On approximations of the sixth order with the smooth polynomial and non-polynomial splines

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***Abstract*—This paper discusses twice continuously differentiable and three times continuously differentiable approximations with polynomial and non-polynomial splines. To construct the approximation, a polynomial and non-polynomial local basis of the second level and the sixth order approximation is constructed. We call the approximation a second level approximation because it uses the first and the second derivatives of the function. The non-polynomial approximation has the properties of polynomial and trigonometric functions. Here we have also constructed a non-polynomial interpolating spline which has the first, the second and the third continuous derivative. This approximation uses the values of the function at the nodes, the values of the first derivative of the function at the nodes and the values of the second derivative of the function at the ends of the interval $[a, b]$. The theorems of the approximations are given. Numerical examples are given.**

***Index Terms*—smooth polynomial splines, smooth non-polynomial splines**

I. INTRODUCTION

It is noted in paper [1] that spline functions are a developing field of the function approximation and digital analysis theory.

Much attention is paid to the construction of smooth splines in the papers by prof. Y.K.Demjanovich (see [2], [3]).

Hermite splines are often used to solve various problems (see [4]–[7]) Currently, there is an increasing interest not only in the use of Hermite-type splines, but also in the design of such splines with new additional properties. Hermite interpolation conditions are used to construct plane curves in paper [8]. Cubic Hermitian splines are widely known and often used. Cubic Hermite curves are adopted in conjunction with the level set method to represent curved interfaces in paper [9]. The third-order Hermite interpolation is used in [10].

In paper [11] the authors construct spline interpolation with the property of monotonicity and convexity preservation, using two types of splines: the Cubic Spline (CS), and the Hermite Cubic Rational polynomial Spline (CRS). Both curves are based on the shape preserving the Hermite Variable Degree Spline (VDS).

Of particular interest are monotonicity-preserving interpolants. They are used in engineering or computer aided design applications. Some new methods to design the monotone cubic Hermite interpolants for uniform and non-uniform grids are presented and analyzed (see Aràndiga (2013)). These methods consist of calculating the derivative values, introducing the weighted harmonic mean and a non-linear variation. With these changes, the methods obtained are third-order accurate, except in extreme situations. In paper [12] a new general mean is used and a third-order interpolant for all cases is gained.

In 1964 Schoenberg introduced trigonometric spline functions and proved the existence of locally supported trigonometric spline and B-spline functions [13]. In some cases, the use of trigonometric splines is preferable to the use of polynomial splines.

This paper continues the series of papers on approximation with local polynomial and non-polynomial splines (see [14]–[18]). The proposed paper offers non-polynomial splines of the Hermite type with the sixth order approximation of the second level (height), as well as smooth non-polynomial splines. The construction of these splines uses the functions of the Chebyshev system. These non-polynomial splines solve the Hermite interpolation problem.

These local basis functions can be used in solving problems of the mean-square approximation, solving boundary value problems by the variational-difference method, and solving integral equations.

II. SIXTH-ORDER SPLINE APPROXIMATION

Let n be a positive number, $n > 3$, and a, b real numbers. Let function $u(x)$ be such that $u \in C^6([a, b])$. The nodes $x_j \in [a, b]$, $j = 0, \dots, n$, such that $a \leq \dots < x_{j-1} < x_j < x_{j+1} < \dots \leq b$. The formulas of the basis splines of the second level and the sixth order of approximation $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$, $w_{j,2}(x)$, $w_{j+1,2}(x)$ on an interval $[x_j, x_{j+1}]$ are obtained by solving the following system of equations:

$$\varphi_i(x_j)w_{j,0}(x) + \varphi_i(x_{j+1})w_{j+1,0}(x) + \varphi_i'(x_j)w_{j,1}(x) + \varphi_i'(x_{j+1})w_{j+1,1}(x) + \varphi_i''(x_j)w_{j,2}(x) + \varphi_i''(x_{j+1})w_{j+1,2}(x) = \varphi_i(x), i = 0, 1, 2, 3, 4, 5. \quad (1)$$

