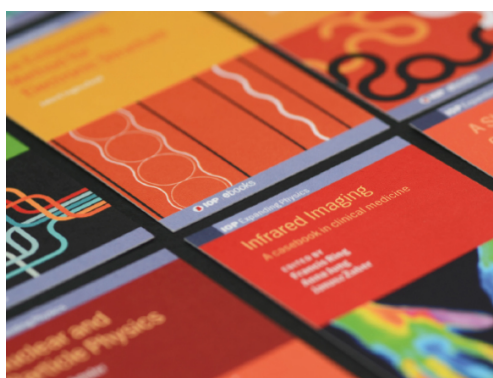


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Vibrations of cylindrical shells with rectangular cross-section

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Abstract. The free vibrations of cylindrical shells with rectangular cross-section are studied in the work. The precision of Rayleigh method for various admissible functions and different boundary conditions is evaluated. Using finite element method, the natural frequencies in the case of deviation of the cross-section shape from square to rectangular are examined, and the vibration frequencies and modes for the shell joined with the plate are studied for various plate thicknesses. Additionally, fundamental frequency behavior is analyzed in the case of fixed support, as one of the cross-section sides approaches zero.

1. Introduction

This paper research subject is the free vibrations of cylindrical shells with rectangular cross-section. This kind of shell could serve as a model for a great variety of thin-walled box-shaped constructions which are used in mechanical engineering, chemical industry, and other technical areas. Different kinds of cabins (i.e. elevator cabins, sound booths, etc) belong to these constructions as well.

This issue is the next step in further studying of the vibrations of cylindrical shells with square cross-section which was considered in [1, 2]. Apart from that, the vibrations of the cylindrical shells with rectangular cross-section and simply-supported edges were examined in paper [3], meanwhile in [4] buckling of simply supported shell with square cross-section was analyzed, and buckling of plates was deeply studied in [5, 6]. In the case of square cross-section one can get approximate analytical solution by numerical solving of the transcendental equation which is obtained after separation of variables in Germain-Lagrange equation. Moreover, if the shell is simply supported, the exact formula for frequencies is known [1]. However, rectangular cross-section case makes such calculations much more sophisticated. One can still use numerical methods to look upon the shell's vibrations, for instance, Rayleigh method and finite element method (FEM). Step-by-step exposition how to apply the methods mentioned so as to deal with this obstacle is what this article is dedicated to. Usage of approximate solutions is well-known for similar problems. Numerical solution based on Bubnov-Galerkin method was obtained for cylindrical shell's frequencies in [7, 8].

In the present paper, vibrations of cylindrical shell with rectangular cross-section are examined using said methods. Precision of the solutions obtained by Rayleigh method is analyzed for various admissible functions under different boundary conditions. FEM is used to investigate shell's vibration frequencies and modes when cross-section shape deviates from



square to rectangular. It is also applied to study vibration frequencies and modes for the shell joined with the plate for various plate thicknesses. Additionally, fundamental frequency of the shell with clamped edges, as one of the cross-section sides approaches zero, is analyzed.

2. Materials and methods

2.1. Formulation of the problem

We take the cylindrical shell with rectangular cross-section. Its four faces are rectangular plates of width a or b and length c , as shown in figure 1.

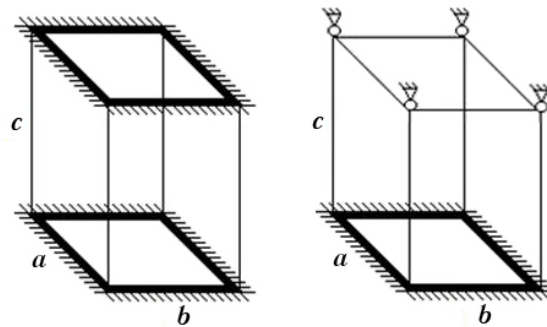


Figure 1. Cylindrical shell with rectangular cross-section under different boundary conditions (a, b, c are the shell sizes).

