$, $s_2 = u(3u - 2)(1 - 2u)^{-1}s_1$ *to* $SF_{16}^{4,0}$; c) *at* $v = -16u^2(u - 1)^{-1}(3u + 1)^{-3}$ *is reduced by the substitution with* $r_2 = (u - 1)(3u + 1)(4u)^{-1}r_1$, $s_2 = -(3u + 1)s_1/2$ *to* $SF_{17}^{4,0}$;

13) $NSF_{25}^{4,0}$: a) *at* $u = 1/3$, $v = 3^{2/3}$ *is reduced by the substitution with* $r_1 = -3^{1/3}r_2$, $s_2 = 0$ *to* $SF_2^{4,0}$; b) *at* $u = 5/9$, $v = 3^{1/3}$, *is reduced by the substitution with* $r_1 = 3^{2/3}r_2$, $s_1 = -3^{2/3}s_2/2$ *to* $SF_3^{4,0}$; c) *at* $u = (\theta_*^3 + 1)\theta_*^{-3}/2$, $v = (\theta_*^3 - 3)(2\theta_*)^{-1}$, $\theta_*: \theta^9 - \theta^6 + 15\theta^3 + 9$ *is reduced by the substitution with* $r_1 = \theta_* r_2$, $s_2 = -(\theta_*^3 + 1)(2\theta_*)^{-1}s_1$ *to* $SF_{10}^{4,0}$;

14) $NSF_{26}^{4,0}$: a) *at* $u = 9/8$, $v = -27/32$ *is reduced by the substitution with* $r_1 = 3r_2/2$, $s_1 = -3s_2/4$ *to* $SF_{10}^{4,0}$; b) *at* $u = -18$, $v = 216$ *is reduced by the substitution with* $r_1 = 3r_2$, $s_1 = -6s_2$ *to* $SF_{25}^{4,0}$;

15) $NSF_{28}^{4,0}$: a) *at* $u = 3(2 \pm \sqrt{2})^{2/3}/2$, $v = \mp 2^{-1/2}(2 \pm \sqrt{2})^{1/3}$ *is reduced by the substitution with* $r_1 = (2 \pm \sqrt{2})^{1/3}r_2$, $s_2 = -(2 \pm \sqrt{2})^{2/3}s_1$ *to* $SF_3^{4,0}$; b) *at* $u = 3 \times 2^{-1/3}(\pm\sqrt{5} - 1)^{1/3}$, $v = (28 \pm 12\sqrt{5})^{1/3}/2$ *is reduced by the substitution with* $r_1 = -(28 \pm 12\sqrt{5})^{1/3}r_2/2$, $s_1 = 2^{-1/3}(3 \mp \sqrt{5})^{1/3}s_2$ *to* $SF_{13}^{4,0}$; c) *at* $u = 3^{2/3}$, $v = -3^{1/3}$ *is reduced by the substitution with* $r_1 = -3^{1/3}r_2$, $s_1 = 0$ *to* $SF_{25}^{4,0}$;

16) $NSF_{31}^{4,0}$: a) *at* $u = -2^{1/3}/6$, $v = -2^{1/3}/3$ *is reduced by the substitution with* $r_2 = 2^{2/3}r_1$, $s_1 = -2^{1/3}s_2$ *to* $SF_{10}^{4,0}$; b) *at* u, $v = 2^{1/3}/3$ *is reduced by the substitution with* $r_1 = 2^{1/3}r_2$, $s_2 = -2^{2/3}s_1$ *to* $SF_{25}^{4,0}$;

17) $NSF_{32}^{4,0}$ *at* $u = 3$, $v = -1$ *is reduced by the substitution with* $r_1 = r_2$, $s_1 = -s_2$ *to* $SF_6^{4,0}$;

18) $NSF_{34,\kappa}^{4,0}$: a) *at* $u = -\kappa$, $v = 1/3$ *is reduced by the substitution with* $r_1 = (6\sqrt{3} - 9\kappa)^{1/2}r_2/3$, $s_1 = -(6\sqrt{3} - 9\kappa)^{1/2}(2 + \sqrt{3}\kappa)s_2/3$ *to* $SF_{15}^{3,0}$; b) *at* $u > \kappa/2$, $v = -(2u - \kappa)(u - 2\kappa)/9$ *is reduced by the substitution with* $r_1 = (6u - 3\kappa)^{1/2}r_2/3$, $s_1 = -(6u - 3\kappa)^{1/2}s_2/3$ *to* $SF_1^{4,0}$; c) *at* $u = (\sqrt{5} + 3\kappa)/2$, $v = -(3 + \sqrt{5}\kappa)/18$ *is reduced by the substitution with* $r_1 = (6\sqrt{5} - 6\kappa)^{1/2}(3 + \sqrt{5}\kappa)r_2/12$, $s_1 = -(6\sqrt{5} - 6\kappa)^{1/2}s_2/6$ *to* $SF_3^{4,0}$; d) *at* $u = \kappa$, $v > 0$ *is reduced by the substitution with* $r_1 = v^{1/4}r_2$, $s_1 = -v^{1/4}s_2$ *to* $SF_8^{4,0}$; e) *at* $u = 1$, $v = -1/9$, $\kappa = -1$ *is reduced by the substitution with* $s_1 = \sqrt{3}(1 + \sqrt{2})s_2/3$, $r_2 = -\sqrt{3}(1 + \sqrt{2})r_1$ *to* $SF_8^{4,0}$; f) *at* $u = 2\kappa + \sqrt{3}$, $v = -(2 + \sqrt{3}\kappa)/3$ *is reduced by the substitution with* $r_1 = (6\sqrt{3} + 9\kappa)^{1/2}r_2/3$, $s_2 = -(2\sqrt{3} + 3\kappa)^{1/2}s_1$ *to* $SF_{16}^{4,0}$; g) *at* $v = u/9$, $\kappa = 1$ *is reduced by the substitution with* $r_2 = \sqrt{3}r_1$, $s_2 = -\sqrt{3}s_1$ *to* $SF_{21}^{4,0}$; h) *at* $u = -1$, $v \geq -1/9$, $\kappa = 1$ *is reduced by the substitution with* $r_1 = (6v + 2 + 2\ell)^{1/2}r_2/2$, $s_2 = -(6v + 2 + 2\ell)^{1/2}(3v - 1 + \ell)(8v)^{-1}s_1$ *to* $SF_{32}^{4,0}$, $\ell = ((v + 1)(9v + 1))^{1/2}$; i) *at* $u = 1$, $v \in (-\infty, -1) \cup (0, +\infty)$, $\kappa = -1$ *is reduced by the substitution with* $r_1 = (-6v - 2 + 2\ell)^{1/2}r_2/2$, $s_2 = -(-6v - 2 + 2\ell)^{1/2}(-3v + 1 + \ell)(8v)^{-1}s_1$ *to* $SF_{32}^{4,0}$, $\ell = ((v + 1)(9v + 1))^{1/2}$;

19) $NSF_{35}^{4,0}$: a) *at* $u = 2 \times (3/5)^{1/3}$, $v = 2 \times 75^{-1/3}/3$ *is reduced by the substitution with* $r_1 = -15^{-1/3}(5 + 2\sqrt{5})^{1/3}r_2$, $s_1 = 15^{-1/3}(2\sqrt{5} - 5)^{1/3}s_2$ *to* $SF_1^{4,0}$; b) *at* $u = 3^{1/6}(\sqrt{3} \pm 1)/2$, $v = 3^{-1/6}(\pm 3 - \sqrt{3})/6$ *is reduced by the substitution with* $r_1 = \mp 3^{-1/6}r_2$, $s_1 = -3^{1/3}(3 \mp \sqrt{3})s_2/6$ *to* $SF_{10}^{4,0}$; c) *at* $v = (3u)^{-1}$ *is reduced by the substitution with* $s_1 = 0$, $r_2 = -ur_1$ *to* $SF_{21}^{4,0}$;

20) $NSF_{36}^{4,0}$: a) *at* $u = -2\theta_*^4$, $v = (1 - 3\theta_*^3)\theta_*^{-1}$ *is reduced by the substitution with* $r_1 = \theta_* r_2$, $s_1 = 2\theta_*(3\theta_*^3 - 1)s_2$, $\theta_*: 54\theta^9 - 18\theta^6 + 1$ *to* $SF_3^{4,0}$; b) *at* $u = (2/81)^{1/3}$, $v = -(3/2)^{1/3}$ *is reduced by the substitution with* $r_1 = 3^{-1/3}(4 \pm 2\sqrt{5})^{1/3}r_2$, $s_1 = 3^{-1/3}(4 \mp 2\sqrt{5})^{1/3}s_2$ *to* $SF_8^{4,0}$; c) *at* $u = -2^{-5/3}$, $v = 2^{-4/3}$ *is reduced by the substitution with* $s_1 = -2^{1/3}s_2$, $r_2 = 2^{2/3}r_1$ *to* $SF_{10}^{4,0}$; d) *at* $u = (8/3 \pm 2\sqrt{2})^{1/3}/3$, $v = (3 \pm 3\sqrt{2})^{1/3}$ *is reduced by the substitution with* $r_1 = (36 \mp 18\sqrt{2})^{1/3}r_2/3$, $s_1 = (-1 \mp 2\sqrt{2}/3)^{1/3}s_2$ *to* $SF_{13}^{4,0}$; e) *at* $u = 2^{2/3}$, $v = -2^{1/3}$ *is reduced by the substitution with* $r_1 = 2^{2/3}r_2$, $s_2 = -2^{-1/3}s_1$ *to* $SF_{25}^{4,0}$;

21) $NSF_{37}^{4,0}$: a) *at* $u = 1/9$, $v = -1/81$ *is reduced by the substitution with* $r_1 = (-1 + \sqrt{5})r_2/6$, $s_1 = (-1 - \sqrt{5})s_2/6$ *to* $SF_1^{4,0}$; b) *at* $v = u^2$, $u > 0$ *is reduced by the substitution* $r_1 = -u^{1/2}r_2$, $s_1 = u^{1/2}s_2$ *to* $SF_8^{4,0}$; c) *at* $v = u(4u \mp (3u - 1)(-u)^{1/2})(9u + 1)^{-1}$, $u < 0$ *is reduced by the substitution* $r_1 = \pm(-u)^{1/2}r_2$, $s_1 = 2u(\pm(-u)^{1/2} - 3u)^{-1}s_2$ *to* $SF_{28}^{4,0}$; d) *at* $v = -u^2$, $u < 0$ *is reduced by the substitution with* $r_1 = (-u)^{1/2}r_2$, $s_1 = -(-u)^{1/2}s_2$ *to* $SF_{34}^{4,0}$.

