PAIRS OF MICROWEIGHT TORI IN GL_n

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Abstract. In the present note we prove a reduction theorem for subgroups of the general linear group $\mathrm{GL}(n,T)$ over a skew-field T, generated by a pair of microweight tori of the same type. It turns out, that any pair of such tori of residue m is conjugate to such a pair in $\mathrm{GL}(3m,T)$, and the pairs that cannot be further reduced to $\mathrm{GL}(3m-1,T)$ form a single $\mathrm{GL}(3m,T)$-orbit. For the case $m=1$ it leaves us with the analysis of $\mathrm{GL}(2,T)$, which was thoroughly studied some two decades ago by the second author, Cohen, Cuypers and Sterk. For the next case $m=2$ this means that the only cases to be considered are $\mathrm{GL}(4,T)$ and $\mathrm{GL}(5,T)$. In these cases the problem can be fully resolved by (direct but rather lengthy) matrix calculations, which are relegated to a forthcoming paper by the authors.

Introduction

The present paper opens a major cycle of joint papers by the authors dedicated to the geometry of *microweight tori* and *long root tori* in Chevalley groups that was announced in [13]. In the present paper we make one of the first steps towards description of orbits and spans for pairs of microweight tori in the simplest case of the group $\mathrm{GL}(n,K)$. Namely we prove a reduction theorem for subgroups generated by a pair of such tori. However, in this case such a reduction can be established by elementary linear algebra (rather than representation theory of algebraic groups), and can be stated in a more general setting. Thus, we decided to publish this case separately.

Recall that as of today, the geometry of microweight tori is fully understood only for the simplest possible case (A_l,ϖ_1). From the elementary viewpoint these are 1-*tori* also called *reflection tori* in $\mathrm{GL}(n,K)$, $n=l+1$, in other words, the one-parameter groups of pseudo-reflections. The second author, Cohen, Cuypers and Sterk [11, 2] completely described orbits of $\mathrm{GL}(n,K)$ on pairs of such tori, and the corresponding spans. One important corollary of these results is that for $|K|\geq 7$ the span $\langle X,Y\rangle$ of two non-commuting 1-tori X and Y contains a unipotent root subgroup.

In our forthcoming papers [8, 9] we do the same for the next case of (A_l,ϖ_2), in other words, for the 2-*tori* also called *bireflection tori* in $\mathrm{GL}(n,K)$ which are one-parameter subgroups of dilations $\mathrm{diag}(\varepsilon,\varepsilon,1,\dots,1)$ of residue 2. This case naturally occurs in the analysis of the microweight cases (D_l,ϖ_1), (E_6,ϖ_1) and (E_7,ϖ_7), and that of the semi-simple root elements for all simply-laced types, including the immensely interesting exceptional cases (E_6,ϖ_2), (E_7,ϖ_1) and (E_8,ϖ_8).

For $m\geq 3$ the m-tori in $\mathrm{GL}(n,R)$, could be fun in themselves, but they play no such special role in the investigation of other cases. Also, explicit description of orbits and spans, or even extraction of unipotents, become progressively harder for larger values of m. However, the *parametrisation* of the m-tori themselves, and *reduction theorems* for such tori the case $m=2$ are not any easier than for the general case. In the present paper

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we introduce the obvious geometric invariants for pairs X and Y of m-tori, and bound their span $\langle X, Y\rangle$.

Observe that our main result are closely related to the classification of subgroups generated by semisimple elements of a given type. Originally, one would mostly consider *finite* such groups. Of course, classically one would think of finite groups generated by reflections and pseudo-reflections, which over fields of characteristic 0 were classified by Coxeter, and Shephard—Todd, and which arise in many contexts, such as Chevalley theorem. Subsequently, Wagner, Zalessky, Serezhkin and others generalised these results to fields of positive characteristic.

However, further geometric applications required classification of finite groups generated by semisimple elements with *two* non-trivial eigenvalues. After initial successes, mostly due to Huffmann and Wales, the subject lay dormant for couple decades, but recently there is a surge of activity, in the works of Lange, Mikhailova, Blum-Smith, and others, see [5, 4, 1], and references there.

The present paper is a part of a major project whose goal is, in particular, to obtain similar results in much more general contexts, removing the condition that $\mathrm{char}(K) = 0$ and relaxing the assumption of finiteness in such similar results.