The differential equation for the deflection $w^{(k)}$ of the k^{th} plate has the form [9]:

$$D\Delta\Delta w^{(k)} - \rho t w^{(k)} \omega^2 = 0, \quad k = \overline{1, 4}, \tag{1}$$

where $w^{(k)} = w^{(k)}(x, y)$ is deflection of k^{th} plate, depending on local coordinates x and y of k^{th} plate (fig. 2), $\Delta w = w_{xx} + w_{yy}$, $D = \frac{Et^3}{12(1-\nu^2)}$ is bending stiffness; E is Young's modulus, ν is Poisson's ratio, ρ is the plate mass density, t is the plate thickness, ω is the natural frequency.

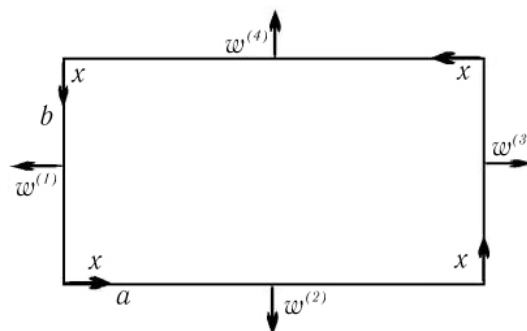


Figure 2. Local coordinates of the shell's surfaces (view from above).

2.2. Imposing boundary conditions

We consider the problem under the following assumptions. First, we neglect the longitudinal compression-stretching of the plates and suppress any rigid body motion of the shell. Then, we suppose that the plates are rigidly joined on the line where they meet each other. Finally, we assume that the bending moments on the line where the plates meet are equal to each other. These assumptions result in the following boundary conditions for each plate:

$$\begin{aligned} w^{(k)}(0, y) &= w^{(k)}(\chi, y) = 0, \\ w_x^{(k)}(\chi, y) &= w_x^{(k+1)}(0, y), \\ w_{xx}^{(k)}(\chi, y) &= w_{xx}^{(k+1)}(0, y). \end{aligned} \quad (2)$$

All index arithmetic is mod 4, i.e., for example, $4 + 1 = 1 \pmod{4}$.

In the case of square cross-section, i. e. if $a = b$, the solution

$$w^{(k)}(x, y) = (-1)^k W^{(k)}(y) \sin \frac{m\pi x}{a}, \quad (3)$$

where $m = 1, 2, \dots$ is a number of waves across the x coordinate of the k^{th} plate, satisfies conditions (2).

We suppose that $y = 0$ edge is clamped and edge $y = c$ is joined with a plate. If the shell is joined with a plate and its thickness differs significantly from the shell's, then the plate can be replaced by boundary conditions of clamped edge or simply supported edge. This allows us to consider the shell's vibrations and the plate's vibrations separately from each other. Hereinafter we refer to these conditions coupled with clamped edge $y = 0$ as "clamped-clamped" and "clamped-simple support", respectively.

The "clamped-clamped" boundary conditions are as follows:

$$\begin{aligned} W^{(k)}(x, 0) &= W^{(k)}(x, c) = 0, \\ \frac{\partial w^{(k)}}{\partial y}(x, 0) &= \frac{\partial w^{(k)}}{\partial y}(x, c) = 0. \end{aligned} \quad (4)$$

The "clamped-simple support" boundary conditions are as follows:

$$\begin{aligned} w^{(k)}(x, 0) &= w^{(k)}(x, c) = 0, \\ \frac{\partial w^{(k)}}{\partial y}(x, 0) &= \frac{\partial^2 w^{(k)}}{\partial y^2}(x, c) = 0. \end{aligned} \quad (5)$$

For rectangular cross-section, separation of variables is not applicable. However, one still can use Rayleigh method and finite element method (FEM) for approximate calculation of the shell's frequencies and modes. These methods are also used to study the vibrations of the shell joined with the plate for various plate thicknesses, as well as to consider the frequencies' behavior, as one of the cross-section sides approaches zero. Hereinafter we use the frequency parameter $f = \tilde{\nu} \sqrt{\frac{\rho t}{D}}$, where $\tilde{\nu} = \frac{\omega}{2\pi}$.

To simplify further calculations, we limit our considerations to the shells of middle length, therefore we take a limited set of sizes to illustrate obtained results. One can get similar outcome for other sizes by scaling respective results.

