Waring Problem as an Issue of Polynomial Computer Algebra

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Abstract. In its original XVIII century form the classical Waring problem consisted in finding for each natural k the smallest such $s = g(k)$ that all natural numbers n can be written as sums of s non-negative k-th powers, $n = x_1^k + \ldots + x_s^k$. In the XIX century the problem was modified as the quest of finding such minimal $s = G(k)$ that *almost all* n can be expressed in this form. In the XX century this problem was further specified, as for finding such $G(k)$ *and* the precise list of exceptions. In the present talk I sketch the key steps in the solution of this problem, with a special emphasis on algebraic and computational aspects. I describe various connections of this problem, and its modifications, such as the rational Waring problem, the easier Waring problem, etc., with the current research in polynomial computer algebra, especially with identities, symbolic polynomials, etc. and promote several outstanding computational challenges.

Introduction

In this talk I plan to describe the status of the classical Waring problem, its versions and variants. The XVIII century Waring problem has been mostly solved. Not by Hilbert in 1909, of course, as many people misguidedly believe, but mostly by Dickson in 1936 (the outstanding small cases $k = 6, 5, 4$ were then settled in 1940, 1964 and 1984, respectively, see § 1 and § 6 for details). But already its XIX century version suggested by Jacobi, not to say all other major XIX and XX century variations, are widely open, as of today.

My objective is to attract attention to some algebraic and computational aspects of the Waring problem in the spirit of reconnecting with the goddess Namakkal, as described in [39]. Here I focus mostly on the related polynomial and rational identities, conjectural answers, and explicit lists of exceptions, many more details and further aspects can be found in [40, 41].

1. Waring problem

Here we take a quick glance at some facets of what is known as the Waring problem. There are many further aspects to be featured in a more systematic treatment, as well as oodles of various generalisations and related problems, some of them mentioned towards the end of the present abstract and discussed in [40, 41].

1.1. Original Waring problem

Guided by the analogy with Lagrange's four square theorem and scarce numerical evidence in 1770–1772 Waring and J. A. Euler (= Euler jr.) proposed what later became known as the [classical] Waring problem, see [13].

• **Waring problem**. Find for each natural k the smallest $s = g(k)$ such that every natural number n can be expressed as the sum of k-th powers of *non-negative* integers

$$n = x_1^k + \ldots + x_s^k,$$

with s summands.

Actually, Waring conjectured that $g(3) = 9$ and $g(4) = 19$, while J. A. Euler made similar prediction for *all* values of $g(k)$:

$$g(k) = 2^k + q + 2,$$

where $3^k = q \cdot 2^k + r$, $1 \leqslant r \leqslant 2^k - 1$, = the **ideal Waring theorem**.

In this form Waring problem was *essentially* solved in 1909–1984.

▪ In 1909 Wieferich [47] established that $g(3) = 9$, gaps in his proof were later filled up by Kempner [23] in 1912 and by Dickson in 1927.

▪ For $k \geqslant 7$ the problem was solved by Dickson [14, 15] and Pillai in 1936, modulo the **Pillai conjecture** that $q + r \leqslant 2^k$. They also compute the precise value of $g(k)$ when Pillai conjecture fails. But there is every reason to believe that Pillai conjecture holds for all k. Firstly, it may fail at most for finitely many values of k. Secondly, it holds for all $k < 5 \cdot 10^8$. And there is much more compelling evidence than that.

▪ The three remaining values $g(6) = 73$, $g(5) = 37$ and $g(4) = 19$ were computed by Pillai [27] in 1940, by Chen Jing-run [4] in 1964, and by Balasubramanian, Deshouillers and Dress [1] in 1984, respectively.

1.2. Asymptotic Waring problem

However, in XIX–XX centuries this problem was remodeled as follows.

• **Asymptotic Waring problem**. Find for each natural k the smallest s such that *almost all* natural numbers n can be expressed as the sum of k-th powers of *non-negative* integers $n = x_1^k + \ldots + x_s^k$, with s summands.

Clearly, the specific purport of this problem depends on the precise meaning of the expression *almost all*. The two most common interpretations are as follows:

▪ **According to Jacobi** as "all, except a finite number" = "all starting from a certain value". The corresponding minimal s is denoted by $G(k)$.

▪ **According to Hardy—Littlewood** in the sense of *natural density*. There can be inifinitely many exceptions, but they become progressively more rare, their number grows as $o(n)$. The corresponding minimal s is denoted by $G^+(k)$.

However apart from the case of squares $G^+(2) = G(2) = g(2) = 4$ that was already known to Lagrange, and a few more values such as $G(4) = 16$, as of today, the asymptotic Waring problem is very far from being solved in either sense.