The system of functions φ_i should be the Chebyshev system on the interval $[\alpha, \beta]$, where α, β are real numbers, $\beta > \alpha$. Based on different systems φ_i we will obtain different basis functions $w_{j,0}(x)$, $w_{j+1,0}(x)$, $w_{j,1}(x)$, $w_{j+1,1}(x)$, $w_{j,2}(x)$, $w_{j+1,2}(x)$. We construct the approximation of function $u(x)$ with these splines on the interval $[x_j, x_{j+1}]$ in the form:

$$U(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x)+$$

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$$u'(x_j)w_{j,1}(x)+u'(x_{j+1})w_{j+1,1}(x)$$
$$+u''(x_j)w_{j,2}(x)+u''(x_{j+1})w_{j+1,2}(x). \qquad (2)$$

The following theorem is valid if $\varphi_i(x) = x^i$, $i = 0,1,2,3,4,5$.

Theorem 1. Let function $u(x)$ be such that $u \in C^6([a,b])$, $\varphi_i(x) = x^i$, $i = 0,1,2,3,4,5$. Suppose the ordered distinct nodes $\{x_k\}$ are $x_{j+1} - x_j = h$. Then for $x \in [x_j, x_{j+1}]$ we have

$$|\, u(x) - U(x)\,| \le K_1 h^6 \|u^{VI}\|_{[x_j,x_{j+1}]}, \quad K_1 \approx 0.015625/6!.$$

Proof. On the interval $[x_j, x_{j+1}]$ we have the following relations $U(x_j) = u(x_j)$, $U'(x_j) = u'(x_j)$, $U''(x_j) = u''(x_j)$, $U(x_{j+1}) = u(x_{j+1})$, $U'(x_{j+1}) = u'(x_{j+1})$, $U''(x_{j+1}) = u''(x_{j+1})$. Thus, we can construct the Hermite interpolation polynomial $U(x)$ on the interval $[x_j, x_{j+1}]$ using (2) with the basis functions: $w_{j,0}(x_j + th) = -(6t^2 + 3t + 1)(t-1)^3$, $w_{j+1,0}(x_j + th) = t^3(10 - 15t + 6t^2)$, $w_{j,1}(x_j + th) = -th(3t+1)(t-1)^3$, $w_{j+1,1}(x_j + th) = -ht^3(t-1)(3t-4)$. $w_{j,2}(x_j + th) = -(1/2)t^2h^2(t-1)^3$, $w_{j+1,2}(x_j + th) = (1/2)h^2t^3(t-1)^2$. These basis functions can be obtained from (1) when $\varphi_i(x) = x^i$ and $x = x_i + th$, $t \in [0,1]$. Using the theorem of the error of Hermite interpolation we get that the error of the interpolation will be the following: $U(x) - u(x) = \frac{u^{VI}(\xi)}{6!}(x - x_j)^3(x - x_{j+1})^3$. Here $\xi = \xi(x)$, $\xi \in [x_j, x_{j+1}]$. If we put $x = x_j + th$, $t \in [0,1]$, we obtain $(x - x_j)^3(x - x_{j+1})^3 = h^6t^3(t-1)^3$. It is easy to obtain $\max\limits_{t\in[0,1]} |t^3(t-1)^3| = 0.015625$. The proof is complete.

Remark. Note that the system of equations (1) is ill-posed. The condition number increases rapidly with decreasing distance between grid nodes. For example, let M be the matrix of the system (1). Simple calculations give the following values for the polynomial case when $j = 0$:

1) when $h = 1$, $ConditionNumber(M) = 832$;

2) when $h = 0.01$ we have $ConditionNumber(M) = 0.247 \cdot 10^{12}$;

3) when $h = 0.00001$ we have $ConditionNumber(M) = 0.240 \cdot 10^{27}$.

Let us take the non-polynomial functions as follows: $\varphi_s(x) = \cos(sx)$, $s = 0,1,2$, $\varphi_{2+s}(x) = \sin(sx)$, $s = 1,2$, $\varphi_5(x) = x$. Simple calculations give the following values for this case when $j = 1$:

1) when $h = 1$ we have $ConditionNumber(M) = 7033.562$;

2) when $h = 0.01$, we have $ConditionNumber(M) = 0.462 \cdot 10^{14}$;

3) when $h = 0.001$ we have $ConditionNumber(M) = 0.451 \cdot 10^{19}$.