The proof is presented in the file statement.mw of the archive (see Introduction).

Now let us find the linear substitutions (1.2) that, preserving the investigated form $CF_i^{m,0}$ ($m = 2, 3, 4$), change in it the values of parameters in order to maximally limit their possible values.

Statement 2.3. *There are only the following cases in which it is possible to change in* $CF_i^{m,0}$ *the value of* σ *from* -1 *to* $+1$ *by means of substitution* (1.2) *with preserving the values of other parameters*:

$CF_{1,\kappa}^{2,0}$ *with* $\kappa = -1$, *the substitution with* $r_1, s_2 = 0, s_1, r_2 = 1$;

$CF_{10,\kappa}^{2,0}$, *the substitution with* $-r_1, s_2 = -1, s_1, r_2 = 0$;

$CF_{15}^{3,0}$ *with* $u = -1/3$, *the substitution with* $r_1, -s_2 = 3^{-1/2}, s_1 = 2^{1/3}3^{-1/6}, r_2 = 2^{2/3}3^{-5/6}$;

$CF_{23,\kappa}^{3,0}$, *the substitution with* $r_1, -s_2 = 1, s_1, r_2 = 0$;

$CF_3^{4,0}$ *with* $u = -2/9$, $v = 3$, *the substitution with* $-r_1, s_2 = 5^{-1/2}, s_1 = -6 \times 5^{-1/2}, r_2 = -2 \times 5^{-1/2}/3$;

$CF_{34,\kappa}^{4,0}$ *with* $u = -\kappa$, $v = -1$, *the substitution with* $r_1, s_2 = 0, s_1, r_2 = 1$;

$CF_{35}^{4,0}$ *with* $u = -(3/2)^{1/3}$, $v = 2^{1/3}3^{-4/3}$, *the substitution with* $r_1, s_2 = 1, s_1 = -(2/3)^{1/3}, r_2 = 12^{1/3}$.

Statement 2.4. *In four forms* $CF_i^{m,0}$, *it is possible to change the values of parameters* u, v *by using substitution* (1.2):

1) $CF_1^{4,0}$ *with* $\sigma = \sigma_*$, $u = u_*$, $v = v_*$, *by the substitution with* $r_1, s_2 = 0, s_1 = |u_*|^{1/2}u_*^{-1}, s_2 = |u_*|^{1/2}v_*^{-1}$, *is reduced to itself with* $\sigma = \sigma_* \operatorname{sign} u_*$, $u = u_* v_*^{-2}$, $v = v_*^{-1}$; *in particular, at* $|v_*| > 1$, *it is possible to obtain* $|v| < 1$;

2) $CF_{8,\kappa}^{4,0}$ *with* $\kappa = \kappa_*$, $\sigma = \sigma_*$, $u = u_*$, $v = v_*$, *by the substitution with* $r_1, s_2 = 0, s_1 = |u_*|^{-1/2}, r_2 = |v_*|^{-1/2}$, *is reduced to itself with* $\kappa = \operatorname{sign}(u_* v_*)$, $\sigma = \sigma_* \operatorname{sign} v_*$, $u = \kappa_* v_*^{-1}$, $v = \kappa_* u_*^{-1}$; *in particular, at* $|u_*|, |v_*| > 1$, *it is possible to obtain* $|u|, |v| < 1$;

3) $CF_{21}^{4,0}$ *with* $\sigma = \sigma_*$, $u = u_*$, $v = v_*$, *by the substitution with* $r_1, s_2 = 0, s_1 = |u_*|^{-1/2}, r_2 = u_*|u_*|^{-7/6}|v_*|^{-1/3}$ *is reduced to itself with* $\sigma = \sigma_* \operatorname{sign} u_*$, $u = u_*^{-1/3}v_*^{-2/3}$, $v = v_*^{1/3}u_*^{-4/3}$; *in particular, at* $|u_*| > 1$, $1 < |v_*| < u_*^4$, *it is possible to obtain* $|u|, |v| < 1$;

4) $CF_{34,\kappa}^{4,0}$ *with* $u = u_*$, $v = v_*$, $\kappa = \kappa_*$ *by the substitution with* $r_1, s_2 = 0, s_1 = v|uv|^{-3/4}, r_2 = |uv|^{-1/4}$ *is reduced to itself with* $\kappa = \operatorname{sign}(u_* v_*)$, $u = u_*^{-1}\kappa\kappa_*$, $v = v_* u_*^{-2}$ *and the same* σ; *in particular, at* $|u_*| > 1$ *it is possible to obtain* $|u| < 1$.

3. REDUCTION OF THE INITIAL SYSTEM TO EACH OF $CF^{m,0}$ AT $m = 2, 3$

Collection 3.1. Constants and linear nonsingular substitutions used below are as follows:

$\psi_1 = 3\tilde{a}_1^2 + 4\tilde{b}_1$, $\psi_2 = 27\tilde{d}_2^2 - 4\tilde{b}_1^3$, $\psi_3 = 3\tilde{a}_1 + (36\tilde{c}_1)^{1/3}$, $\psi_4 = \tilde{d}_1 + 2^{2/3}\tilde{d}_2^{4/3}$, $\psi_5 = (-3\tilde{c}_1\tilde{a}_1^{-1})^{1/2}$;

$L_1 = \{r_1 = 0, s_1 = 3^{3/2}d_1, r_2 = 3^{1/2}, s_2 = -3^{1/2}c_1\}$, $L_2 = \{r_1 = 3^{1/2}, s_1 = -3^{1/2}b_2, r_2 = 0, s_2 = 3^{3/2}a_2\}$, $L_3 = \{r_1 = \zeta, s_1 = (-b_2\zeta^2 + (b_1 - 2c_2)\zeta + 2c_1 - 3d_2)\zeta/3, r_2 = 1, s_2 = ((2b_2 - 3a_1)\zeta^2 + (c_2 - 2b_1)\zeta - c_1)/3\}$, where ζ is any real nonzero number that is not a zero of polynomial $Q = (a_1 - b_2)\zeta^2 + (b_1 - c_2)\zeta + c_1 - d_2$, since $\det L_3 = -\zeta Q(\zeta)$;

$L_{1;1}^{2,0} = \{r_1 = (3\tilde{a}_1 - (3\psi_1)^{1/2})r_2/6, s_1 = (3\tilde{a}_1 + (3\psi_1)^{1/2})s_2/6, r_2 = 3\sqrt{2}\psi_1^{-1/2}|3\tilde{a}_1 - (3\psi_1)^{1/2}|^{-1/2}, s_2 = 3\sqrt{2}\psi_1^{-1/2}|3\tilde{a}_1 + (3\psi_1)^{1/2}|^{-1/2}\}$,

$L_{10;1}^{2,0} = \{r_1 = |\tilde{d}_1|^{-1/8}, s_1, r_2 = 0, s_2 = r_1^3\}$, $L_{10;2}^{2,0} = \{r_1 = (9\tilde{d}_2 + (3\psi_2)^{1/2})(6\tilde{b}_1)^{-1}r_2, s_1 = (9\tilde{d}_2 - (3\psi_2)^{1/2})(6\tilde{b}_1)^{-1}s_2, s_2 = \psi_2(9\tilde{d}_2 + (3\psi_2)^{1/2})\tilde{b}_1^{-3}r_2^3/18, r_2 = 2^{1/4}\psi_2^{-1/2}|9\tilde{d}_2 + (3\psi_2)^{1/2}|^{-1/4}|3\tilde{b}_1|^{7/8}\}$,

$L_{1;1}^{3,0} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = (\tilde{a}_1 - \eta_*)(\eta_* - \theta_*)^{-1}(\eta_* + \theta_* - \tilde{a}_1)^{-1}|\tilde{a}_1 - \eta_*|^{-1/2}/3, s_2 = (\eta_* - \theta_*)^{-1}|\tilde{a}_1 - \eta_*|^{-1/2}\}$,

$L_{2;1}^{3,0} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = \sqrt{2}(\eta_* - \theta_*)^{-1}|\tilde{a}_1 - 3\eta_* - \theta_*|^{-1/2}, s_2 = (\eta_* - \theta_*)^{-1}|\tilde{a}_1 - \eta_*|^{-1/2}\}$,