1. Notation

Let T be a skew-field, in deeper results and actual applications it will be commutative, in which case it is denoted by K. Further, let $V = T^n$ be the *right* vector space of columns of height n over T, and let $e_1, \dots, e_n$ be the standard base of T^n. Here e_i is the column, whose i-th component equals 1, whereas all other components are equal to 0.

The dual vector space $V^* = {}^nT$ is a *left* vector space over T. It can be interpreted as the space of rows of length n with components in T. By $f_1, \dots, f_n$ we denote the standard base of ${}^nT$. It is dual to $e_1, \dots, e_n$ with respect to the standard pairing, $V^* \times V \longrightarrow T$, $(u, v) \longrightarrow uv$.

For a subspace $U \leq T^n$ we denote by

$$^{\perp}U = \big\{x \in T^n \mid \forall u \in U,\ xu = 0\big\}.$$

Dually, for a subspace $W \leq {}^nT$ we denote by

$$W^{\perp} = \big\{y \in {}^nT \mid \forall v \in W,\ vy = 0\big\}.$$

As usual, $M(m, n, T)$ denotes the left/right vector space of matrices of size $m \times n$ over T, and $M(n, T) = M(n, n, T)$ is the full matrix ring of degree n over T. Further, $G = \mathrm{GL}(n, T) = M(n, T)^*$ is the general linear group of degree n over T. Sometimes we identify a matrix $g \in G$ with the corresponding linear map $T^n \longrightarrow T^n$, $v \longrightarrow gv$. Here g acts *on the left*. Similarly, transformations of left vector spaces are written on the right. To stress that we are using this geometric viewpoint, in such cases we call elements of G *transformations*.

For a matrix $g \in \mathrm{GL}(n, T)$ we denote by g_{ij} its entry in the position (i, j), so that $g = (g_{ij})$, $1 \leq i, j \leq n$. As usual, $g^{-1} = (g'_{ij})$ denotes the inverse of g, e denotes the identity matrix and e_{ij} is a standard matrix unit, i.e. the matrix whose entry in the position (i, j) is 1 and all the remaining entries are zeroes. Thus $g = \sum g_{ij} e_{ij}$. By g^t we denote the *formal* transpose of g, whose entry in the position (i, j) equals g_{ji} considered as an element of T. (In the correct definition of a transpose g_{ji} should be considered an element of the *opposite* skew-field T^0).

Let $D = D(n, T)$ be the group of diagonal matrices, and $N = N(n, T)$ be the group of monomial matrices. The quotient group N/D is isomorphic to S_n, the symmetric group on n letters. Denote by $W = W_n$ the group of permutation matrices in G. We identify S_n and W_n via the isomorphism $\pi \mapsto w_\pi$, where w_π is the matrix whose entry in the position (i,j) is $\delta_{i,\pi j}$.

By $t_{ij}(\xi) = e + \xi e_{ij}$ for $\xi \in T$ and $1 \leq i \neq j \leq n$ we denote an *elementary transvection*. For given $i \neq j$ we consider the corresponding unipotent *root subgroup* $X_{ij} = \{t_{ij}(\xi), \xi \in T\}$. The subgroup $E(n,T)$ of G, generated by all X_{ij}, $1 \leq i \neq j \leq n$, is called the *elementary* subgroup of G. When $T = K$ is commutative, it coincides with the special linear group $\mathrm{SL}(n,K)$. Similarly, by $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$ we denote an *elementary pseudo-reflection*. For a given i we consider the corresponding 1**-torus** $Q_i = \{d_i(\varepsilon),\ \varepsilon \in T^*\}$. Clearly, $\mathrm{GL}(n,T)$ is generated by $E(n,T)$ and Q_1.

2. One-dimensional transformations

Recall that a transformation $g \in G$ is called m*-dimensional*, if $\mathrm{rk}(g-e) = m$. An alternative terminology is to call $\mathrm{res}(g) = \mathrm{rk}(g-e)$ the *residue* of g, and speak of m-dimensional transformations as transformations of residue m. The largest subspace $W \leq V$ such that $g|_W = \mathrm{id}$ is called the *axis* of g. Similarly, the subspace $U = \{gv - v \mid v \in T^n\}$ is called the *residual space* of g or, alternatively, the *centre* of g. Clearly, $\dim U = m$ and $\dim W = n - m$. Many useful properties of residues and residual spaces can be found in [3].