2.3. Rayleigh method calculations

We demonstrate an approach, which involves use of the adjustment factor coupled with the admissible functions. As a result, it allows us to find approximate formula for the fundamental frequency of the shell in spite of absence of the symmetry.

When developing approximate formulae with Rayleigh method [10], one can consider just two adjacent shell's surfaces due to the symmetry. We use the following admissible functions:

$$W_{\chi_i} = C\chi_i^{\frac{11}{10}} \sin \frac{\pi x}{\chi_i} \left(1 - \cos \frac{2\pi y}{c}\right) \text{--- for "clamped-clamped" boundary conditions,} \quad (6)$$

$$W_{\chi_i} = C\chi_i^{\frac{5}{4}} \sin \frac{\pi x}{\chi_i} (y^4 - 2cy^3 + c^2y^2) \text{--- for "clamped-clamped" boundary conditions,} \quad (7)$$

$$W_{\chi_i} = C\chi_i \sin \frac{\pi x}{\chi_i} \left(y^4 - \frac{5c}{2}y^3 + \frac{3c^2}{2}y^2\right) \text{--- for "clamped-simple support" boundary conditions,} \quad (8)$$

$$W_{\chi_i} = C\chi_i^{\frac{5}{4}} \sin \frac{\pi x}{\chi_i} \left(y^4 - \frac{5c}{2}y^3 + \frac{3c^2}{2}y^2\right) \text{--- for "clamped-simple support" boundary conditions,} \quad (9)$$

where χ_i^α is adjustment factor, $i = 1, 2$, $\chi_i = a$ if $i = 1$ and $\chi_i = b$ if $i = 2$. α was selected in order to ensure minimal error.

Potential energy of strain and maximal kinetic energy are calculated upon the following formulae:

$$\Pi = \int_0^c \left(\int_{-\chi_i}^0 \Pi_{\chi_i} dx + \int_0^{\chi_{i+1}} \Pi_{\chi_{i+1}} dx \right) dy, \quad (10)$$

$$T = \int_0^c \left(\int_{-\chi_i}^0 T_{\chi_i} dx + \int_0^{\chi_{i+1}} T_{\chi_{i+1}} dx \right) dy, \quad (11)$$

where

$$\Pi_{\chi_i} = D \left(\left(\frac{d^2 W_{\chi_i}}{dx^2} + \frac{d^2 W_{\chi_i}}{dy^2} \right)^2 + 2(1 - \nu) \left(\left(\frac{d^2 W_{\chi_i}}{dx dy} \right)^2 - \frac{d^2 W_{\chi_i}}{dx^2} \frac{d^2 W_{\chi_i}}{dy^2} \right) \right), \quad (12)$$

$$T_{\chi_i} = \rho t W_{\chi_i}^2, \quad (13)$$

whereupon they are being substituted into the expression for the fundamental frequency:

$$f = \tilde{\nu} \sqrt{\frac{\rho t}{D}} = \frac{\omega}{2\pi} \sqrt{\frac{\rho t}{D}} = \frac{1}{2\pi} \sqrt{\frac{2\Pi \rho t}{T D}}. \quad (14)$$

2.4. Finite-element method evaluations

Finite-element method evaluations were made using package ANSYS. The shell was modeled with the partitioning of 400 elements SHELL63 per m^2 . The values of obtained frequencies were rounded to the third place of the decimal point. The shell surfaces' thicknesses were assumed to be equal to $t = 0,015$ m.

FEM is used to study the frequencies' and modes' behavior both under "clamped-clamped" and "clamped-simple support" boundary conditions when square cross-section is deviated from its initial shape to rectangular preserving its perimeter. We also apply FEM to consider the vibrations of the shell with one edge clamped and another one joined with plate of thickness t_c in order to compare its frequencies and modes with ones in "clamped-clamped" and "clamped-simple support" cases at different values of t_c .

2.5. The case of $b \rightarrow 0$

We can examine the frequency behavior as one of the cross-section sides approaches zero. For the sake of brevity, we hereinafter assume $b \rightarrow 0$.

Considering the value of the limit as $b \rightarrow 0$, we study, for what values of b the frequency of the shell with clamped edges is close to the frequency of the clamped plate.