We will now find the basis splines formulas in symbolic form. Nevertheless, in some cases, it is necessary to apply this approximation with an increased number of characters in the mantissa when solving various problems with a small step.

It is easy to see that the Vronsky determinant (for the system $\varphi_s(x) = \cos(sx)$, $s = 0,1,2$, $\varphi_{2+s}(x) = \sin(sx)$, $s = 1,2$, $\varphi_5(x) = x$) is nonzero:

$$\begin{vmatrix} 1 & \sin(x) & \cos(x) & \sin(2x) & \cos(2x) & x \\ 0 & \cos(x) & -\sin(x) & 2\cos(2x) & -2\sin(2x) & 1 \\ 0 & -\sin(x) & -\cos(x) & -4\sin(2x) & -4\cos(2x) & 0 \\ 0 & -\cos(x) & \sin(x) & -8\cos(2x) & 8\sin(2x) & 0 \\ 0 & \sin(x) & \cos(x) & 16\sin(2x) & 16\cos(2x) & 0 \\ 0 & \cos(x) & -\sin(x) & 32\cos(2x) & -32\sin(2x) & 0 \end{vmatrix} =$$

=288.

It follows that the considered non-polynomial system is the Chebyshev system on any interval of the entire real axis.

Using the Maple package we will find formulas for basis functions $\tilde{w}_{j,i}$. The simplest form they have when $x_{j+1} = x_j + h$, $x = x_j + th$, $t \in [0,1]$:

$\tilde{w}_{j,0}(x_j + th) = (15\sin(h) + 18h\cos(h) + 3\sin(3h) - 18\cos(h)th + 16th - 8\sin(th) - 2h\cos(3h) - 4\sin(2th) + \sin(-3h + 2th) + 12\sin(th + h) - 8\sin(th - h) + \sin(2th + h) + 6\sin(2th - h) - 4\sin(2th - 2h) - 4\sin(2h + th) + 12\sin(-2h + th) + 2th\cos(3h) - 4\sin(th - 3h) - 16h - 12\sin(2h))/(30\sin(h) + 18h\cos(h) - 2h\cos(3h) + 6\sin(3h) - 16h - 24\sin(2h))$;

$\tilde{w}_{j+1,0}(x_j + th) = (15\sin(h) + 4\sin(2th - 2h) - \sin(2th + h) - 6\sin(2th - h) + 4\sin(2h + th) - 12\sin(-2h + th) + 18\cos(h)th - \sin(-3h + 2th) + 4\sin(th - 3h) - 16th + 8\sin(th) + 4\sin(2th) - 12\sin(th + h) + 8\sin(th - h) - 2th\cos(3h) + 3\sin(3h) - 12\sin(2h))/(30\sin(h) + 18h\cos(h) - 2h\cos(3h) + 6\sin(3h) - 16h - 24\sin(2h))$;

$\tilde{w}_{j,1}(x_j + th) = (9\cos(h) - 3\cos(-3h + 2th) + 6\cos(2th - h) - 3\cos(2th + h) + 2h\sin(-3h + 2th) - 12\cos(-2h + th) + 12\cos(2h + th) + 6th\sin(3h) - 24th\sin(2h) - 12\cos(th + h) + 24h\sin(th + h) - 8h\sin(th - 3h) + 6h\sin(2th - h) - 18h\sin(h) - 16h\sin(th) + 12\cos(th - 3h) - 6h\sin(3h) - 9\cos(3h) - 8h\sin(2th) + 30th\sin(h))/(60\sin(h) + 36h\cos(h) - 4h\cos(3h) + 12\sin(3h) - 32h - 48\sin(2h))$;

$\tilde{w}_{j+1,1}(x_j + th) = (-9\cos(h) + 9\cos(3h) + 6th\sin(3h) - 8h\sin(2th - 2h) - 24th\sin(2h) + 12\cos(th + h) + 12\cos(-2h + th) - 12\cos(2h + th) + 3\cos(-3h + 2th) + 3\cos(2th + h) - 6\cos(2th - h) - 12\cos(th - 3h) + 2h\sin(2th + h) + 6h\sin(2th - h) + 30th\sin(h) - 16h\sin(th - h) + 24h\sin(2h) - 12h\sin(h) + 24h\sin(-2h + th) - 8h\sin(2h + th))/(60\sin(h) + 36h\cos(h) - 4h\cos(3h) + 12\sin(3h) - 32h - 48\sin(2h))$;