The most important individual elements of $\mathrm{GL}(n,T)$ are the 1-dimensional tranformations, also called *elementary transformations* of the first/second kind. The general form of an 1-dimensional transformation is $x_{vu}(\xi) = e + v\xi u$, where $v \in T^n$, $u \in {}^nT$, and $\xi \in T$. In this case the centre of $x_{vu}(\xi)$ is the space generated by v, whereas its axis is the hyperplane orthogonal to u. Let $uv = \delta$. If $\delta = 0$, the tranformation $x_{vu}(\xi)$ is a *transvection* for all $\xi \in T$. If $\delta \neq 0$, then replacing ξ, if necessary, we an assume that $\delta = 1$. In this case $x_{vu}(\xi)$ is a *pseudo-reflection* for all $\xi \in K \setminus \{-1\}$.

For ensuing reference, let us reproduce one of the principal results of our paper [11], Theorem 1. The geometric invariants occuring here are explained in a more general context in the next section.

Lemma 1. *Assume that* $|T| \geq 7$*. Then for any* $n \geq 3$ *there are the following orbits of* $\mathrm{GL}(n,T)$ *acting by simultaneous conjugation on pairs* (X,Y) *of* 1*-tori. These orbits can be distinguished by the values of* l, m, p, q *and* c*. The values of these invariants on orbits and the corresponding spans are identified in the following table.*

$NN.$	l	m	p	q	c	$\langle X,Y\rangle$
1.	1	1	1	1	1	Q_1
2.	1	2	1	1	1	Q_1X_{12}
3.	2	1	1	1	1	Q_1X_{21}
4.	2	2	0	0	$-$	Q_1Q_2
5.	2	2	0	1	$-$	$Q_1Q_2X_{12}$
6.	2	2	1	0	$-$	$Q_1Q_2X_{12}$
7.	2	2	1	1	1	$Q_2X_{12}X_{13}X_{23}$
$8*$.	2	2	1	1	$\neq 1$	$\mathrm{GL}(2,T)$

Our immediate goal is to obtain a similar result for the next case of 2-tori, which is crucial for the analysis of the exceptional microweight cases. However, already in this case the lists are conspicuously longer, and the identification of spans is significantly more involved. Nevertheless, the initial warm-up fragments of the proof, namely the reduction to $\mathrm{GL}(3,T)$ and the analysis of those orbits in $\mathrm{GL}(3,T)$ that do not occur in $\mathrm{GL}(2,T)$ (roughly corresponding to §§ 2 and 3 of [11]), readily generalise to m-tori over skew-fields. Predictably, in this case $\mathrm{GL}(3,T)$ should be replaced by $\mathrm{GL}(3m,T)$. This is precisely what we carry out in this note.

3. m-DIMENSIONAL TRANSFORMATIONS

Our goal is to study orbits of $\mathrm{GL}(n,T)$ for the conjugation action on the pairs of m-tori

$$(X,Y)\mapsto(gXg^{-1},gYg^{-1}),\qquad g\in G,$$

and to identify the corresponding spans. In the present section we introduce the obvious invariants of such pairs, and prove a reduction theorem that for the case of $m=2$ reduces analysis to the *three* cases, of degrees 4,5 and 6, respectively.

Observe that any m-torus is conjugate to the elementary torus Q, consisting of diagonal matrices whose first m entries at the principal diagonal are $\varepsilon\in T^*$, whereas all other diagonal entries are 1:

$$Q=\{\mathrm{diag}(\varepsilon,\dots,\varepsilon,1,\dots,1),\ \varepsilon\in T^*\}.$$

The elementary torus $Q=Q_{U_0,W_0}$ corresponds to the subspaces $U_0=\langle e_1,\dots,e_m\rangle$ and $W_0=\langle f_1,\dots,f_m\rangle$ generated by the first

$$d_0(\varepsilon)=e+e_1(\varepsilon-1)f_1+\ldots+e_n(\varepsilon-1)f_n,$$

Then the elements of an arbitrary m-torus can be expressed as

$$d(\varepsilon)=e+v_1(\varepsilon-1)u_1+\ldots+v_m(\varepsilon-1)u_m.$$

where $e_i=gv_i$, $f_i=u_ig^{-1}$, $1\le i\le n$, for some matrix $g\in\mathrm{GL}(n,T)$. At that, $U=\langle u_1,\dots,u_n\rangle$ and $W=\langle v_1,\dots,v_n\rangle$.