Firstly, we use Rayleigh method to find approximate formula for the plate's frequency, using admissible function $W = C \left(1 - \cos \frac{2\pi x}{a}\right) \left(1 - \cos \frac{2\pi y}{c}\right)$:

$$f = \frac{2\pi}{3a^2c^2} \sqrt{3a^4 + 3c^4 + 2a^2c^2}. \quad (15)$$

Furthermore, assuming a and c to be fixed, we equate the frequencies obtained by formulae (15) and (16), thereby we get an equation for the value of b , at which approximate value of the shell's frequency is equal to that of the plate.

3. Results and discussion

3.1. Results obtained with using Rayleigh method

Calculations using (10)-(14) were performed for each of the admissible functions (6)-(9) with respect to the boundary conditions they satisfy.

We will use the relative error $J = \frac{f_R - f_N}{f_N} 100\%$, where f_R is the frequency found via Rayleigh method, and f_N is the frequency found via FEM, to make conclusions on convenience of use for each considered case.

The following approximate solutions were obtained:

- "clamped-clamped" boundary conditions, admissible function (6):

$$f = \frac{\pi\sqrt{3}}{6a^{\frac{2}{5}}b^{\frac{2}{5}}c^2} \sqrt{\frac{3a^{\frac{4}{5}}c^4 + 3b^{\frac{4}{5}}c^4 + 8a^2b^{\frac{4}{5}}c^2 + 8a^{\frac{4}{5}}b^2c^2 + 16a^{\frac{4}{5}}b^4 + 16a^4b^{\frac{4}{5}}}{a^{\frac{16}{5}} + b^{\frac{16}{5}}}}. \quad (16)$$

For this solution J varies in the range from 2,1 % to 9,9 %.

- "clamped-clamped" boundary conditions, admissible function (7):

$$f = \frac{1}{2\pi\sqrt[4]{abc^2}} \sqrt{\frac{-(A_1 + A_2)}{B}} \quad (17)$$

where

$$\begin{aligned} A_1 &= \pi^4 c^4 + 24a^{\frac{3}{2}}\pi^2\sqrt{bc^2} - 24\pi^2 abc^2 + 24b^{\frac{3}{2}}\pi^2\sqrt{ac^2}, \\ A_2 &= 504a^{\frac{7}{2}}\sqrt{b} + 504a^{\frac{5}{2}}b^{\frac{3}{2}} - 504a^3b - 504a^2b^2 - 504ab^3 + 504a^{\frac{3}{2}}b^{\frac{5}{2}} + 504\sqrt{ab^{\frac{7}{2}}}, \\ B &= a^{\frac{5}{2}}\sqrt{b} + a^{\frac{3}{2}}b^{\frac{3}{2}} + \sqrt{ab^{\frac{5}{2}}} - a^3 - a^2b - ab^2 - b^3. \end{aligned}$$

For this solution J varies in the range from 0,1 % to 3,4 %.

- "clamped-simple support" boundary conditions, admissible function (8):

$$f = \frac{\sqrt{19}}{38\pi c^2} \sqrt{\frac{19\pi^4 c^4 + 432\pi^2 abc^2 + 4536a^3b - 4536a^2b^2 + 4536ab^3}{ab(a^2 - ab + b^2)}}. \quad (18)$$

For this solution J varies in the range from 0,1 % to 15,7 %.

- "clamped-simple support" boundary conditions, admissible function (9):

$$f = \frac{\sqrt{19}}{38\pi\sqrt[4]{abc^2}} \sqrt{\frac{-(A_1 + A_2)}{B}}, \quad (19)$$

where

$$\begin{aligned} A_1 &= 19\pi^4 c^4 + 432\pi^2 a^{\frac{3}{2}} \sqrt{bc^2} - 432\pi^2 abc^2 + 432\pi^2 \sqrt{ab^{\frac{3}{2}}} c^2, \\ A_2 &= 4536a^{\frac{7}{2}} \sqrt{b} - 4536a^3 b + 4536a^{\frac{5}{2}} b^{\frac{3}{2}} - 4536a^2 b^2 + 4536a^{\frac{3}{2}} b^{\frac{5}{2}} - 4536ab^3 + 4536\sqrt{ab^{\frac{7}{2}}}, \\ B &= a^{\frac{5}{2}} \sqrt{b} + a^{\frac{3}{2}} b^{\frac{3}{2}} + \sqrt{ab^{\frac{5}{2}}} - a^3 - a^2 b - ab^2 - b^3. \end{aligned}$$

For this solution J varies in the range from 0,1 % to 3,7 %.