$L^{3,0}_{4;1} = \{r_1, s_2 = 0, s_1 = |\tilde a_1|^{-1/2}, r_2 = \tilde a_1^{-1}|\tilde a_1|^{-1/2}\}$, $L^{3,0}_{4;2} = \{r_1 = -3\tilde d_1 r_2 \tilde c_1^{-1}, s_1 = 0, r_2 = -s_2, s_2 = \sqrt{3}|\tilde c_1|^{-1/2}\}$, $L^{3,0}_{4;3} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = (\eta_* + \theta_* - \tilde a_1)(\tilde a_1 - \eta_*)^{-1} s_2, s_2 = (\eta_* - \theta_*)^{-1}|\tilde a_1 - \eta_*|^{-1/2}\}$,

$L^{3,0}_{9;1} = \{r_1, s_2 = 0, s_1 = \tilde d_2 |\tilde d_2|^{-1/2} \tilde b_1^{-1}, r_2 = |\tilde d_2|^{-1/2}\}$, $L^{3,0}_{9;2} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = \sqrt{3}(\eta_* - \theta_*)|\tilde a_1 + 2\eta_*|^{-1/2}$, $s_2 = -(\tilde a_1 + 2\eta_*)(\tilde a_1 - \eta_*)^{-1} r_2/3\}$, $L^{3,0}_{9;3} = \{r_1 = -\theta_* r_2/2, s_1 = \theta_* s_2, r_2 = 6^{1/3}\tilde c_1^{-1/3}|\psi_3|^{-1/2}, s_2 = 2\psi_3(\psi_3 - 9\tilde a_1)^{-1}/3\}$,

$L^{3,0}_{15;1} = \{r_1, s_2 = 0, s_1 = \tilde c_1^{1/3}|\tilde c_1|^{-1/2}, r_2 = |\tilde c_1|^{-1/2}\}$, $L^{3,0}_{15;2} = \{r_1 = 0, s_1 = -\tilde c_1 \tilde c_2^{-1} s_2, r_2 = |\tilde c_1|^{-1/2}, s_2 = -3^{1/3}\tilde c_2(3\tilde c_1^2 + \tilde c_2^3)^{-1/3}|\tilde c_1|^{-1/2}\}$, $L^{3,0}_{15;3} = \{r_1 = |\tilde c_1|^{1/2}\tilde c_2^{-1}, s_1 = 0, r_2 = \tilde c_2 \tilde c_1^{-1} r_1, s_2 = -3^{1/3}|\tilde c_1|^{1/2}\tilde c_1^{-1}\}$, $L^{3,0}_{15;4} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = |\tilde a_1 - \eta_*|^{-1/2}(\eta_* - \theta_*)^{-1}, s_2 = (3\tilde a_1 - 3\eta_*)^{1/3}(\eta_* + 3\theta_* - \tilde a_1)^{-1/3} r_2\}$, $L^{3,0}_{15;5} = \{r_1 = \pm(-\tilde c_2/3)^{1/2} r_2, s_1 = \pm 3^{1/2}(\tilde b_1 + \tilde c_2)(-\tilde c_2)^{-1/2} s_2/6, r_2 = 2(-3\tilde c_2)^{1/4}(\tilde b_1 + 3\tilde c_2)^{-1}, s_2 = -2^{4/3}(-3\tilde c_2)^{7/12}(\tilde b_1 - \tilde c_2)^{-1/3}(\tilde b_1 + 3\tilde c_2)\}$, $L^{3,0}_{15;6} = \{r_1 = -(4\tilde d_2)^{1/3} r_2, s_1 = (\tilde d_2/2)^{1/3} s_2, r_2 = 2^{1/3}\tilde d_2^{-1/3}|\tilde a_1 + (4\tilde d_2)^{1/3}|^{-1/2}/3, s_2 = 6^{1/3}(\tilde a_1 + (4\tilde d_2)^{1/3})^{1/3}((4\tilde d_2)^{1/3} - 2\tilde a_1)^{-1/3} r_2\}$,

$L^{3,0}_{20;1} = \{r_1 = \eta_* s_2, s_1 = \theta_* s_2, r_2 = |\tilde a_1|^{-1/2}(\eta_* - \theta_*)^{-1}, s_2 = (\eta_* - \tilde a_1)\tilde a_1^{-1} r_2\}$, $L^{3,0}_{20;2} = \{r_1 = |\tilde a_1|^{-1/2}, s_1, r_2 = 0, s_2 = |\tilde a_1|^{-1/2}\tilde a_1^{-1}\}$, $L^{3,0}_{20;3} = \{r_1, s_2 = 0, s_1 = \tilde d_1 |\tilde d_2|^{-1/2}\tilde d_2^{-1}, r_2 = |\tilde d_2|^{-1/2}\}$, $L^{3,0}_{20;4} = \{r_1 = (4\tilde d_2)^{1/3} r_2/2, r_2 = 2^{1/3}|\tilde a_1|^{-1/2}\tilde d_2^{-1/3}/3, s_1 = -(4\tilde d_2)^{1/3} s_2, s_2 = (1 - 2^{1/3}\tilde a_1 \tilde d_2^{-1/3})\tilde a_1^{-1}|\tilde a_1|^{-1/2}/3\}$, $L^{3,0}_{20;5} = \{r_1 = -\theta_* r_2/2, s_1 = \theta_* s_2, r_2 = 36|(\tilde b_1\theta_*^2 - 4\tilde d_1)\theta_*|^{1/2}, s_2 = 972\tilde b_1(\tilde b_1\theta_*^2 - 4\tilde d_1)\theta_*^2 r_2\}$, $L^{3,0}_{20;6} = \{r_1 = 0, s_1 = -\tilde d_1^{1/3}\tilde a_1^{-1/3}, r_2 = \tilde d_1^{-1/3}|\tilde a_1|^{-1/6}, s_2 = -r_2\}$,

$L^{3,0}_{23;1} = \{r_1, s_2 = 0, s_1 = \tilde d_1|\tilde d_1|^{-9/8}, r_2 = |\tilde d_1|^{-3/8}\}$, $L^{3,0}_{23;2} = \{r_1 = |\tilde d_1|^{-1/8}, s_1, r_2 = 0, s_2 = |\tilde d_1|^{-3/8}\}$, $L^{3,0}_{23;3} = \{r_1 = 2^{5/8}|\tilde d_2|^{1/2}|\psi_4|^{-1/2}/3, s_1 = (\tilde d_2/2)^{1/3} s_2, r_2 = -r_1 s_2(2s_1)^{-1}, s_2 = -2^{5/24}|\tilde d_2|^{3/2}\psi_4\tilde d_2^{-4/3}|\psi_4|^{-3/2}/3\}$, $L^{3,0}_{23;4} = \{r_1 = 0, s_1 = \tilde d_1 r_2^3, r_2 = 2^{1/8}|2\tilde d_1^3 - \tilde d_2^4|^{1/8}, s_2 = \tilde d_2\tilde d_1^{-1} s_1\}$, $L^{3,0}_{23;5} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, r_2 = 2^{1/8}(\eta_* - \theta_*)^{-1}|\tilde a_1 + 2\theta_*|^{-1/8}|\tilde a_1 - \eta_*|^{-3/8}, s_2 = (\eta_* - \tilde a_1)(\eta_* - \theta_*)^2 r_2^3\}$, $L^{3,0}_{23;6} = \{r_1 = -2(\tilde c_1/18)^{1/3} r_2, r_2 = \sqrt{6}|\tilde c_1|^{-1/4}|(9\tilde a_1 + 18^{2/3}\tilde c_1^{1/3})^3(18^{1/3}\tilde a_1\tilde c_1^{2/3} - 2\tilde c_1)|^{-1/8}, s_1 = -(\tilde c_1/18)^{1/3} s_2, s_2 = -\tilde c_1^{2/3}(9 \times 18^{-2/3}\tilde a_1 + \tilde c_1) r_2^3/9\}$, $L^{3,0}_{23;7} = \{r_1 = \mp(-\tilde d_1)^{1/4} r_2, s_1 = \pm(-\tilde d_1)^{1/4} s_2, s_2 = 4(\pm\tilde d_1 + (-\tilde d_1)^{1/4}\tilde d_2)(-\tilde d_1)^{-1/4} r_2^3, r_2 = (-2\tilde d_1)^{1/8}|2\tilde d_1 \pm (-\tilde d_1)^{1/4}\tilde d_2|^{-1/8}|\tilde d_1 \pm (-\tilde d_1)^{1/4}\tilde d_2|^{-3/8}/2\}$, $L^{3,0}_{23;8} = \{r_1 = -(4\tilde d_2)^{1/3} r_2, s_1 = (\tilde d_2/2)^{1/3} s_2, r_2 = 2^{11/24}\tilde d_2^{-1/3}|\tilde a_1 + (4\tilde d_2)^{1/3}|^{-1/2}/3, s_2 = -9 \times 2^{-2/3}\tilde d_2^{2/3}(\tilde a_1 + (4\tilde d_2)^{1/3}) r_2^3\}$, $L^{3,0}_{23;9} = \{r_1 = \pm(-\tilde c_2/3)^{1/2} r_2, s_1 = \mp\tilde b_1(-3\tilde c_2)^{-1/2} s_2, r_2 = 3^{3/4}|\tilde c_2|^{3/8}|\tilde b_1 + \tilde c_2|^{-1/8}(\tilde b_1 - \tilde c_2)^{-1}, s_2 = \mp(\tilde b_1 - \tilde c_2)^2(-3\tilde c_2)^{-1/2} r_2^3/3\}$, $L^{3,0}_{23;10} = \{r_1 = \pm(-3\tilde d_1\tilde c_2)^{1/2}\tilde c_2^{-1} r_2, s_1 = 0, r_2 = 3^{1/4}|\tilde c_2|^{1/2}|\tilde d_1|^{-1/4}|2\tilde c_2^2 - 9\tilde d_1|^{-3/8}, s_2 = \mp\tilde d_1(2\tilde c_2^2 - 9\tilde d_1)(-3\tilde d_1\tilde c_2)^{-1/2}\tilde c_2^{-1} r_2^3\}$,