The subspace U is precisely the *centre* of Q_{UW}, in the sense of being the centre of every $d(\varepsilon)\in Q_{UW}$, $\varepsilon\neq1$. Similarly, the subspace $W^\perp$ orthogonal to $W\le{}^nT$ with respect to the canonical pairing ${}^nT\times T^n\longrightarrow T$, is precisely the *axis* of Q_{UW}, in the above sense. Oftentimes we loosely refer to W itself as the axis of Q_{UW}. The following two observations are obvious.

Lemma 2. *Every m-torus $Q=Q_{UW}$ is completely determined by the subspaces $U\le T^n$, $W\le{}^nT$ such that*

$$\dim(U)=\dim(W)=m,\qquad T^n=U\oplus W^\perp.$$

Lemma 3. *For any $g\in\mathrm{GL}(n,T)$ we have $gQ_{UW}g^{-1}=Q_{gU,Wg^{-1}}$.*

Lemma 4. *For any subspace $U\le T^n$ and any $g\in\mathrm{GL}(n,T)$ one has ${}^\perp(gU)={}^\perp Ug^{-1}$.*

Dually, for any subspace $W\le{}^nT$ and any $g\in\mathrm{GL}(n,T)$ one has $(Wg)^\perp=g^{-1}W^\perp$.

Proof. To prove the first claim, recall that ${}^\perp(gU)$ consists of all $x\in{}^nT$ such that $x(gu)=0$ for all $u\in U$. This equality can be rewritten as $(xg)u$ for all $u\in U$. Thus, $xg\in{}^\perp U$, or, what is the same, $x\in{}^\perp Ug^{-1}$, as claimed. The second claim can be established similarly (and, in fact, follows by duality). □

Now we are in a position to construct some obvious invariants of a pair of m-tori.

4. Obvious invariants

Now, let X and Y be two m-tori with centres U_1 and U_2 and axes W_1 and W_2, respectively. We introduce the following notation.

- $r = r(X,Y) = \dim(U_1+U_2)$,
- $s = s(X,Y) = \dim(W_1+W_2)$.

Clearly, the parameters r and s take their values in the interval $m \le r,s \le 2m$.

Further, we introduce the following notation

- $p = p(X,Y) = \dim(U_1 \cap W_2^{\perp})$,
- $q = q(X,Y) = \dim(U_2 \cap W_1^{\perp})$.

It is easy to see that the parameters p and q take their values in the interval $0 \le p,q \le m$.

Lemma 5. *The above parameters r, s, p and q are not changed under simultaneous conjugation.*

Proof. For r and s this is obvious. To prove the invariance of p, recall that by Lemma 4 one has

$$p(gXg^{-1}, gYg^{-1}) = \dim(gU_1 \cap (W_2 g^{-1})^{\perp}) = \dim(gU_1 \cap gW_2^{\perp}) = \dim(U_1 \cap W_2^{\perp}) = p(X,Y),$$

the invariance of q is verified similarly. □

To classify *orbits* on pairs of 1-tori, in [11] we introduced yet another invariant of a pair of tori. However, the *span* of such a pair was only influenced by whether that invariant was equal to 1 or distinct from 1. Since we are interested in classifying possible spans spans much more than in classifying orbits, here we limit ourselves to the discrete part of that invariant. Namely, we set

- $t = t(X,Y) = \max\Big(\dim\big((U_1+U_2)\cap(W_1+W_2)^{\perp}\big), \dim\big({}^{\perp}(U_1+U_2)\cap(W_1+W_2)\big)\Big)$

Clearly, the parameter t takes values in the interval $0 \le t \le m$ and, by the same token as in Lemma 5, it is not changed under simultaneous conjugation.