As follows from the results, selection of the admissible functions can affect both precision of the solution and its usability. However, in all the cases J is minimal as the cross-section shape tends to square.

One might notice that the polynomial as a admissible function for "clamped-clamped" boundary conditions leads to more precise solution, although it appears to be cumbersome. Trigonometric function, on the contrary, gives more usable yet more inaccurate result. Selection of the adjustment factor has impact too, as shown in "clamped-simple support" case with using polynomials.

However, one can go beyond these cases to further examine influence of the adjustment factor on obtained formulas and its accuracy.

3.2. Results obtained with using FEM

With simultaneous k -fold varying of a , b , c , t , the frequency f is changed inversely proportional to changing of k (fig. 3). If also t is fixed, then dependency of f on k is close to inversely quadratic.

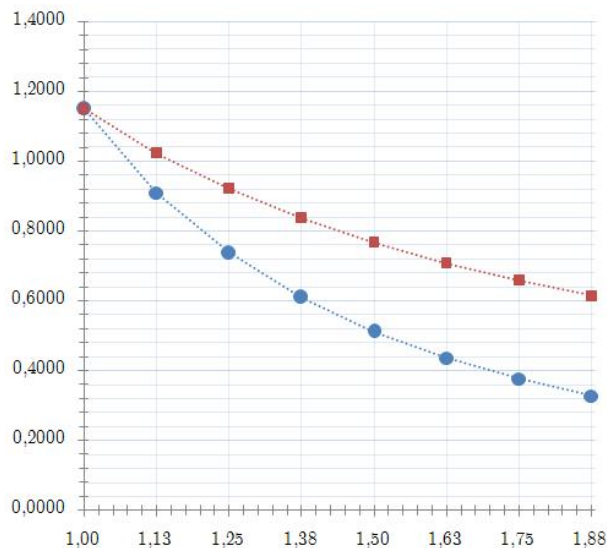


Figure 3. Examples of dependency of f on k at initial sizes $a = b = c = 2$, $t = 0,015$.

Red plot corresponds to the increase of all sizes of the shell in k times, blue plot corresponds to the increase of all sizes of the shell except for t in k times.

The analysis of higher frequencies shows that in the case of square cross-section second and third frequency are multiple.

When the cross-section shape deviates from square to rectangular having its perimeter preserved, the frequency splits into two different frequencies. With growth of $\varepsilon = \frac{M}{m} - 1$, where M and m are the maximum and the minimum of the cross-section sides' lengths respectively, the 2^{nd} frequency is decreasing and the 3^{rd} one is increasing. For the vibration mode corresponding to the 2^{nd} frequency, the longer plates of the shell have one wave in x direction and deflect codirectionally, meanwhile the shorter ones have two waves and smaller deflection compared to the longer plates. The 3^{rd} frequency vibration mode is similar, but shorter and longer plates are swapped (fig. 4).

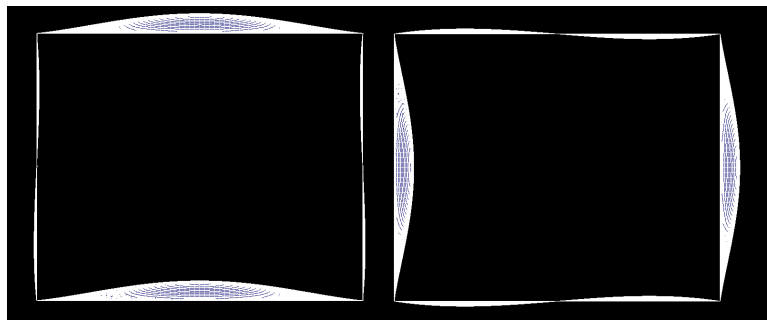


Figure 4. The 2^{nd} and 3^{rd} vibration modes at $\varepsilon = 0.3$.

Figure 5 represents examples of the frequency split at different values of ε parameter. Similar phenomenon occurs, when the cross-section shape is deviated from square to rectangular having its area preserved.