$\tilde{w}_{j,2}(x_j + th) = (21\sin(h) + 18h\cos(h) + 2h\cos(3h) - 3\sin(3h) - 30\cos(h)th + 20th - 2\sin(2th) - 16\sin(th) + 5\sin(-3h + 2th) + 8\sin(th - h) - \sin(2th + h) + 12\sin(2th - h) - 14\sin(2th - 2h) + 4\sin(2h + th) + 12\sin(-2h + th) - 8\sin(th - 3h) - 2th\cos(3h) + 16\cos(th)h + 4\cos(2th)h + 12th\cos(2h) + 2h\cos(-3h + 2th) - 6h\cos(2th - h) - 12h\cos(th + h) - 4h\cos(th - 3h) - 20h - 6\sin(2h))/(60\sin(h) + 36h\cos(h) - 4h\cos(3h) + 12\sin(3h) - 32h - 48\sin(2h))$;

$\tilde{w}_{j+1,2}(x_j + th) = (21\sin(h) - 12h\cos(h) - 3\sin(3h) + 4h\cos(2th - 2h) - 12h\cos(-2h + th) - 4h\cos(2h + th) + 16h\cos(th - h) + 2h\cos(2th + h) + 30\cos(h)th - 20th + 14\sin(2th) - 8\sin(th) + \sin(-3h + 2th) - 12\sin(th + h) + 16\sin(th - h) - 5\sin(2th + h) - 12\sin(2th - h) + 2\sin(2th - 2h) + 8\sin(2h + th) - 4\sin(th - 3h) + 2th\cos(3h) - 12th\cos(2h) - 6h\cos(2th - h) + 12h\cos(2h) -$

$6\sin(2h))/(60\sin(h)+36h\cos(h)-4h\cos(3h)+12\sin(3h)-32h-48\sin(2h))$.

The following theorem is valid if $\varphi_s(x)=\cos(sx)$, $s=0,1,2$, $\varphi_{2+s}(x)=\sin(sx)$, $s=1,2$, $\varphi_5(x)=x$. Let $x\in[x_j,x_{j+1}]$. Using (1) we construct the approximation in the form (2).

Theorem 2. Let function $u(x)$ be such that $u\in C^6([a,b])$. Suppose the ordered distinct nodes $\{x_k\}$ are $x_{j+1}-x_j=h$, $h<1.5$. Then for $x\in[x_j,x_{j+1}]$ we have

$$\mid u(x)-U(x)\mid \le K_2h^6\|5u^{IV}+u^{VI}+4u''\|_{[x_j,x_{j+1}]}, K_2>0.$$

Proof. In the non-polynomial case when $x\in[x_j,x_{j+1}]$ we have $w_{j,i}$ which are given above. The method for finding the estimate is described in detail in [14]. Here we briefly dwell on the main points of the proof. Decomposing the determinant $Lu=$

$$\begin{vmatrix} 1 & \sin(x) & \cos(x) & \sin(2x) & \cos(2x) & x & u \\ 0 & \cos(x) & -\sin(x) & 2\cos(2x) & -2\sin(2x) & 1 & u' \\ 0 & -\sin(x) & -\cos(x) & -4\sin(2x) & -4\cos(2x) & 0 & u'' \\ 0 & -\cos(x) & \sin(x) & -8\cos(2x) & 8\sin(2x) & 0 & u''' \\ 0 & \sin(x) & \cos(x) & 16\sin(2x) & 16\cos(2x) & 0 & u^{IV} \\ 0 & \cos(x) & -\sin(x) & 32\cos(2x) & -32\sin(2x) & 0 & u^{V} \\ 0 & -\sin(x) & -\cos(x) & -32\sin(2x) & -32\cos(2x) & 0 & u^{VI} \end{vmatrix}$$

into elements of the last column, we obtain $Lu=5u^{IV}+u^{VI}+4u''$. Thus we have found a homogeneous equation $Lu=5u^{IV}+u^{VI}+4u''=0$. Now we need to find a solution to the inhomogeneous equation $Lu=f(x)$ by the method of varying arbitrary constants.