$L^{3,0}_{24;1} = \{r_1 = 0, s_1 = \tilde d_1\tilde d_2^{-1} s_2, r_2 = -3^{-1/6}|\tilde d_2|^{-1/2}, s_2 = |3\tilde d_2|^{-1/2}\}$, $L^{3,0}_{24;2} = \{r_1 = \tilde c_1^{1/3}|\tilde c_1|^{-1/2}, s_1, r_2 = 0, s_2 = |\tilde c_1|^{-1/2}\}$, $L^{3,0}_{24;3} = \{r_1 = \tilde a_1 r_2, s_1 = |3\tilde a_1|^{-1/2}, r_2 = 3^{-1/6}\tilde a_1^{1/3}|\tilde a_1|^{-1/2}(\tilde d_1 - 2\tilde a_1^4)^{-1/3}, s_2 = 0\}$, $L^{3,0}_{24;4} = \{r_2 = 2(4\tilde d_2)^{-1/3} r_1, r_1 = (2/3)^{7/6}\tilde d_2^{1/3}|\tilde d_2|^{1/6}(2^{1/3}\tilde d_1 + 3\tilde d_2^{4/3})^{1/3}|2^{1/3}\tilde d_1 + 3\tilde d_2^{4/3}|^{-1/2}(\tilde d_2^{4/3} - 2^{1/3}\tilde d_1)^{-1/3}, s_1 = -(4\tilde d_2)^{1/3} s_2, s_2 = (2/3)^{3/2}|\tilde d_2|^{1/6}|2^{1/3}\tilde d_1 + 3\tilde d_2^{4/3}|^{-1/2}\}$, $L^{3,0}_{24;5} = \{r_1 = \eta_* r_2, s_1 = \theta_* s_2, s_2 = |3\tilde a_1|^{-1/2}(\eta_* - \theta_*)^{-1}, r_2 = 3^{-1/6}\tilde a_1^{1/3}|\tilde a_1|^{-1/2}(\eta_* - \theta_*)^{-1}(\eta_* - \tilde a_1)^{-1/3}\}$, $L^{3,0}_{24;6} = \{r_1 = \mp(-\tilde c_1(3\tilde a_1)^{-1})^{1/2} r_2, s_1 = \pm(-\tilde c_1(3\tilde a_1)^{-1})^{1/2} s_2, s_2 = |\tilde c_1|^{-1/2}/2, r_2 = \mp 3^{2/3}\tilde a_1^{1/3}|\tilde c_1|^{-1/2}(\psi_5 \pm 3\tilde a_1)^{-1/3}/2\}$.

Let us divide the set of systems (1.1) with $R \neq 0$ into three non-overlapping classes:

1] $d_1 \neq 0$; 2] $d_1 = 0$, $a_2 \neq 0$; 3] $d_1 = a_2 = 0$.

Lemma 3.1. *Any system* (1.1) *from the class k*] *is reduced by the substitution* L_k *to the system*

$$\tilde A = \begin{pmatrix} \tilde a_1 & \tilde b_1 & \tilde c_1 & \tilde d_1 \\ 1 & 0 & \tilde c_2 & \tilde d_2 \end{pmatrix}, \tag{3.1}$$

in which

in 1]: $\tilde{b}_1 = -9c_1d_2 + 9c_2d_1$, $\tilde{c}_1 = 9b_1c_1d_1 + 27b_2d_1^2 - 3c_1^3 + 9c_1^2d_2 - 18c_1c_2d_1$,

$\tilde{d}_1 = c_1(27a_1d_1^2 - 9b_1c_1d_1 - 27b_2d_1^2 + 2c_1^3 - 3c_1^2d_2 + 9c_1c_2d_1) + 81a_2d_1^3$,

$\tilde{a}_1 = c_1 + 3d_2$, $\tilde{c}_2 = 9b_1d_1 - 3c_1^2$, $\tilde{d}_2 = 27a_1d_1^2 - 9b_1c_1d_1 + 2c_1^3$ *at* $d_1 \neq 0$;

in 2]: $\tilde{b}_1 = -9a_1b_2 + 9b_1a_2$, $\tilde{c}_1 = 9a_1b_2^2 - 18b_1a_2b_2 + 27c_1a_2^2 + 9a_2b_2c_2 - 3b_2^3$,

$\tilde{a}_1 = 3a_1 + b_2$, $\tilde{d}_1 = b_2(27a_2^2d_2 - 3a_1b_2^2 + 9b_1a_2b_2 - 27c_1a_2^2 - 9a_2b_2c_2 + 2b_2^3)$,

$\tilde{c}_2 = 9a_2c_2 - 3b_2^2$, $\tilde{d}_2 = 27a_2^2d_2 - 9a_2b_2c_2 + 2b_2^3$ *at* $d_1 = 0$, $a_2 \neq 0$;

in 3]: $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{d}_1, \tilde{c}_2, \tilde{d}_2$ *at* $d_1 = a_2 = 0$ *are presented in the file named lemma.mw of the archive*.

The proof of Lemma 3.1 is also presented in the file lemma.mw of the archive.

Theorem 3.1. *For any* $CF_i^{m,0}$ (m = 2, 3) *from list* 2.1, *the following is indicated*: a) *the conditions on the coefficients of system* (3.1), b) *the substitution* $L_i^{m,0}$ *that transforms* (3.1) *under the indicated conditions to* $CF_i^{m,0}$, c) *the values obtained of parameters from* $cs_i^{m,0}$:

$CF_{1,\kappa}^{2,0}$: a) $\tilde{c}_1 = 0$, $\tilde{d}_1 = \tilde{b}_1^2/9$, $\tilde{c}_2 = \tilde{b}_1$, $\tilde{d}_2 = -\tilde{a}_1\tilde{b}_1/3$, $\psi_1 > 0$, b) $L_1^{2;1}$, c) $\kappa = -\operatorname{sign}\tilde{b}_1$, $\sigma = \operatorname{sign}(3\tilde{a}_1 + (3\psi_1)^{1/2})$;

$CF_{10,\kappa}^{2,0}$: 1) a) $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{c}_1 = 0$, $\tilde{c}_2 = 0$, $\tilde{d}_2 = 0$, b) $L_{10;1}^{2,0}$, c) $\kappa = \operatorname{sign}\tilde{d}_1$, $\sigma = 1$;

2) a) $\tilde{a}_1 = 0$, $\tilde{c}_1 = -3\tilde{d}_2$, $\tilde{d}_1 = (27\tilde{d}_2^2 - \tilde{b}_1^3)(9\tilde{b}_1)^{-1}$, $\tilde{c}_2 = -\tilde{b}_1$, $\psi_2 > 0$, b) $L_{10;2}^{2,0}$, c) $\sigma = 1$, $\kappa = \operatorname{sign}\tilde{b}_1$;

$CF_1^{3,0}$: a) $\psi_1 \geq 0$, $3\eta_*^2(\tilde{a}_1 - \eta_*)^2 - 4\tilde{c}_1\eta_* \geq 0$, $\tilde{d}_1 = \theta_*\eta_*(2\theta_*\eta_* + \eta_*\tilde{a}_1 + 2\theta_*^2 - \eta_*^2 - 2\theta_*\tilde{a}_1)$, $\tilde{c}_2 = 3(\tilde{a}_1 - \eta_*)(\theta_* - \eta_*) - 3\theta_*^2$, $\tilde{d}_2 = 2\theta_*^3 + (\tilde{a}_1 - \eta_*)(\eta_*^2 + \eta_*\theta_* - 2\theta_*^2)$, *where* θ_*: $3\eta_*\theta^2 - 3\eta_*(\tilde{a}_1 - \eta_*)\theta + \tilde{c}_1 = 0$, $\eta_* = (3\tilde{a}_1 \pm (3\psi_1)^{1/2})/6$, b) $L_{1;1}^{3,0}$, c) $\sigma = \operatorname{sign}(\tilde{a}_1 - \eta_*)$, $u = (3\eta_* + 2\theta_* - 2\tilde{a}_1)(\tilde{a}_1 - \eta_*)(\eta_* + \theta_* - \tilde{a}_1)^{-2}/9$;

$CF_{2,\kappa}^{3,0}$: a) $\psi_1 \geq 0$, $(6\eta_* - 3\tilde{a}_1)^2 - 12(3\tilde{a}_1\eta_* - 3\eta_*^2 + 2\tilde{c}_2) \geq 0$, $\tilde{a}_1 \neq \eta_* + \theta_*$, $\tilde{c}_1 = 3\eta_*(\eta_* + \theta_*)(\tilde{a}_1 - \eta_* - \theta_*)/2$, $\tilde{d}_1 = \theta_*\eta_*(\eta_*^2 + 4\eta_*\theta_* + \theta_*^2 - \tilde{a}_1(\eta_* + \theta_*))/2$, $d_2 = ((2\eta_* + \theta_*)(\eta_* - \theta_*)\tilde{a}_1 + (\eta_* + \theta_*)(\theta_*^2 + 3\eta_*\theta_* - 2\eta_*^2))/2$, *where* θ_*: $3\theta^2 + (6\eta_* - 3\tilde{a}_1)\theta + 3\tilde{a}_1\eta_* - 3\eta_*^2 + 2\tilde{c}_2$, $\eta_* = (3\tilde{a}_1 \pm (3\psi_1)^{1/2})/6$, b) $L_{2;1}^{3,0}$, c) $\kappa = -\sigma\operatorname{sign}(\tilde{a}_1 - 3\eta_* - \theta_*)$, $\sigma = \operatorname{sign}(\tilde{a}_1 - \eta_*)$, $u = 3(\tilde{a}_1 - \eta_* - \theta_*)(\tilde{a}_1 - \eta_*)^{-1}/2$;