1.3. Algorithmic Waring problem

However, with the advent of computers this problem was reformulated once again as something terribly much more ambitious.

• **Waring problem, XX century version**. Find $G(s)$ as above and the explicit list of exceptions. Construct an algorithm that for a given n finds a shortest expression of n as the sum of k-th powers (or, preferably, all such expressions).

In this form the problem seems to be quite recalcitrant. The *only* non-trivial case, for which Waring problem is fully *solved* in this form is that of biquadrates, see § 6. The only other case, for which the problem is fully *stated* in this form is that of cubes. However, for cubes we are nowhere near its solution and even the statement itself required thumping calculations, see § 5. For fifth powers it seems we are not even close to being able to *state* the problem in this form, see § 7.

1.4. Easier Waring problem

In the 1930-ies several mathematicians started to systematically consider the following version of Waring problem, which turned out to be *much* harder than the original Waring problem and is still unsolved even today.

• **Easier Waring problem**. Find for each natural k the smallest $s = v(k)$ such that all natural numbers n can be expressed as sums/differences of k-th powers of integers

$$n = \pm x_1^m \pm x_2^m \pm \ldots \pm x_s^m.$$

This is what Hardy and Wright call "sums affected with signs" and what Habsieger renamed **signed Waring problem**. They prove an obvious bound $v(k) \leqslant 2^k + (k!)/2$, [20], Theorems 400 and 401. There is a much better upper bound $v(k) \leqslant G(k) + 1$, of course. However, the explicit value of $v(k)$ is not known even for $k = 3$.

1.5. Rational Waring problem

Actually, there are further versions of Waring problem, also known since the early XIX century.

• **Rational Waring problem**. Find for each natural k the smallest $s = \rho(k)$ such that every rational number x can be expressed as sums/differences of k-th powers of rational numbers $n = x_1^k \pm \ldots \pm x_s^k$.

• **Positive rational Waring problem**. Find for each natural k the smallest s such that every positive rational number x can be expressed as the sum of k-th powers of non-negative rational numbers $x = x_1^k + \ldots + x_s^k$.

These problems are closely related to another classical problem.

• **Waring problem at zero**. Find for each natural k the smallest $s = \theta(k)$ such that 0 can be *non-trivially* expressed as sums/differences of k-th powers of integers $\pm x_1^m \pm x_2^m \pm \ldots \pm x_s^m = 0$.

The existence of Pythagorean triples implies that $\theta(2) = 3$. The great Fermat theorem is the claim that $\theta(k) \geqslant 4$ for all $k \geqslant 3$. Using the geometry of elliptic curves, Fermat and Euler have proven that indeed Fermat equation $x^3 + y^3 = z^3$ has no non-trivial solutions, and thus $\theta(3) = 4$. Euler even made a much stronger claim that $\theta(k) \geqslant k + 1$, but that turned out to be both wrong and false. In particular, already $\theta(4) = 4$. Similarly, $\theta(5) \leqslant 5$, but it is unknown, whether the precise value is 4 or 5.

2. Polynomial identities in the classical Waring problem

Here we display some assorted classical identities used to estimate $g(k)$. In my view, they deserve a serious further scrutiny, and with the tools of polynomial computer algebra we can now start a systematic search for new such identities.

2.1. Tardy type identities

Already in Euclid's "Elements" one can find the identity $4xy = (x+y)^2 - (x-y)^2$, later reproduced by Diophantus. Gauss generalised it to cubes

$$24xyz = (x + y + z)^3 - (x + y - z)^3 - (x - y + z)^3 + (x - y - z)^3.$$

In 1851 Tardy observed a similar identity for biquadrates

$$\begin{aligned}192xyzw = {} & (x + y + z + w)^4 - (x + y + z - w)^4 - (x + y - z + w)^4 \\ & - (x - y + z + w)^4 + (x + y - z - w)^4 + (x - y + z - w)^4 \\ & + (x - y - z + w)^4 - (x - y - z - w)^4\end{aligned}$$

and all further powers, and thus gave the first solution of the *rational* Waring problem. This was clearly the starting point for Liouville and all subsequent development (Tardy was his student in Paris). Tardy identities were then rediscovered by Boutin in 1910.