Let $u(x)=\sum_1^6 C_i(x)\varphi_i(x)$. To determine the coefficients C_i, we should solve a system of linear algebraic equations:

$$\sum_1^6 C_i'(x)\varphi_i(x)=0,$$

$$\sum_1^6 C_i'(x)\varphi_i^{(k)}(x)=0,\ k=1,\dots,4,$$

$$\sum_1^6 C_i'(x)\varphi_i^{(5)}(x)=f(x).$$

Solving this system of equations, we obtain

$$C_i'(x)=W_{6,i}(x)f(x)/W(x).$$

Here $W_{6,i}$ is the algebraic complement of the elements of the i-th column of the 7-th row of the determinant W.

Using the results from paper [14] we get $u(x)=\int_{x_j}^x(5u^{IV}(t)+u^{VI}(t)+4u''(t))(-\sin(t-x)/3+\sin(2t-2x)/24+(t-x)/4)dt+c_1+c_2\sin(x)+c_3\cos(x)+c_4\sin(2x)+c_5\cos(2x)+c_6x$, where c_i, $i=1,2,3,4,5,6$, are some arbitrary constants. Using the expression $u(x)$ and derivative of it, expression (2) with the $\tilde{w}_{j,i}$ written above, we receive the estimation of the error of the approximation with the non-polynomial splines. The proof is complete.

Remark 2. It can be obtain, that when $h\to 0$
$\tilde{w}_{j,0}(x_j+th)=-(6t^2+3t+1)(t-1)^3+O(h)$,
$\tilde{w}_{j+1,0}(x_j+th)=t^3(10-15t+6t^2)+O(h)$,
$\tilde{w}_{j,1}(x_j+th)=-th(3t+1)(t-1)^3+O(h^2)$,
$\tilde{w}_{j+1,1}(x_j+th)=-ht^3(t-1)(3t-4)+O(h^2)$,
$\tilde{w}_{j,2}(x_j+th)=-t^2h^2(t-1)^3/2+O(h^3)$,
$\tilde{w}_{j+1,2}(x_j+th)=h^2t^3(t-1)^2/2+O(h^3)$.

We have obtained the function $U(x)$, $x\in[x_j,x_{j+1}]$, such that $u(x_j)=U(x_j)$, $u'(x_j)=U'(x_j)$, $u''(x_j)=U''(x_j)$ $u(x_{j+1})=U(x_{j+1})$, $u'(x_{j+1})=U'(x_{j+1})$, $u''(x_{j+1})=U''(x_{j+1})$, using formula
$U(x)=u(x_j)\tilde{w}_{j,0}(x)+u(x_{j+1})\tilde{w}_{j+1,0}(x)+u'(x_j)\tilde{w}_{j,1}(x)+u'(x_{j+1})\tilde{w}_{j+1,1}(x)+u''(x_j)\tilde{w}_{j,2}(x)+u''(x_{j+1})\tilde{w}_{j+1,2}(x)$
on every interval $[x_j,x_{j+1}]$. Now we can construct the piecewise function $\tilde{U}(x)$, $x\in[a,b]$, such that $\tilde{U}(x)=U(x)$ for $x\in[x_j,x_{j+1}]$. This piecewise function $\tilde{U}(x)$ interpolates the function u and u', u'' at the nodes. Thus, $\tilde{U}(x)$ is a continuous function and its the first and the second derivatives are also continuous ones. Our aim is to construct a piecewise function $\tilde{\tilde{U}}(x)$ so that it will not only have the first and the second continuous derivative but also the third continuous derivative. Moreover, it will interpolate the function u in the nodes x_j. The way of constructing such piecewise functions in a polynomial case is known (see, for example, papers of Kvasov B.I., Zavyalov Yu.S., Miroshnichenko V.L). We shall construct a piecewise function $\tilde{\tilde{U}}(x)$, $x\in[a,b]$, which is equal to $u(x_j)\tilde{w}_{j,0}(x)+u(x_{j+1})\tilde{w}_{j+1,0}(x)+u'(x_j)\tilde{w}_{j,1}(x)+u'(x_j)\tilde{w}_{j+1,1}(x)+c_j\tilde{w}_{j,2}(x)+c_{j+1}\tilde{w}_{j+1,2}(x)$ on every $[x_j,x_{j+1}]$. The parameters c_j, c_{j+1} are defined by the condition that the third derivative of $\tilde{\tilde{U}}(x)$ is continuous.