$CF_4^{3,0}$: 1) a) $\tilde{b}_1 = 0$, $\tilde{c}_1 = 0$, $\tilde{d}_1 = 0$, $\tilde{c}_2 = 0$, b) $L_{4;1}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{a}_1$, $u = \tilde{d}_2\tilde{a}_1^{-3}$; 2) a) $\tilde{a}_1 = (\tilde{c}_1^4 - 81\tilde{d}_1^3)(27\tilde{d}_1^2\tilde{c}_1)^{-1}$, $\tilde{b}_1 = \tilde{c}_1^2\tilde{d}_1^{-2}/3$, $\tilde{c}_2, \tilde{d}_2 = 0$, b) $L_{4;2}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{c}_1$, $u = -81\tilde{d}_1^3\tilde{c}_1^{-4}$;

3) a) $\tilde{a}_1 \neq \eta_* + \theta_*$, $\tilde{b}_1 = 3\eta_*^2 - 3\tilde{a}_1\eta_*$, $\tilde{d}_1 = \eta_*^4 + (\theta_* - \tilde{a}_1)\eta_*^3 - \theta_*^2\eta_*^2$, $\tilde{d}_2 = \eta_*\theta_*(\eta_* + \theta_*)$, *where* η_*: $3\eta^3 - 3\tilde{a}_1\eta^2 - \tilde{c}_2\eta + \tilde{c}_1$, $\theta_* = -\tilde{c}_2\eta_*^{-1}/3$, b) $L_{4;3}^{3,0}$, c) $\sigma = \operatorname{sign}(\tilde{a}_1 - \eta_*)$, $u = \eta_*(\eta_* + \theta_* - \tilde{a}_1)^2(\tilde{a}_1 - \eta_*)^{-3}$;

$CF_9^{3,0}$: 1) a) $\tilde{a}_1 = 0$, $\tilde{c}_1 = 0$, $\tilde{d}_1 = 0$, $\tilde{c}_2 = 0$, b) $L_{9;1}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{d}_2$, $u = \tilde{d}_2^2\tilde{b}_1^{-3}$; 2) a) $\tilde{c}_1 = \eta_*\theta_*(\tilde{a}_1 - 4\eta_*) + 2\eta_*^2(\tilde{a}_1 - \eta_*)$, $\tilde{d}_1 = (\eta_*(\tilde{a}_1 + 2\eta_*)\theta_*^2 - \eta_*^2(2\tilde{a}_1 - 5\eta_*)\theta_* - 2\eta_*^3(\tilde{a}_1 - \eta_*))/3$, $\tilde{d}_2 = ((\tilde{a}_1 + 2\eta_*)\theta_*^2 + \eta_*(\tilde{a}_1 + 2\eta_*)\theta_* - 2\eta_*^2(\tilde{a}_1 - \eta_*))/3$, *where* η_*: $3\eta^3 - (3\tilde{a}_1^2 + 2\tilde{b}_1 + \tilde{c}_2)\eta - \tilde{a}_1(\tilde{b}_1 - \tilde{c}_2)$, $\theta_* = -(\eta_*^2 - \tilde{a}_1\eta_* + \tilde{c}_2)(\tilde{a}_1 + 2\eta_*)^{-1}$, b) $L_{9;2}^{3,0}$, c) $\sigma = \operatorname{sign}(\tilde{a}_1 + 2\eta_*)$, $u = (\tilde{a}_1 + 2\eta_*)^2(2\tilde{a}_1 - 2\eta_* - 3\theta_*)(\tilde{a}_1 - \eta_*)^{-3}/27$;

3) a) $\tilde{b}_1 = 0$, $\tilde{d}_1 = \tilde{a}_1\tilde{c}_1/3$, $\tilde{c}_2 = 3\theta_*(\theta_* - 2\tilde{a}_1)/4$, $\tilde{d}_2 = \tilde{c}_1/3$, *where* $\theta_* = -(4\tilde{c}_1/3)^{1/3}$, b) $L_{9;3}^{3,0}$, c) $\sigma = \operatorname{sign}\psi_3$, $u = 16\psi_3^3(9\tilde{a}_1 - \psi_3)^{-3}/27$;

$CF_{15}^{3,0}$: 1) a) $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{d}_1 = 0$, $\tilde{c}_2 = 0$, b) $L_{15;1}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{c}_1$, $u = \tilde{d}_2\tilde{c}_1^{-1}$;

2) a) $\tilde{a}_1 = \tilde{c}_2^2\tilde{c}_1^{-1}$, $\tilde{b}_1 = 2\tilde{c}_2$, $\tilde{d}_1 = 0$, $\tilde{d}_2 = 2\tilde{c}_1/3$, b) $L_{15;2}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{c}_1$, $u = 2/3$;

3) a) $\tilde{a}_1 = (\tilde{c}_1^2 + \tilde{c}_2^3)(\tilde{c}_1\tilde{c}_2)^{-1}$, $\tilde{b}_1 = -\tilde{c}_2$, $\tilde{d}_1 = -\tilde{c}_1^2\tilde{c}_2^{-1}/3$, $\tilde{d}_2 = -\tilde{c}_1/3$, b) $L_{15;3}^{3,0}$, c) $\sigma = \operatorname{sign}\tilde{c}_1$, $u = (3\tilde{c}_1^2 + 2\tilde{c}_2^3)\tilde{c}_2^{-3}/3$;

4) a) $(3\tilde{a}_1^2+2\tilde{b}_1+2\tilde{c}_2)^2+12\tilde{a}_1^2(\tilde{b}_1-2\tilde{c}_2)\ge 0$, $\tilde{c}_1=(\tilde{a}_1-\eta_*)\theta_*^2+(\tilde{a}_1-4\eta_*)\eta_*\theta_*+(\tilde{a}_1-\eta_*)\eta_*^2$, $\tilde{d}_1=-((\tilde{a}_1-4\eta_*)(\eta_*+\theta_*)\eta_*\theta_*+(\tilde{a}_1-\eta_*)\eta_*^3)/3$, $\tilde{d}_2=((2\tilde{a}_1+\eta_*)\theta_*^2-(\tilde{a}_1-4\eta_*)\eta_*\theta_*-(\tilde{a}_1-\eta_*)\eta_*^2)/3$, *where* η_*: $3\tilde{a}_1\eta^2-(3\tilde{a}_1^2+2\tilde{b}_1+2\tilde{c}_2)\eta-\tilde{a}_1(\tilde{b}_1-2\tilde{c}_2)$, $\theta_*=-(\tilde{b}_1+\tilde{c}_2)(3\tilde{a}_1)^{-1}$, b) $L^{3,0}_{15;4}$, c) $\sigma=\operatorname{sign}(\tilde{a}_1-\eta_*)$, $u=(2\tilde{a}_1+\eta_*)(\tilde{a}_1-\eta_*)^{-1}/3$;

5) a) $\tilde{a}_1=\mp 2(-\tilde{c}_2/3)^{1/2}$, $\tilde{c}_1=\mp 3^{1/2}(b_1-\tilde{c}_2)^2(-\tilde{c}_2)^{-1/2}/12$, $\tilde{d}_1=(\tilde{b}_1^2+\tilde{c}_2^2)/18$, $\tilde{d}_2=\mp 3^{1/2}(\tilde{b}_1^2+6\tilde{b}_1\tilde{c}_2+\tilde{c}_2^2)(-\tilde{c}_2)^{-1/2}/36$, b) $L^{3,0}_{15;5}$, c) $\sigma=\mp 1$, $u=1/3$;

6) a) $\tilde{b}_1=0$, $\tilde{c}_1=3\tilde{d}_2^{2/3}(2^{-2/3}\tilde{a}_1-\tilde{d}_2^{1/3})$, $\tilde{d}_1=\tilde{a}_1\tilde{d}_2$, $\tilde{c}_2=-3\times 2^{-1/3}\tilde{a}_1\tilde{d}_2^{1/3}$, b) $L^{3,0}_{15;6}$, c) $\sigma=\operatorname{sign}(\tilde{a}_1+(4\tilde{d}_2)^{1/3})$, $u=(2\tilde{a}_1-(4\tilde{d}_2)^{1/3})(\tilde{a}_1+(4\tilde{d}_2)^{1/3})^{-1}/3$;

$CF^{3,0}_{20}$: 1) a) $\tilde{b}_1\ne 0$, $\tilde{c}_1=3(\tilde{a}_1-\eta_*)\theta_*^2-3\eta_*^2\theta_*$, $\tilde{d}_1=(\eta_*-\tilde{a}_1)\theta_*^3+\eta_*^2\theta_*^2+\eta_*^3\theta_*$, $\tilde{d}_2=\eta_*\theta_*(\eta_*+\theta_*)$, *where* $\eta_*=\tilde{a}_1\tilde{c}_2(\tilde{b}_1+\tilde{c}_2)^{-1}$, $\theta_*=-(\tilde{b}_1+\tilde{c}_2)\tilde{a}_1^{-1}/3$, b) $L^{3,0}_{20;1}$, c) $\sigma=\operatorname{sign}\tilde{a}_1$, $u=\tilde{b}_1^3\tilde{a}_1^{-2}(\tilde{b}_1+\tilde{c}_2)^{-2}/3$;