2.2. Liouville type identities

The first non-trivial estimate for $g(k)$ for any $k \geqslant 3$ in Waring problem was obtained by Liouville some time before 1859, who proved that $g(4) \leqslant 53$. His proof begins with the following identity. Let $2n = x^2 + y^2 + z^2 + w^2$, then

$$\begin{aligned}6n^2 = {} & x^4 + y^4 + z^4 + w^4 + \Big((x + y + z + w)/2\Big)^4 + \Big((x + y + z - w)/2\Big)^4 + \\ & \Big((x + y - z + w)/2\Big)^4 + \Big((x + y - z - w)/2\Big)^4 + \Big((x - y + z + w)/2\Big)^4 + \\ & \Big((x - y + z - w)/2\Big)^4 + \Big((x - y - z + w)/2\Big)^4 + \Big((x - y - z - w)/2\Big)^4.\end{aligned}$$

Later, Hurwitz and Venkov gave an interpretation of this identity in terms of integral quaternions, whereas Lucas has rewritten it in the form

$$6\big(x_1^2+x_2^2+x_3^2+x_4^2\big)^2=\sum(x_i+x_j)^4+\sum(x_i-x_j)^4,$$

where both sums in the right-hand-side are taken over all $1\leqslant i<j\leqslant 4$. Clearly, Lucas identity readily generalises:

$$6\big(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\big)^2=\sum(x_i+x_j)^4+\sum(x_i-x_j)^4-2\sum x_h^4,$$

$$6\big(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2\big)^2=\sum(x_i+x_j)^4+\sum(x_i-x_j)^4-4\sum x_h^4,$$

and similarly for any m, where the sums are taken over $1\leqslant i<j\leqslant m$, $1\leqslant h\leqslant m$.

2.3. Maillet and Wieferich identities

To give a first non-trivial estimate of $g(3)$ Maiilet used the following identity:

$$6x(x^2+y^2+z^2+w^2)=(x+y)^3+(x-y)^3+(x+z)^3+(x-z)^3+(x+w)^3+(x-w)^3.$$

He himself derived this identity differently, but retrospectively, it is simply the derivative of the Liouville identity in Lucas form. Later, Linnik has used a more general identity

$$\begin{aligned}4(x_1^3+y_1^3+x_2^3+y_2^3+x_3^3+y_3^3)&=(x_1+y_1)^3+(x_2+y_2)^3+(x_3+y_4)^3+\\&3\big((x_1+y_1)(x_1-y_1)^2+(x_2+y_2)(x_2-y_2)^2+(x_3+y_3)(x_3-y_3)^2\big).\end{aligned}$$

in his proof of the seven cube theorem.

Later, Maillet obtained a similar estimate for *fifth* powers, and Wieferich [46] explicitly produced the corresponding identity

$$\begin{aligned}&2x\Big(2^2\cdot 3\cdot 5(43x^2+y^2+z^2+w^2)^2-2^2\cdot 1579x^4\Big)=\\&(8x+y)^5+(8x-y)^5+(8x+z)^5+(8x-z)^5+(8x+w)^5+(8x-w)^5+\\&(x+y+z+w)^5+(x+y+z-w)^5+(x+y-z+w)^5+(x-y+z+w)^5+\\&(x+y-z-w)^5+(x-y+z-w)^5+(x-y-z+w)^5+(x-y-z-w)^5.\end{aligned}$$

In the same paper Wieferich used also a similar identity for *seventh* powers, which we do not reproduce here.

2.4. Fleck, Hurwitz and Schur identities

In 1907 Fleck came up with a similar identity for the 6-th powers,

$$\begin{aligned}60(x^2+y^2+z^2+w^2)^3&=36(x^6+y^6+z^6+w^6)+\\&2\big((x+y)^6+(x-y)^6+\ldots+(z+w)^6+(z-w)^6\big)+\\(x+y+z)^6+(x-y+z)^6&+(x+y-z)^6+(x-y-z)^6+\ldots+(y-z-w)^6,\end{aligned}$$

there are 12 summands in the second line (the choice of a pair, and a sign), and 16 summands in the third line (the choice of a triple and two *independent* choices of signs), 32 summands in total.

The same year Hurwitz has discovered the identity for 8-th powers,

$$5040(x^2+y^2+z^2+w^2)^4 = 6\big((2x)^8+(2y)^8+(2z)^8+(2w)^8\big)+$$
$$60\big((x+y)^8+(x-y)^8+\ldots+(z+w)^8+(z-w)^8\big)+$$
$$(2x+y+z)^8+(2x-y+z)^8+(2x+y-z)^8+(2x-y-z)^8+\ldots+(-y-z+2w)^{10}+$$
$$6\big((x+y+z+w)^8+(x+y+z-w)^8+\ldots+(x-y-z-w)^8\big).$$