The piecewise approximation $\tilde{\tilde{U}}(x)$ will be such that it is continuous and the first two derivatives of the piecewise interpolation will also be continuous. Let c_{j-1}, c_j, c_{j+1} be some parameters to be determined, and $x=x_j+th$, $t\in[0,1]$. On every interval $[x_j,x_{j+1}]$, $j=0,\dots,n-1$, we construct the approximation in the form:

$$V(t)=c_j\tilde{w}_{j,2}(t)+c_{j+1}\tilde{w}_{j+1,2}(t)+u(x_j)\tilde{w}_{j,0}(t)+$$
$$u(x_{j+1})\tilde{w}_{j+1,0}(t)+u'(x_j)\tilde{w}_{j,1}(t)+u'(x_{j+1})\tilde{w}_{j+1,1}(t). \quad (3)$$

We differentiate three times this expression and the similar one when $x\in[x_{j-1},x_j]$. After that, set them equal to each other in the common node x_j, $j=1,\dots,n-1$. Thus we construct the equation:

$$c_{j-1}\tilde{w}'''_{j-1,2}(1)+c_j\tilde{w}'''_{j,2}(1)+u(x_{j-1})\tilde{w}'''_{j-1,0}(1)+$$
$$u(x_{j+1})\tilde{w}'''_{j,0}(1)+u'(x_{j-1})\tilde{w}'''_{j-1,1}(1)+u'(x_{j+1})\tilde{w}'''_{j,1}(1)$$
$$=c_jw'''_{j,1}(0)+c_{j+1}w'''_{j+1,1}(0)+u(x_j)w'''_{j,0}(0)+$$
$$u(x_{j+1})w'''_{j+1,0}(0)+u'(x_j)w'''_{j,1}(0)+u'(x_{j+1})w'''_{j+1,1}(0),$$

where $\tilde{w}_{j-1,2}(1)=\tilde{w}_{j-1,2}(x_{j-1}+h)$, $\tilde{w}_{j,2}(1)=\tilde{w}_{j,2}(x_{j-1}+h)$, $\tilde{w}_{j-1,1}(1)=\tilde{w}_{j-1,1}(x_{j-1}+h)$, $\tilde{w}_{j,1}(1)=\tilde{w}_{j,1}(x_{j-1}+h)$, $\tilde{w}_{j-1,0}(1)=\tilde{w}_{j-1,0}(x_{j-1}+h)$, $\tilde{w}_{j,0}(1)=\tilde{w}_{j,0}(x_{j-1}+h)$ constructed when $x_{j-1}+th=x\in[x_{j-1},x_j]$. We need two extra conditions at the ends of the interval $[a,b]$. Let $V''(a)$=$u''(a)$ and $V''(b)$=$u''(b)$. Taking into account the interpolation conditions $V(x_j)=u(x_j)$, $V'(x_j)=u'(x_j)$, $j=0,\dots,n$, and the boundary conditions $V''(x_j)$=$u''(x_j)$,

$j = 0, n$, we construct the piecewise function $\tilde{\tilde{U}}(x)$. This function and its first two derivatives will be continuous. It interpolates the function u in the nodes x_j, $j = 0, \ldots, n$. So we need the expressions for the third derivative of the basis functions $\tilde{w}_{j,i}$. The third derivative of the basis functions $\tilde{w}_{j,i}(x)$ can be easily obtained.