2) a) $\tilde{b}_1=0$, $\tilde{c}_1=0$, $\tilde{c}_2=0$, $\tilde{d}_2=0$, $\tilde{d}_1\ne 0$, b) $L^{3,0}_{20;2}$, c) $\sigma=\operatorname{sign}\tilde{a}_1$, $u=\tilde{d}_1\tilde{a}_1^{-4}$;

3) a) $\tilde{a}_1=0$, $\tilde{b}_1=0$, $\tilde{c}_1=0$, $\tilde{c}_2=0$, $\tilde{d}_1\ne 0$, b) $L^{3,0}_{20;3}$, c) $\sigma=\operatorname{sign}\tilde{d}_2$, $u=\tilde{d}_1^3\tilde{d}_2^{-4}$;

4) a) $\tilde{d}_2\ne 2\tilde{a}_1^3$, $\tilde{b}_1=3(\tilde{d}_2/2)^{1/3}(2\tilde{a}_1-(4\tilde{d}_2)^{1/3})$, $\tilde{c}_1=3(\tilde{a}_1(4\tilde{d}_2)^{2/3}-3\tilde{d}_2)$, $\tilde{d}_1=\tilde{d}_2(8\tilde{a}_1-3(4\tilde{d}_2)^{1/3})/2$, $\tilde{c}_2=3(4\tilde{d}_2)^{2/3}/2$, b) $L^{3,0}_{20;4}$, c) $u=(2^{1/3}\tilde{a}_1-\tilde{d}_2^{1/3})^3(\tilde{d}_2/2)^{1/3}\tilde{a}_1^{-4}$, $\sigma=\operatorname{sign}\tilde{a}_1$;

5) a) $\tilde{a}_1=-(3\theta_*^2+2\tilde{b}_1)(6\theta_*)^{-1}$, $\tilde{c}_1=(\tilde{b}_1\theta_*^2-6\tilde{d}_1)\theta_*^{-1}$, $\tilde{c}_2=-4(\tilde{b}_1\theta_*^2-3\tilde{d}_1)\theta_*^{-2}$, $\tilde{d}_2=2(\tilde{b}_1\theta_*^2-3\tilde{d}_1)(3\theta_*)^{-1}$, $\tilde{b}_1\theta_*^2\ne 3\tilde{d}_1, 4\tilde{d}_1$; $4\tilde{b}_1^2+18\tilde{d}_1\ge 0$, *where* θ_*: $3\theta^4+8\tilde{b}_1\theta^2-24\tilde{d}_1$, b) $L^{3,0}_{20;5}$, c) $\sigma=\operatorname{sign}(\tilde{b}_1\theta_*^2-4\tilde{d}_1)\theta_*$, $u=-2^{11}3^{20}(\tilde{b}_1\theta_*-3\tilde{d}_1)\theta_*^6\tilde{b}_1^3(\tilde{b}_1\theta_*^2-4\tilde{d}_1)^4$;

6) a) $\tilde{b}_1=3\tilde{a}_1^{2/3}\tilde{d}_1^{1/3}$, $\tilde{c}_1=3\tilde{a}_1^{1/3}\tilde{d}_1^{2/3}$, $\tilde{c}_2=0$, $\tilde{d}_2=0$, b) $L^{3,0}_{20;6}$, c) $\sigma=\operatorname{sign}\tilde{a}_1$, $u=\tilde{d}_1^{1/3}\tilde{a}_1^{-4/3}$;

$CF^{3,0}_{23,\kappa}$: 1) a) $\tilde{a}_1=0$, $\tilde{b}_1=0$, $\tilde{c}_1=0$, $\tilde{c}_2\ne 0$, $\tilde{d}_2=0$, b) $L^{3,0}_{23;1}$, c) $\sigma=1$, $\kappa=\operatorname{sign}\tilde{d}_1$, $u=\tilde{d}_1\tilde{c}_2\left|\tilde{d}_1\right|^{-3/2}$;

2) a) $\tilde{a}_1=0$, $\tilde{b}_1\ne 0$, $\tilde{c}_1=0$, $\tilde{c}_2=0$, $\tilde{d}_2=0$, b) $L^{3,0}_{23;2}$, c) $\sigma=1$, $\kappa=\operatorname{sign}\tilde{d}_1$, $u=\tilde{b}_1\left|\tilde{d}_1\right|^{-1/2}$;

3) a) $\tilde{a}_1=-2^{2/3}(2^{1/3}\tilde{d}_1+3\tilde{d}_2^{4/3})\tilde{d}_2^{-1}$, $\tilde{a}_1\ne 0$, $\tilde{b}_1=3\times 2^{2/3}(\tilde{d}_1+2\tilde{d}_2^{4/3})\tilde{d}_2^{-2/3}$, $\tilde{c}_1=3(2^{4/3}\tilde{d}_1+5\tilde{d}_2^{4/3})\tilde{d}_2^{-1/3}$, $\tilde{c}_2=-3(3\times 2^{1/3}\tilde{d}_1+7\tilde{d}_2^{4/3})(2\tilde{d}_2)^{-2/3}$, b) $L^{3,0}_{23;3}$, c) $\sigma=1$, $\kappa=-1$, $u=3\times 2^{-5/6}(2^{1/3}\tilde{d}_1+3\tilde{d}_2^{4/3})\psi_4^{-1}$;

4) a) $\tilde{a}_1=-\tilde{d}_2^3\tilde{d}_1^{-2}$, $\tilde{b}_1=3\tilde{d}_2^2\tilde{d}_1^{-1}$, $\tilde{c}_1=-3\tilde{d}_2$, $\tilde{c}_2=-3\tilde{d}_2^2\tilde{d}_1^{-1}/2$, $\tilde{d}_2\ne 0$, b) $L^{3,0}_{23;4}$, c) $\sigma=1$, $\kappa=\operatorname{sign}(2\tilde{d}_1^3-\tilde{d}_2^4)$, $u=3\tilde{d}_2^2|4\tilde{d}_1^3-2\tilde{d}_2^4|^{-1/2}$;

5) a) $\tilde{d}_1=(\eta_*-\tilde{a}_1)\theta_*^3+(\eta_*-\tilde{a}_1)\eta_*\theta_*^2+(2\eta_*+\tilde{a}_1)\eta_*^2\theta_*/2+\tilde{a}_1\eta_*^3/2$, $\tilde{c}_2=3\tilde{a}_1(\theta_*-\eta_*)/2-3\eta_*\theta_*$, $\tilde{d}_2=(\eta_*-\tilde{a}_1)\theta_*^2+(2\eta_*+\tilde{a}_1)\eta_*\theta_*/2+\tilde{a}_1\eta_*^2/2$, *where* $\eta_*=(3\tilde{a}_1\theta_*+\tilde{b}_1)(3\theta_*)^{-1}$, $\theta_*:(9\tilde{a}_1^2+6\tilde{b}_1)\theta^3+(9\tilde{a}_1^3+9\tilde{a}_1\tilde{b}_1+6\tilde{c}_1)\theta^2+(6\tilde{a}_1^2\tilde{b}_1+2\tilde{b}_1^2)$, b) $L^{3,0}_{23;5}$, c) $\sigma=1$, $\kappa=\operatorname{sign}(\eta_*-\tilde{a}_1)(\tilde{a}_1+2\theta_*)$, $u=3\tilde{a}_1(\tilde{a}_1-\eta_*)|\tilde{a}_1-\eta_*|^{-3/2}|2\tilde{a}_1+4\theta_*|^{-1/2}$;

6) a) $\tilde{a}_1\ne 0$, $\tilde{b}_1=18^{1/3}(18^{1/3}\tilde{a}_1\tilde{c}_1^{1/3}+2\tilde{c}_1^{2/3})/6$, $\tilde{d}_1=\tilde{c}_1(7\times 18^{2/3}\tilde{c}_1^{1/3}-27\tilde{a}_1)/162$, $\tilde{c}_2=18^{2/3}\tilde{c}_1^{1/3}(9\tilde{a}_1-2\times 18^{2/3}\tilde{c}_1^{1/3})/108$, $\tilde{d}_2=(18^{1/3}\tilde{a}_1\tilde{c}_1^{2/3}-3\tilde{c}_1)/9$, b) $L^{3,0}_{23;6}$, c) $\sigma=1$, $\kappa=-\operatorname{sign}(9\tilde{a}_1+18^{2/3}\tilde{c}_1^{1/3})(18^{1/3}\tilde{a}_1\tilde{c}_1^{2/3}-2\tilde{c}_1)$, $u=3\times 18^{2/3}\tilde{a}_1\tilde{c}_1^{4/3}(9\tilde{a}_1+18^{2/3}\tilde{c}_1^{1/3})|\tilde{c}_1|^{-1}|(9\tilde{a}_1+18^{2/3}\tilde{c}_1^{1/3})^3(18^{1/3}\tilde{a}_1\tilde{c}_1^{2/3}-2\tilde{c}_1)|^{-1/2}/2$;

7) a) $\tilde{a}_1=-\tilde{d}_2(-\tilde{d}_1)^{-1/2}$, $\tilde{b}_1=3(\tilde{d}_1\pm(-\tilde{d}_1)^{1/4}\tilde{d}_2))(-\tilde{d}_1)^{-1/2}$, $\tilde{c}_1=0$, $\tilde{c}_2=-\tilde{b}_1$, $\tilde{d}_1<0$, b) $L^{3,0}_{23;7}$, c) $\sigma=1$, $\kappa=-\operatorname{sign}(\tilde{d}_1\pm(-\tilde{d}_1)^{1/4}\tilde{d}_2)(2\tilde{d}_1\pm(-\tilde{d}_1)^{1/4}\tilde{d}_2)$, $u=3(-\tilde{d}_1)^{1/4}(\pm\tilde{d}_1+(-\tilde{d}_1)^{1/4}\tilde{d}_2)\tilde{d}_2|\tilde{d}_1\pm(-\tilde{d}_1)^{1/4}\tilde{d}_2|^{-3/2}|4\tilde{d}_1\pm 2(-\tilde{d}_1)^{1/4}\tilde{d}_2|^{-1/2}$;