and conjectured the existence of such similar identities expressing [some multiple of] $(x^2+y^2+z^2+w^2)^k$ as the sum of $2k$-th powers of linear forms in x,y,z,w for all k. The next such identity was indeed constructed the same year by Schur,

$$22680(x^2+y^2+z^2+w^2)^5 = 9\big((2x)^{10}+(2y)^{10}+(2z)^{10}+(2w)^{10}\big)+$$
$$180\big((x+y)^{10}+(x-y)^{10}+\ldots+(z+w)^{10}+(z-w)^{10}\big)+$$
$$(2x+y+z)^{10}+(2x-y+z)^{10}+(2x+y-z)^{10}+(2x-y-z)^{10}+\ldots+(-y-z+2w)^{10}+$$
$$9\big((x+y+z+w)^{10}+(x+y+z-w)^{10}+\ldots+(x-y-z-w)^{10}\big),$$

Observe that these identities have 12 summands in the second line (the choice of a pair and a sign), 48 summands in the third line (the choice of one position out of four for the coefficient 2, the choice of one of the three remaining positions for the coefficient 0 and two *independent* choices of signs), and, finally, 8 summands in the last line (three *independent* choices of signs for all positions other than the first one), 72 summands in total.

2.5. Hilbert type identities

In 1909 Hilbert [21] solved a cheap version of the classical Waring problem = mere finiteness of $g(k)$, without computing the actual value, or actually providing *any* estimate of $g(k)$. As part of his solution, Hilbert verified the above Hurwitz conjecture. In fact, he has proven that there *exist* identities expressing k-th power of the sum of m squares as *positive* linear combinations of $q=\binom{2k+1}{m}$ expressions which are $(2k)$-th powers of *linear forms* :

$$a(x_1^2+\ldots+x_m^2)^k = a_1(b_{11}x_1+\ldots+b_{1m}x_m)^{2k}+\ldots+a_q(b_{q1}x_1+\ldots+b_{qm}x_m)^{2k},$$

where $a, a_i \in \mathbb{N}$ and $b_{ij}\in\mathbb{Z}$, for $1\leqslant i\leqslant q$, $1\leqslant j\leqslant m$.

Actually in his solution of the cheap Waring problem Hilbert only used the identities for $m=5$, but his method is quite general and allows to prove the *existence* of similar identities **Hilbert identities** for arbitrary m and k. His proof is a pure existence proof and, in its original form, does not give any estimate on the size of the coefficients.

A posteriori, many further such identities were explicitly written. Say, by Kürschak [24] in 1911, for $k=2$ and $m\equiv 1 \pmod 3$:

$$60\big(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2\big)^2 = \sum(x_i\pm x_j\pm x_h)^4,$$
$$672\big(x_1^2+x_2^2+x_3^2+\ldots+x_9^2+x_{10}^2\big)^2 = \sum(x_i\pm x_j\pm x_h\pm x_l)^4,$$

etc. By Kempner [23] in 1912, for $m = 4$ and $k = 6, 7$. Note also the next Fleck identity

$$60\big(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\big)^3 = \sum (x_i \pm x_j \pm x_h)^6 + 36 \sum x_l^6,$$

and the like. Some *estimates* on the size of coefficients in Hilbert identities were later produced by Rieger, Pollack and Nesterenko [34, 28, 26] in the process of effectivisation of Hilbert's proof, but they have not attempted to come up with the actual coefficients.

The following problem seems to be extremely significant not just as a direct mathematical and computational challenge, but also as a methodological, historical and philosophical issue.

Problem 1. *Can one solve the original Waring problem with Hilbert's approach*?

If mathematics is what we think it is, this should be possible. Personally, I would feel very disappointed should Wieferich [47] proof of the equality $g(3) = 9$ and the estimate $g(4) \leqslant 30$ Dress [16] remain the best and only partial solutions obtained along these lines.

However, it will be by no means easy.

Problem 2. *Implement a systematic computer search of Hilbert type and similar identities with small coefficients.*

3. Polynomial identities in the easier and rational Waring problems

Identities that allow to give estimates of $v(k)$ and $\rho(k)$ are shorter and [in a sense] easier, but much less understood, than the identities used to estimate $g(k)$.

3.1. Richmond identity, Norrie identity, and beyond

Actually, Tardy identities and all such further uniform series of identities give vastly exaggerated upper bounds for $\rho(k)$ in the rational Waring problem. So far, getting the best possible estimate required separate clever identities in each individual case.