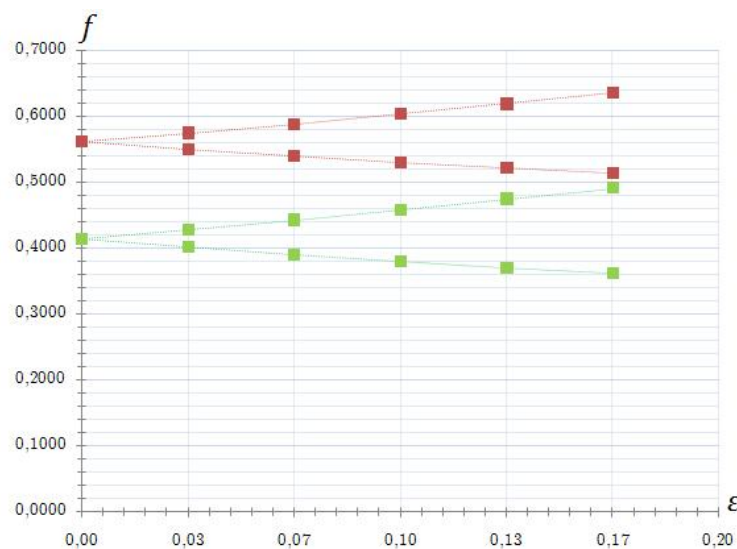


Figure 5. Frequency split as ε grows ($a + b = const$). Red dots correspond to the frequency split for the shell at initial sizes $a = 3, b = 3, c = 3$, green — for the shell at initial sizes $a = 3, b = 3, c = 4$.

Therefore, for the shell with square cross-section having small deviation from square to rectangular, the 2^{nd} and the 3^{rd} frequencies are approximately equal.

According to the plots, it can be suggested that for the shells with square cross-section there exist such c_1 , c_2 , ε , that the 2nd frequency of the shell of length c_1 is equal to the 3rd frequency of the shell of length c_2 . However, explicit finding of them is not studied in this paper.

For the case of the joined plate, we will introduce several new denotements. Let t_c is the plate thickness, f_1 , f_2 , f_3 are the first respective frequencies of the shell joined with plate, f_{fixed} is the frequency of the shell with clamped edges, f_{simple} is the frequency of the shell with clamped and simply supported edges, and f_c is the frequency of the plate with clamped edges calculated by formula (15) using Rayleigh method with due regard for difference of the thicknesses of the plate and the shell:

$$f_c = f \frac{t_c}{t} = \frac{2\pi}{3a^2b^2} \frac{t_c}{t} \sqrt{3a^4 + 3b^4 + 2a^2b^2}. \quad (20)$$

We will now compare specified frequencies at different values of t_c . A numerical example at $a = 2, b = 3, c = 3$ and $t = 0,015$ is presented in table 1.

Table 1. Frequencies of the shell joined with the plate of thickness t_c .

| t_c | 5t | 2t | t | 0.5t | 0.3t | 0.2t |
|--------------|-------|-------|-------|-------|-------|-------|
| f_1 | 0.542 | 0.531 | 0.492 | 0.444 | 0.315 | 0.213 |
| f_2 | 0.572 | 0.565 | 0.536 | 0.500 | 0.460 | 0.330 |
| f_3 | 0.789 | 0.784 | 0.766 | 0.526 | 0.491 | 0.456 |
| f_c | 5.548 | 2.219 | 1.110 | 0.555 | 0.333 | 0.222 |
| f_{fixed} | | | 0.543 | | | |
| f_{simple} | | | 0.455 | | | |

At $t_c = 5t$ (thick plate) the frequency f_1 is close to the frequency $f_{fixed} = 0,543$ of the shell with clamped edges. At $t_c = 0,2t$ the frequency f_1 differs slightly from the frequency $f_c = 0,222$. One can also note closeness of the frequencies f_2 with $t_c = 0,3t$ and f_3 with $t_c = 0,2t$ to the frequency $f_{simple} = 0,455$ of the shell having one edge clamped and another one simply supported.