8) a) $\tilde{a}_1 \neq 0$, $\tilde{b}_1 = -3 \times 2^{-1/3}\tilde{d}_2^{1/3}(\tilde{a}_1 + (4\tilde{d}_2)^{1/3})$, $\tilde{c}_1 = -3\tilde{d}_2^{2/3}(2^{1/3}\tilde{a}_1 + \tilde{d}_2^{1/3})$, $\tilde{c}_2 = 3 \times 2^{2/3}\tilde{d}_2^{1/3}(3\tilde{a}_1 + 2^{5/3}\tilde{d}_2^{1/3})/4$, $\tilde{d}_1 = -\tilde{d}_2(\tilde{a}_1 + 3(4\tilde{d}_2)^{1/3})/2$, b) $L^{3,0}_{23;8}$, c) $\sigma = 1$, $\kappa = -1$, $u = 3 \times 2^{-1/2}\tilde{a}_1(\tilde{a}_1 + (4\tilde{d}_2)^{1/3})^{-1}$;

9) a) $\tilde{a}_1 = \pm 2(-\tilde{c}_2/3)^{1/2}$, $\tilde{c}_1 = \pm(\tilde{b}_1^2 - \tilde{c}_2^2)(-3\tilde{c}_2)^{-1/2}$, $\tilde{d}_1 = -(\tilde{b}_1^3 + \tilde{b}_1^2\tilde{c}_2 - 2\tilde{b}_1\tilde{c}_2^2 - \tilde{c}_2^3)(9\tilde{c}_2)^{-1}$, $\tilde{c}_2 < 0$, $\tilde{d}_2 = \mp(\tilde{b}_1^2 - 2\tilde{b}_1\tilde{c}_2 - \tilde{c}_2^2)(-3\tilde{c}_2)^{-1/2}/3$, b) $L^{3,0}_{23;9}$, c) $\sigma = 1$, $\kappa = -\operatorname{sign}\tilde{c}_2(\tilde{b}_1 + \tilde{c}_2)$, $u = -3|\tilde{c}_2|^{3/2}\tilde{c}_2^{-1}|\tilde{b}_1 + \tilde{c}_2|^{-1/2}$;

10) a) $\tilde{a}_1 = \mp 2\tilde{c}_2^2(-3\tilde{d}_1\tilde{c}_2)^{-1/2}/3$, $\tilde{c}_1 = -3\tilde{d}_2$, $\tilde{d}_2 = \pm\tilde{d}_1\tilde{c}_2(-3\tilde{d}_1\tilde{c}_2)^{-1/2}$, $\tilde{d}_1\tilde{c}_2 < 0$, b) $L^{3,0}_{23;10}$, c) $\sigma = 1$, $\kappa = -\operatorname{sign}(2\tilde{c}_2^2 - 9\tilde{d}_1)$, $u = -3\tilde{d}_1\tilde{c}_2(2\tilde{c}_2^2 - 9\tilde{d}_1)|2\tilde{c}_2^2 - 9\tilde{d}_1|^{-3/2}|\tilde{d}_1|^{-1}$;

$CF^{3,0}_{24}$: 1) a) $\tilde{d}_2 \neq 0$, $\tilde{a}_1 = -\tilde{d}_2^3\tilde{d}_1^{-2}$, $\tilde{b}_1 = 3\tilde{d}_2^2\tilde{d}_1^{-1}$, $\tilde{c}_1 = -3\tilde{d}_2$, $\tilde{c}_2 = -3\tilde{d}_2^2\tilde{d}_1^{-1}$, b) $L^{3,0}_{24;1}$, c) $\sigma = -\operatorname{sign}\tilde{d}_2$, $u = 9^{-2/3}(\tilde{d}_1^3 - 2\tilde{d}_2^4)\tilde{d}_2^{-4}$;

2) a) $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{d}_1 \neq 0$, $\tilde{c}_2 = 0$, $\tilde{d}_2 = 0$, b) $L^{3,0}_{24;2}$, c) $\sigma = \operatorname{sign}\tilde{c}_1$, $u = \tilde{d}_1\tilde{c}_1^{-4/3}$;

3) a) $\tilde{b}_1 = 0$, $\tilde{c}_1 = -3\tilde{a}_1^3$, $\tilde{c}_2 = -3\tilde{a}_1^2$, $\tilde{d}_2 = 2\tilde{a}_1^3$, b) $L^{3,0}_{24;3}$, c) $\sigma = \operatorname{sign}\tilde{a}_1$, $u = (\tilde{d}_1 - 2\tilde{a}_1^4)^{1/3}(3\tilde{a}_1)^{-4/3}$;

4) a) $\tilde{a}_1 = (2\tilde{d}_1 + 3 \times 4^{1/3}\tilde{d}_2^{4/3})(8\tilde{d}_2)^{-1}$, $\tilde{b}_1 = 3 \times 2^{-5/3}(2^{1/3}\tilde{d}_1 - \tilde{d}_2^{4/3})\tilde{d}_2^{-2/3}$, $\tilde{c}_1 = 24(2^{1/3}\tilde{d}_1 - 5\tilde{d}_2^{4/3})\tilde{d}_2^{-1/3}$, $\tilde{c}_2 = -3 \times 2^{-8/3}(3 \times 2^{1/3}\tilde{d}_1 + \tilde{d}_2^{4/3})\tilde{d}_2^{-2/3}$, $\tilde{d}_1 \neq 2^{-1/3}\tilde{d}_2^{4/3}$, b) $L^{3,0}_{24;4}$, c) $\sigma = \operatorname{sign}\tilde{d}_2(2^{1/3}\tilde{d}_1 + 3\tilde{d}_2^{4/3})$, $u = -2 \times 3^{-4/3}(2^{1/3}\tilde{d}_1 - \tilde{d}_2^{4/3})^{4/3}(2^{1/3}\tilde{d}_1 + 3\tilde{d}_2^{4/3})^{-4/3}$;

5) a) $\tilde{c}_1 = 3(\tilde{a}_1 - \eta_*)(\eta_* + \theta_*)\theta_* - 3\tilde{a}_1\eta_*^2$, $\tilde{d}_1 = (\eta_* - \tilde{a}_1)(\eta_*^2 + \eta_*\theta_* + \theta_*^2)\theta_* + 2\tilde{a}_1\eta_*^3$, $\tilde{d}_2 = (\eta_* - \tilde{a}_1)(\eta_* + \theta_*)\theta_* + 2\tilde{a}_1\eta_*^2$, *where* $\eta_* = -(\tilde{b}_1 + \tilde{c}_2)(3\tilde{a}_1)^{-1}$, $\theta_* = -\tilde{a}_1\tilde{b}_1(3\tilde{a}_1^2 + \tilde{b}_1 + \tilde{c}_2)^{-1}$, b) $L^{3,0}_{24;5}$, c) $\sigma = \operatorname{sign}\tilde{a}_1$, $u = 3^{-4/3}(\eta_* - \tilde{a}_1)^{1/3}(2\tilde{a}_1 + \theta_*)^{-2/3}\tilde{a}_1^{-4/3}$;

6) a) $\tilde{b}_1 = \tilde{c}_1\tilde{a}_1^{-1} \mp \psi_5\tilde{a}_1$, $\tilde{d}_1 = \tilde{c}_1(\pm 3\psi_5\tilde{a}_1^2 - \tilde{c}_1)\tilde{a}_1^{-2}/9$, $\tilde{c}_2 = \pm 2\psi_5\tilde{a}_1 - \tilde{c}_1\tilde{a}_1^{-1}$, $\tilde{d}_2 = -2\tilde{c}_1/3$, $\tilde{a}_1\tilde{c}_1 < 0$, b) $L^{3,0}_{24;6}$, c) $\sigma = -\operatorname{sign}\tilde{c}_1$, $u = -3^{-5/3}(2\psi_5\tilde{a}_1^2 - \tilde{c}_1)(\psi_5 + 3\tilde{a}_1)^{1/3}\tilde{a}_1^{-7/3}\psi_5^{-1}$.

The proof is presented in the file theorem.mw of the archive (see Introduction).

As a result, for any system (1.1) with $R \neq 0$, using Lemma 3.1 and Theorem 3.1, it is possible to establish if it is equivalent to some form $CF_i^{m,0}$ ($m = 2, 3$) from list 2.1 and, in the case of a positive response, indicate the compositions of substitutions that reduce it to the linearly equivalent form $CF_i^{m,0}$, as well as the concrete values of parameters from $cs_i^{m,0}$.

4. APPLICATION OF THE RESULTS

In [6], A. Cima and J. Llibre have divided the set of systems (1.1) into ten linearly nonequivalent canonical classes (CC) to obtain a complete topological classification of the phase portraits in the Poincare circle.

The work allows one to draw the phase portrait of the selected system with $l = 0$, if the CC to which it belongs is established and if the linear nonsingular substitution (1.2) reducing (1.1) to some representative of the established CC is found. For the systems with $l > 0$, only the set of possible phase portraits in the Poincare circle is indicated.

More precisely, in [6], to arbitrary system (1.1) $\dot{x} = P(x_1, x_2)$, the binary form $f = x_1P_2(x_1, x_2) - x_2P_1(x_1, x_2) = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$ was put into correspondence.