Below we reproduce two such classical identities, stemming from 1920-ies, **Richmond identity** for cubes

$$x = \left(\frac{x^3 - 3^6}{3^2x^2 + 3^4x + 3^6}\right)^3 + \left(\frac{-x^3 + 3^5x + 3^6}{3^2x^2 + 3^4x + 3^6}\right)^3 + \left(\frac{3^3x^2 + 3^5x}{3^2x^2 + 3^4x + 3^6}\right)^3$$

and **Norrie identity** for biquadrates

$$x = \left(\frac{a^2(a^8 - b^8 + 2x)}{2(a^8 - b^8)}\right)^4 - \left(\frac{a^2(a^8 - b^8 - 2x)}{2(a^8 - b^8)}\right)^4 + \left(\frac{2a^4x - b^4(a^8 - b^8)}{2ab(a^8 - b^8)}\right)^4 - \left(\frac{2a^4x + b^4(a^8 - b^8)}{2ab(a^8 - b^8)}\right)^4.$$

There were similar identities for $k = 5, 6, 7, 8, 9$, but they are constructed ad hoc, and there is no clear pattern as to their shape. Compare, in particular, Choudhry or Reynia [5, 6, 7, 32, 33].

3.2. Rao and Vaserstein identities, and beyond

However, the works by Habsieger [18, 19] give some hope. Imitating the classical Rao identity for *sixth* powers,

$$\begin{aligned}12abcd(c^4-d^4)(a^{24}-b^{24})x = {}&\\ &(a^5c+bdx)^6+(a^5d-bcx)^6+(b^5c-adx)^6+(b^5d+acx)^6\\ &\quad-(a^5c-bdx)^6-(a^5d+bcx)^6-(b^5c+adx)^6-(b^5d-acx)^6.\end{aligned}$$

Vaserstein [37] discovered a similar identity for *eighth* powers. Habsieger [19] has rewritten Vaserstein identity in the following more symmetric form:

$$\begin{aligned}16(uvw)^6(u^{48}v^{64}+v^{48}w^{64}+w^{48}u^{64}-u^{48}w^{64}-v^{48}u^{64}-w^{48}v^{64})y = {}&\\ &(u^7v^{10}+u^5w^6y)^8+(u^7w^{10}-u^5v^6y)^8+(v^7w^{10}+v^5u^6y)^8\\ &+(u^7u^{10}-v^5w^6y)^8+(w^7u^{10}+w^5v^6y)^8+(w^7v^{10}-w^5u^6y)^8\\ &-(u^7v^{10}-u^5w^6y)^8-(u^7w^{10}+u^5v^6y)^8-(v^7w^{10}-v^5u^6y)^8\\ &\quad-(u^7u^{10}+v^5w^6y)^8-(w^7u^{10}-w^5v^6y)^8-(w^7v^{10}+w^5u^6y)^8\end{aligned}$$

At this point the link to the representation theory of finite groups becomes obvious, and Habsieger [18, 19] is able to construct many similar **symmetric identities**.

Problem 3. *Is it possible to construct series of rational identities of all degrees that would give correct bound in the easier Waring problem and in the rational Waring problem*?

3.3. Becker type identities

In 1979 Eberhard Becker constructed analogues of Hilbert identities

$$(x_1^l+\ldots+x_m^l)^k = f_1(x_1,\ldots,x_m)^{lk}+\ldots+f_q(x_1,\ldots,x_m)^{lk}$$

for arbitrary k, l, m. However, for $l \geqslant 4$ the f_j's here have to be rational functions rather than polynomials. I they had been polynomials, they are bound to be linear forms, which immediately leads to a contradiction.

I plan to demonstrate some such explicit identities in my talk.

3.4. Frolov type identities

There are another type of identities that were used in the easier Waring problem, which come from the solution of Prouhet—Tarry—Escott problem, and which oftentimes lead to better bounds for $v(k)$, than the bounds obtained via the above symmetric identities.

Recall that, given natural numbers s and k, the Prouhet—Tarry—Escott problem (or simply PTE for short) asks with $s > k$, whether there are distinct multisets of integers, say $X = [x_1, \ldots, x_s]$ and $Y = [y_1, \ldots, y_s]$, such that

$$x_1^i + \ldots + x_s^i = y_1^i + \ldots + y_s^i, \qquad j = 1, \ldots, k.$$

Hosts of special/partial solutions to this problem were constructed in the late XIX century and in the early XX century.

The relevance of PTE resides in the fact that every such solution leads to the corresponding **Frolov identity**

$$(t + x_1)^k + \ldots + (t + x_s)^k = (t + y_1)^k + \ldots + (t + y_s)^k.$$

These and similar identities were extensively used by Demianenko, Revoy [30, 31] and others to obtain sharper bounds in the easier Waring problem.