Similar behavior of frequencies is present for other shell sizes. This allows us to find approximate value of the fundamental frequency f_1 using variational formulae (16) or (17) in the case of thick plate. For thin plate, the approximate variational formula (20) can be used to find f_1 . Formulae (18) and (19) can be used for approximate calculation of frequencies of the shell joined with the thin plate.

The first modes corresponding to different values of t_c are presented in figure 6. In the case of the thick plate, its deflection is negligible compared to deflections of the shell's surfaces. For the thin plate, on the contrary, the surfaces' deflections are small compared to the plate's.

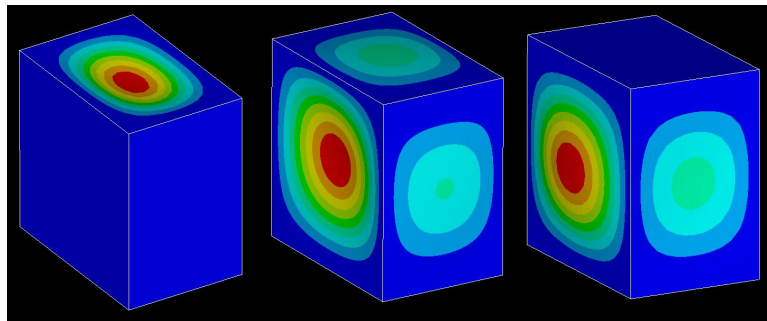


Figure 6. Alteration of the shell's 1st mode as the plate's thickness grows.

3.3. Results obtained for the case of $b \rightarrow 0$

An equation for the value of b , at which the approximate value of the shell's frequency is equal to that of the plate, is as follows:

$$b^4 - A_1 b^2 + A_2 b^{\frac{4}{5}} - A_3 = 0 \tag{21}$$

where

$$A_1 = \frac{24a^{\frac{24}{5}}}{D}, A_2 = \frac{39a^4 c^2 + 8a^6}{D}, A_3 = \frac{9a^{\frac{24}{5}} c^2}{D}, D = 48a^{\frac{4}{5}} c^2 + 32a^{\frac{14}{5}}.$$

This equation can be solved numerically. Its roots at different values of a and c are presented in table 2.

Table 2. Roots of the equation (21).

| | | c | | | |
|---|---|--------|--------|--------|--------|
| | | 1 | 2 | 3 | 4 |
| a | 2 | 0.1525 | 0.2676 | 0.3243 | 0.3299 |
| | 3 | 0.1562 | 0.2949 | 0.4015 | 0.4680 |
| | 4 | 0.1576 | 0.3051 | 0.4334 | 0.5353 |

It's seen that for considered range of values of a and c (approximately 6-8 times less than minimum from a and c) the fundamental frequency of the shell is proximate to that of the plate. Deflection of narrow surfaces of the shell is negligible compared to wide ones (fig. 7).

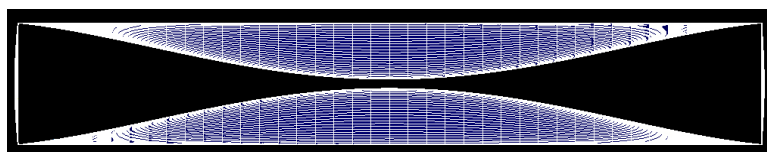


Figure 7. Vibration mode of the shell with two narrow surfaces (view from above).

4. Conclusion

The free vibrations of cylindrical shells with rectangular cross-section are examined. By means of finite element method, the split of 2nd and 3rd frequencies under deviation of the cross-section shape from square to rectangular is investigated.

Precision of Rayleigh method for calculation of the fundamental frequency with using various admissible functions is compared. It is shown that depending on the adjustment factor, precision and convenience of use of approximate formula can change notably.

The vibration frequencies and modes for the shell joined with the plate are studied for various plate thicknesses. For a shell joined with a plate of significantly different thickness compared to the shell, the fundamental frequency may be found by using approximate variational formulae.

For the shell with clamped edges an equation is received, which allows to find out the width of cross-section that makes the shell's frequency close to that of the clamped plate.

The approach on developing method for approximate evaluation of the fundamental frequency may be further considered in application to other cross-section shapes. Dependency of the approximation's precision and usability on the adjustment factor can be examined in future as well.

Acknowledgments

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