Further, based on G. Gurevich's study [8], the binary forms $f(x_1, x_2)$ were classified with respect to substitutions (1.2) by using algebraic invariants: the Hessian $H_f = (f_{x_1x_1}f_{x_2x_2} - f_{x_1x_2}^2)/144$, discriminant $D_f = i_f^3 - 27j_f^2$, as well as $12H_f^2 - i_ff^2$ and $2i_fH_f - 3j_ff$, where $i_f = a_0a_4 - 4a_1a_3 + 3a_2^2$, $j_f = a_0a_2a_4 + 2a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 - a_2^3$. Ten linearly nonequivalent classes were obtained; in each of them the generator—the canonical binary form (CBF) F_i ($i = \overline{1,10}$) was separated (see the list of CBFs in ([6], p.437)).

In addition, for each CBF, the canonical class of the systems, to which it can be put into correspondence, was indicated; this class depends on parameters p_1, p_2, p_3 and, possibly, on α, μ. The list of CC_i

($i = \overline{1,10}$) is presented in ([6], p. 436). It was proved that for any system (1.1), there is the substitution (1.2) that reduces (1.1) to the representative of one of the separated CCs.

It should be noted that the CBFs were selected such that it would be convenient to use them when finding the phase portraits in the Poincare circle. Based on the results of [9, 10], the conditions were obtained on the values of the polynomial $P_1(1, x_2^*)$ from any canonical class, where x_2^* is an arbitrary zero of the corresponding CBF (see [6], p.446), which makes it possible to separate the phase portrait from the list presented in ([6], p. 444).

So, the main technical problem for the practical application of the classification made in [6] is to find the above-mentioned linear substitution in the explicit form.

Let us demonstrate how the results obtained in this study can be used for constructing the phase portraits of system (1.1). For this purpose, les us reduce the initial system with $l = 0$, whose coefficients after the substitution L_k with the appropriate k (see formulas (3.1)) satisfy one of the conditions of Theorem 3.1, to the corresponding canonical form from list 2.1.

Let us assume that system (3.1) obtained after substituting L_k by the substitution indicated in Theorem 3.1 is reduced to $CF_{2,\kappa}^{3,0} = \sigma\begin{pmatrix} \kappa & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $u \neq 3/2$ at $\kappa = -1$.

Let us calculate the binary form of $CF_{2,\kappa}^{3,0}$ and its invariants:

$$f = -\kappa x_1^3 x_2 + (1-u)x_1 x_2^3; \quad D_f = -\kappa(u-1)^3/64, \quad H_f = -(\kappa x_1^2 - (u-1)x_2^2)^2/16,$$
$$12H_f^2 - i_f f^2 = 3x_1^8/64 + \kappa(u-1)x_1^6 x_2^2/16 + 25(u-1)^2 x_1^4 x_2^4/32 + \kappa(u-1)^3 x_1^2 x_2^6/16 + 3(u-1)^4 x_2^8/64$$
$$(i_f = \kappa(1-u)/4, j_f = 0).$$

1. Let $u > 1$, $\kappa = 1$ or $u < 1$, $\kappa = -1$. Then, $D_f < 0$, therefore $CF_{2,1}^{3,0}$ must be reduced in CC_3 with the matrix $\tilde{A} = \begin{pmatrix} p_1 & p_2 & p_3 & \mu \\ \mu & p_1 & p_2 & p_3 \end{pmatrix}$ $(\mu \neq 0)$ and the CBF $F_3 = \mu(x_1^4 - x_2^4)$.

Indeed, using the substitution (1.2) with $\{r_1 = 2^{-1/4}\kappa(\kappa(u-1))^{1/8}$, $s_1 = -r_1$, $r_2 = 2^{-1/4}(\kappa(u-1))^{-3/8}$, $s_2 = r_2\}$ and the time substitution $t = \delta_{rs}\tau$ $(\delta_{rs} = r_1 s_2 - s_1 r_2)$, we obtain $\tilde{A}$ with $p_1, p_3 = u(u-1)^{-1}$, $p_2 = -(u-3)(u-1)^{-1}$, and $\mu = -1$.

The polynomial $F_3(1, x_2)$ has zeros ± 1, then, $P_1(1, 1) = -4(u-1)^{-1}$, $P_1(1, -1) = -4$. Verifying the conditions from ([6], p. 444), we find that at $u > 1$, $\kappa = 1$, this representative of CC_3 has the phase portrait N11 from ([6], p. 444); at $u < 1$, $\kappa = -1$, the phase portrait is N10.

Let us investigate two remaining cases in a similar way.

2. Let $u = 1$, $\kappa = 1$ or $\kappa = -1$. Then, $D_f = 0$, $H_f < 0$ at $x_1 \neq 0$, therefore, by the substitution (1.2) with $\{r_1 = 0, s_1 = 1, r_2 = 4\kappa, s_2 = -4\kappa\}$ and the time substitution, $CF_{2,1}^{3,0}$ is reduced in CC_5 with the matrix $\tilde{A} = \begin{pmatrix} p_1 & p_2 & p_3 + 2 & -4 \\ 0 & p_1 & p_2 & p_3 - 2 \end{pmatrix}$, in which $p_1 = -64$, $p_2 = 128$, $p_3 = -66$, and the CBF $F_5 = 4(x_2 - x_1)x_2^3$. Since $P_1(1, 0) = -64\kappa$, $P_1(1, 1) = -4$, we find that at $\kappa = 1$, this representative of CC_5 has the phase portrait N11; at $\kappa = -1$, it is N10.

3. Let $u < 1$, $\kappa = 1$ or $u > 1$, $\kappa = -1$. Then, $D_f > 0$, $H_f < 0$ at $x_1^2 + x_2^2 \neq 0$ and it is easy to see that $12H_f^2 - i_f f^2 > 0$, therefore, $CF_{2,1}^{3,0}$ is reduced by means of substitution (1.2) with $\{r_1 = -3^{1/4}(1+\sqrt{2})^2(3+2\sqrt{2})^{-1}(\kappa(1-u))^{1/8}$, $s_1 = 3^{1/4}(3+2\sqrt{2})^{1/2}(\kappa(1-u))^{1/8}$, $r_2 = -\kappa(3(17+12\sqrt{2}))^{1/4}(\kappa(1-u))^{-3/8}$, $s_2 = -\kappa \times 3^{1/4}(\kappa(1-u))^{-3/8}\}$ and the time substitution in CC_1 with the matrix $\tilde{A} = \begin{pmatrix} p_1 & p_2 + 3(1+\mu^4) & p_3 & -6\mu^2 \\ 6\mu^2 & p_1 & p_2 - 3(1+\mu^4) & p_3 \end{pmatrix}$ $(\mu > 1)$, in which $p_1 = 6(1+\sqrt{2})(u+3+3\sqrt{2})(1-u)^{-1}$, $p_2 = 12(4+3\sqrt{2})u(1-u)^{-1}$, $p_3 = 6(7+5\sqrt{2})(u+3-3\sqrt{2})(u-1)^{-1}$,

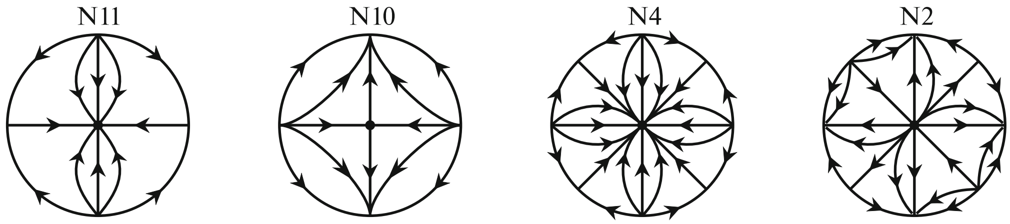

Fig. 1. Phase portraits in the Poincare circle.

$\mu = 1 + \sqrt{2}$, and CBF $F_1 = 6(\mu^2 x_1^4 - (1 + \mu^4)x_1^2 x_2^2 + \mu^2 x_2^4)$. Since $P_1(1, -\mu) = 480 + 336\sqrt{2}$, $P_1(1, -1/\mu) = 24(2 + \sqrt{2})(u - 1)^{-1}$, $P_1(1, 1/\mu) = -48(2 + \sqrt{2})(u - 1)^{-1}$, $P_1(1, \mu) = 24(10 + 7\sqrt{2})(u - 1)^{-1}$, we find that at $u < 1$, $\kappa = 1$, this representative of CC_1 has the phase portrait N4; at $u > 1$, $\kappa = -1$, the phase portrait is N2 (Fig. 1).

Hence, any system that is linearly equivalent to any representative of $CF_{2,\kappa}^{3,0}$ with $u \geq 1$ and $\kappa = 1$, has the phase portrait N11 in the Poincare circle, despite the fact that at $u = 1$ and $u > 1$, $CF_{2,\kappa}^{3,0}$ belongs to different canonical classes, namely, to CC_3 and to CC_5, respectively. The same is true for the values of parameters $u \leq 1$ and $\kappa = -1$ that attribute $CF_{2,\kappa}^{3,0}$ to the phase portrait N10.

Returning to the cases investigated in [2–5], when in system (1.1), there is a common multiplier of degree $l \geq 1$, we note that to find the phase portraits in the Poincare circle, it is necessary to use the time substitution to reduce the $CF^{m,\,l}$ obtained to the two-dimensional homogeneous quadratic systems whose canonical forms are found in [11] and the phase portraits are found in [12, 13].

In addition, let us note that in [14], the classification of two-dimensional quadratic-cubic homogeneous systems with no common multiplier is performed; their phase portraits in the Poincare circle are found.

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