4. Vinogradov's method

In the early 1920-ies Hardy and Littlewood considered the generating function

$$f_{k,N}(z) = 1 + z^{1^k} + z^{2^k} + z^{3^k} + \ldots$$

Then the coefficient $r_{k,s}(n)$ of z^n in the series

$$f_{k,N}(z)^s = 1 + \sum_{n=1}^{\infty} r_{k,s}(n) z^n$$

equals the number of representations of n as the sum of k-th powers of s non-negative integers. In particular, the original Waring conjecture is equivalent to the claim that $r_{3,9}(n) \neq 0$, that $r_{4,19}(n) \neq 0$, that $r_{5,37}(n) \neq 0$, etc., for all natural n.

Side remark. Actually, Hardy and Littlewood considered a slightly different generating function, namely

$$f_{k,N}(z) = 1 + 2z^{1^k} + 2z^{2^k} + 2z^{3^k} + \ldots$$

But this is pure fetishism, explained by the fact that for $k = 2$ such a choice of the generating function leads to the Jacobi theta-function, and explicit computation of $r_{2,s}(n)$. We do not know, what could be a correct choice of the coefficients in the generating function that would produce a similar theory for higher degrees. If we do not attempt to calculate explicit values, but are interested only in the **asymptotic behaviour** of $r_{k,s}(n)$, the specific choice of the generating function does not play any role anyway.

As a function complex variable $z \in \mathbb{C}$ this series converges inside the unit disk, but the circle $|z| = 1$ consists entirely of singular points. The idea of the **circle method** is to use the Cauchy formula

$$r_{k,s}(n) = \frac{1}{2\pi i} \int_C \frac{f_k(z)^s}{z^{n+1}} dz,$$

where C is the circle of radius $0 < \rho < 1$, and then to *estimate* this integral when $\rho \longrightarrow 1$, using the character of singularities on the unit circle.

In the late 1920-ies Vinogradov proposed a radical simplification onf this method. Namely, he noticed that if we are interested in the number of representations of a *specific* n as the sum of s non-negative k-th powers, then we do not have to look at the whole generating function, as Hardy and Littlewood did. In fact, the whole infinite tail of the generating function does not play any role, we can limit ourselves with the *polynomial*

$$f_{k,N}(z) = 1 + z^{1^k} + z^{2^k} + \ldots + z^{N^k}.$$

Then the coefficient $r_{k,s}^N(n)$ of z^n in the *polynomial*

$$f_{k,N}(z)^s = 1 + \sum_{n=1}^{sN} r_{k,s}^N(n) z^n$$

equals the number of representations of n as the sum of k-th powers of $\leqslant s$ integers $1 \leqslant m \leqslant N$.

Clearly, the integers m such that $m^k > n$ cannot occur in such a representation. Thus, for any $N \geqslant \sqrt[k]{n}$ one has $r_{k,s}^N(n) = r_{k,s}(n)$. Thus, in Vinogradov's method the passage to limits still occurs, but now we can from the onset assume that $\rho = 1$ and calculate the limit as $N \longrightarrow \infty$, which is a dramatic technical simplification.

It was precisely this idea that allowed to improve bounds on $G(k)$ from exponential in k to polynomial in k (and, eventually, to almost linear in k). It was precisely the huge gap between the expected exponential bound for $g(k)$ and the polynomial bound for $G(k)$ that allowed to apply Dickson's ascent.

Problem 4. *Can one solve the original Waring problem as a problem of polynomial computer algebra by directly verifying that for any k and n and any $N \geqslant \sqrt[k]{n}$ there exists an s such that $r_{k,s}^N(n) \neq 0$?*

5. Algorithmic Waring problem for cubes

In 1909 Landau proved by the methods of *elementary* analytic number theory that $G(3) \leqslant 8$, in other words, almost all positive integers are sums of $\leqslant 8$ positive cubes. Indeed, in 1939 Dickson established that the only positive integers that require 9 cubes are 23 and 239. In 1943 Linnik proved his famous seven cubes theorem asserting that $G(3) \leqslant 7$. A few years ago this result was made explicit.

5.1. Experimental evidence for cubes.

Based on extensive computer calculations, asymptotics in the Hardy—Littlewood theory, and probabilistic trials Romani stated the following conjectures [35].

- **Problem of seven cubes.** There are exactly 15 natural numbers that can be expressed as sums of *eight*, but not of *seven* non-negative cubes, the largest of them being 454.

▪ **Problem of six cubes.** There are exactly 121 natural numbers that can be expressed as sums of *seven*, but not of *six* non-negative cubes, the largest of them being 8042.