Thus we have to solve the system of algebraic equations $GC = F$, where the square matrix $G = \{g_{ij}\}_{i,j=1}^{n-1}$, $G \in \mathbb{R}^{(n-1)\times(n-1)}$, and the vector $F \in \mathbb{R}^{n-1}$, $F = \{f\}_j$. We have $g_{j,j} = C_j$, $g_{j,j+1} = C_{j+1}$, $g_{j-1,j} = C_{j-1}$,

$f_1 = -(u(x_1)U_j + u(x_2)U_{j+1} + u(x_0)U_{j-1} + u'(x_1)U_j^1 + u'(x_2)U_{j+1}^1 + u'(x_0)U_{j-1}^1) - u''(x_0)C_{j+1}$,

$f_{n-1} = -(u(x_{n-1})U_j + u(x_n)U_{j+1} + u(x_{n-2})U_{j-1} + u'(x_{n-1})U_j^1 + u'(x_n)U_{j+1}^1 + u'(x_{n-2})U_{j-1}^1) - u''(x_n)C_{j+1}$,

$f_j = -(u(x_j)U_j + u(x_{j+1})U_{j+1} + u(x_{j-1})U_{j-1} + u'(x_j)U_j^1 + u'(x_{j+1})U_{j+1}^1 + u'(x_{j-1})U_{j-1}^1)$, $j = 2, \ldots, n-2$,

We obtain the following expressions for the non-polynomial case with equidistant nodes:

$U_{j-1} = -h(4\cos(h) - 2\cos^2(h) - 2)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$U_j = -h(-8\cos(h) + 4\cos^2(h) + 4)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h))$,

$U_{j+1} = -h(4\cos(h) - 2\cos^2(h) - 2)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$U_{j+1}^1 = -h(6\sin(h) - 3h - 6h\cos(h) + 3\sin(h)\cos(h))/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$U_{j-1}^1 = -h(3h - 6\sin(h) + 6h\cos(h) - 3\sin(h)\cos(h))/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$C_j = -h(16\cos^2(h) - 8\cos(h) + 6h\sin(h)\cos(h) + 6h\sin(h) - 8)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$C_{j+1} = -h(4\cos(h) + \cos^2(h) + 3h\sin(h) - 5)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$,

$C_{j-1} = -h(4\cos(h) + \cos^2(h) + 3h\sin(h) - 5)/(4(h\cos^2(h) - 3\sin(h)\cos(h) + h\cos(h) - 2h + 3\sin(h)))$.

TABLE I

THE ACTUAL ERRORS OF THE SMOOTH POLYNOMIAL AND NON-POLYNOMIAL APPROXIMATIONS, $h = 0.4$, $[a, b] = [-1, 1]$.

Function $u(x)$	Polynomial splines	Non-polynomial splines
$\sin(3x)$	$0.772 \cdot 10^{-4}$	$0.387 \cdot 10^{-4}$
$\sin(7x) - \cos(9x)$	$0.249 \cdot 10^{-1}$	$0.236 \cdot 10^{-1}$
$x^7 - x^9$	$0.110 \cdot 10^{-2}$	$0.121 \cdot 10^{-2}$
$1/(1 + 25x^2)$	$0.358 \cdot 10^{-1}$	$0.361 \cdot 10^{-1}$

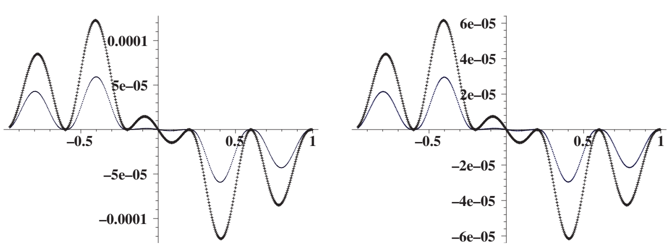

Fig. 1. Plots of the error of the approximation of the function $\sin(3x)$ with the polynomial splines and with the smooth polynomial splines (left), and with the non-polynomial splines and with the smooth non-polynomial splines (right).

Expressions for the polynomial case can be found in a similar way. We do not give them here. The actual errors of the constructed smooth polynomial and non-polynomial approximations, when $h = 0.4$, $[a, b] = [-1, 1]$ are given in Table 1. Plots of the error of the approximation of the function $\sin(3x)$ with the polynomial splines and smooth polynomial splines (thick black line) are given in Fig. 1 (left). Plots of the error of the approximation of the function $\sin(3x)$ with the non-polynomial splines and smooth non-polynomial splines (thick black line) are given in Fig. 1 (right).

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