▪ **Problem of five cubes.** There are exactly 3922 natural numbers that can be expressed as sums of *six*, but not of *five* non-negative cubes, the largest of them being 1290740.

As a further evidence for that in 1999 Bertault, Ramaré and Zimmerman [2] established that all integers between 1290740 and $3.375 \cdot 10^{12}$ can be expressed as sums of five cubes, which by Dickson's ascent implies that all 455 and $2.5 \cdot 10^{26}$ can be expressed as sums of seven cubes. The same year Deshouillers, Hennecart and Landreau [10] extended these calculations to 10^{16}.

The largest natural number known today that requires exactly five cubes is 7373170279850. In 1999 Deshouillers, Hennecart and Landreau [10] stated the ollowing conjecture (l. c., Conjectures 1 and 2):

▪ **Problem of four cubes.** There are exactly 113936676 natural numbers that can be expressed as sums of *five*, but not of *four* non-negative cubes, the largest of them being 7373170279850.

5.2. Problem of seven cubes.

In 2005 Ramaré published yet another effectivisation of Linnik's theorem: all integers

$$n \geqslant e^{205000} \approx 2.3377074809 \cdot 10^{89030}.$$

can be expressed as sums of seven cubes. The improvement was based on the **Bombieri identity**

$$\begin{aligned}2(u^6v^6 + u^6w^6 + v^6w^6)a^3 + 6au^2v^2w^2(x^2 + y^2 + z^2) = \\ (u^2v^2a + wx)^3 + (u^2v^2a - wx)^3 + (u^2w^2a + vy)^3 + \\ (u^2w^2a - vy)^3 + (v^2w^2a + uz)^3 + (v^2w^2a - uz)^3,\end{aligned}$$

In 2007 Ramaré [29] further dramatically improved the bound to

$$n \geqslant e^{524} \approx 3.71799 \cdot 10^{227},$$

after which it became clear that a complete solution was close.

In 2008–2009 Boklan and Elkies [3] proved the seven cube conjecture for numbers divisible by 4, and in 2010 Elkies [17] proved it for all even integers. These results essentially used both the Ramare upper bound, and the Deshouillers—Hennecart—Landreau lower bound. Finally, in 2015 Siksek announced a complete solution of the problem, which was published in 2016 in [36]. The only numbers which cannot be presented in such a form are

15, 22, 23, 50, 114, 167, 175, 186, 212, 231, 238, 239, 303, 364, 420, 428, 454.

Among other things, this work relies on dozens of thousands hours of computer time.

However, the problems of six, five and four cubes are still wi[l]d[e]ly open!

6. Algorithmic Waring problem for biquadrates

Dickson's estimate $g(4) \leqslant 35$ has not been improved for almost 40 years. However, in 1970–1971 Dress had a happy idea to return to the elementary approach with new techniques. In particular, using new polynomial identities that occurred in the solution of the easier Waring problem, and some computer calculations, he improved the bound to $g(4) \leqslant 30$ *by elementary methods*. After that things accelerated, see [40] for a detailed description.

6.1. Nineteen biquadrates.

In 1985 Deshouillers announces a complete solution of the original Waring problem in the last remaining case of biquadrates. Observe the $\geqslant 125$ year gap between the Liouville breakthrough (who proved not mere *finiteness* of $g(4)$, but established a realistic estimate!), and the final solution of the Waring problem $g(4) = 19$, as stated by Waring himself.

In 1985 Balasubramanian, Deshouillers and Dress [1] announce the general plan of such a solution. In [1] it is claimed that all integers $n \geqslant 10^{367}$ are sums of 19 biquadrates, the details were then published in [8]. Moreover in [1] the authors describe a calculation that shows that all natural numbers $n \leqslant 10^{378}$ are also sums of 19 biquadrates. Later in [9] this computation is even extended to $n \leqslant 10^{448}$. Thus, the upper and lower domains overlap by 80 orders of magnitude!

6.2. Sixteen biquadrates.

In 1939 Davenport has proven that $G(4) = 16$. Now we know that 13792 is the largest integer that requires more than 16 biquadrates, all $n \geqslant 13793$ are in fact sums of 16 biquadrates. This was shown in 1999–2005 by Deshouillers, Hennecart, Kawada, Landreau and Wooley.

Namely, in [12] it is proven that all integers $n \geqslant 10^{216}$ not divisible by 16, are sums of 16 biquadrates. The proof of this result uses new polynomial identities. Also, the authors had to rework the estimates in and around the circle method from scratch and *with explicit constants*. On the other hand, in 2000 Deshouillers, Hennecart and Landreau [11] established that all $13793 \leqslant n \leqslant 10^{245}$. are sums of 16 biquadrates. Thus, again the upper and lower domains overlap and for biquadrates we can give a *complete* answer to Waring problem. There are exactly 96 natural numbers that are not sums of 16 biquadrates, here they are:

47, 62, 63, 77, 78, 79, 127, 142, 143, 157, 158, 159, 207, 222, 223, 237,
238, 239, 287, 302, 303, 317, 318, 319, 367, 382, 383, 397, 398, 399, 447,
462, 463, 477, 478, 479, 527, 542, 543, 557, 558, 559, 607, 622, 623, 687,
702, 703, 752, 767, 782, 783, 847, 862, 863, 927, 942, 943, 992, 1007, 1008,
1022, 1023, 1087, 1102, 1103, 1167, 1182, 1183, 1232, 1247, 1248, 1327,
1407, 1487, 1567, 1647, 1727, 1807, 2032, 2272, 2544, 3552, 3568, 3727,
3792, 3808, 4592, 4832, 6128, 6352, 6368, 7152, 8672, 10992, 13792

for each one of them it is very easy to determine, whether it requires 17, 18 or 19 biquadrates.

7. The big computational challenge

As we've seen above, $k = 4$ is *the only* case (apart from that of $k = 2$, known to Lagrange back in 1770), when Waring problem has been completely solved in the XX century sense. Even in the case $k = 3$ there is a *huge* uncertainty $4 \leqslant G(3) \leqslant 7$ as to the actual value of $G(3)$ — not to say the explicit list of exceptions!

To give some idea of the computational immensity of the problem, below we reproduce the table of values of $g(k)$, $5 \leqslant k \leqslant 15$, as confronted with the *conjectural* values of $G(k)$ — with the known upper *estimates* of $G(k)$, coming mostly from the work of Vaughan and Wooley (see, for instance, [38]) somewhere in between.

k	5	6	7	8	9	10	11	12	13	14	15
$g(k)$	37	73	143	279	548	1079	2132	4223	8384	16673	33203
$G(k) \leqslant$	17	24	33	42	50	59	67	76	84	92	100
$G(k) =$	6	9	8	32	13	12	12	16	14	15	16

TABLE 1. Conjectured values of $G(k)$ for $5 \leqslant k \leqslant 15$

It would be a rather ambitious project simply to repeat with the use of computers what Dickson has accomplished *by hand* back in the 1930-ies. But of course, today we should set much higher goals, namely, to try to document the explicit lists of exceptions that require more than $G(k)$ non-negative k-th powers.

Can we do this? Say for the cases $5 \leqslant k \leqslant 20$, with which Dickson *started*? For instance, $g(5) = 37$, while $G(5) = 6$, as everybody believes, so that we have to verify one by one all values $s = 37, 36, \ldots, 7$ and towards the end of this list the possible exceptions are bound to occur well into 10^{hundreds}. So here is the warm up problem, which would show, where we are, as far as the computational power.

Problem 5. *Compute for each $s = 37, \ldots, 7$ the explicit list of natural n which can be expressed as sums of s non-negative fifth powers, and cannot be expressed as shorter such sums.*

If we can do this, about what I have some doubts, we could proceed to higher powers, and see where we have to stop. It seems to me, that the XX century form of Waring problem is well beyond our current grasp — or what's the metaphor.

Conclusion

Poincaré used to say "Il n'y a pas de problèmes résolus, il n'y a que des problèmes plus ou moins résolus". Waring problem is certainly one of the kind. Despite the egregious efforts of many generations of mathematicians, even the XVIII century Waring problem is only 99.9999% solved, and in the meantime we were able to *fully* solve the XIX–XX century forms of the problem (with an explicit list of exceptions) for a single new case, $G(4) = 16$.

Here are my principles — well, problems — if you don't like them, I have other[s]. One can ask the same questions for other fields and rings, in particular, for number rings other than $\mathbb{Z}$, for polynomial rings, fields of rational fractions, etc. (compare the recent papers by Im Bo-Hae, Larsen, and Nguyen Dong Quan Ngoc [22, 25] for a whole new look at Waring type problems, in the context of algebraic groups). There are simultaneous sums of powers, Euler problem, taxicab numbers, PTE and variants, etc. Not to say, the mixed Waring problems, the restricted Waring problems, the Waring—Goldbach problems of all sorts, the Kamke type problems, etc. And, of course, we are still not anywhere close to doing for cubes what Jacobi has done for squares, the explicit formulas for the number of representations.

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