5. Degree reduction

In the next result we denote by H_m the linear group of degree $3m$, generated by Q_1 and by all X_{ij}, $1 \le i \le 2m$, $1 \le j \le 3m$. In other words,

$$H_m = \left\{ \begin{pmatrix} x & y \\ 0 & e \end{pmatrix} \;\middle|\; x \in \mathrm{GL}(2m,T),\ y \in M(2m,m,T) \right\} \le \mathrm{GL}(3m,T).$$

By default, we identify linear groups of different degrees via the stability embedding. In other words, for $m \le n$, we set

$$\mathrm{GL}(m,T) \longrightarrow \mathrm{GL}(n,T), \qquad g \mapsto g \oplus e = \begin{pmatrix} g & 0 \\ 0 & e \end{pmatrix},$$

where e is the identity matrix of degree $n-m$. Let

$$H(n)_m = H_m \cap \mathrm{GL}(n,T),$$

By the very definition $H(n)_m = H_m$ for all $n \ge 3m$.

Now we are all set to start proving our basic reduction to degree 3m.

Lemma 6. *Let X and Y be two m-tori in $\mathrm{GL}(n,T)$, $n \ge m+1$. Then there exists an $g \in \mathrm{GL}(n,T)$ such that $gXg^{-1}, gYg^{-1} \le H(n)_m$.*

Proof. From the very beginning we can assume that $X = Q_{U_0,W_0}$, where $U_0 = \langle e_1, \dots, e_m \rangle$, $W_0 = \langle f_1, \dots, f_m \rangle$. Let $Y = Q_{U_1,W_1}$.

Consider the factor-space V/U_0 and let $\dim(U_0 \cap U_1) = k$, $0 \le k \le m$. We denote $\overline{U_1} = U_1/(U_0 \cap U_1)$. Then there exists an element $\overline{g_1} \in \mathrm{GL}(n-m+k, T)$ such that $\overline{g_1}\overline{U_1}$ is contained in the subspace $\overline{V_1}$, spanned by the projections of the first $2m-k$ vectors of the standard base $e_1, \dots, e_{2m-k}$. Then the matrix $\overline{g_1}$ only differs from the identity matrix in the block g' of size $m-k$, standing in the upper left corner.

Setting $g_1 = e_m \oplus g' \oplus e_{n-2m+k} \in \mathrm{GL}(n,T)$ we get $g_1(U_0 + U_1) \subseteq V_1$, $W_0 g_1^{-1} = W_0$. Now, it remains to repeat the same argument for W's.

Set $U = U_0 + U_1$, $\dim U = 2m-k$, and consider the dual space V^*/U^*. There exists an element $\overline{g_2} \in \mathrm{GL}(n-2m+k, T)$ such that $\overline{W_1}\overline{g_2^{-1}}$ is contained in the subspace generated by the projections of the dual standard base $f_{2m-k+1}, \dots, f_{3m-k}$. The matrix $\overline{g_2}$ only differs from the identity matrix by its block g'' of size $m-k$, standing in the upper left corner.

As above, set $g_2 = e_{2m-k} \oplus g'' \oplus e_{n-3m+k} \in \mathrm{GL}(n,T)$. Then $g = g_1 g_2$ is the required conjugating matrix. □

From now on, we can assume that we are inside $\mathrm{GL}(3m,T)$ — all orbits on pairs of tori have representatives inside this group. Interchanging centres and axes in the above argument, we get a similar reduction inside the transpose of $H(n)^t_m$.

Lemma 7. *Let X and Y be two m-tori in $\mathrm{GL}(n,T)$, $n \ge m+1$. Then there exists an $g \in \mathrm{GL}(n,T)$ such that $gXg^{-1}, gYg^{-1} \le H(n)^t_m$.*

Obviously, any pair of parabolic subgroups is simultaneously conjugate to a pair P_1, wP_2w^{-1}, where P_1 and P_2 are standard parabolic subgroups and w is an element of the Weyl group. Thus, the previous lemma immediately implies the following result.

Theorem 1. *Let X and Y be two m-tori in $\mathrm{GL}(n,T)$, $n \ge m+1$. Then there exists an $g \in \mathrm{GL}(n,K)$ such that*

$$gXg^{-1}, gYg^{-1} \le H(n)_m \cap wH(n)^t_m w^{-1}$$

for some $w \in W_n$.

In particular for $m = 2$, only one of the three possibilities may occur for the intersection of two maximal parabolic subgroups stabilising a 4-subspace and a 2-subspace in $\mathrm{GL}(6,T)$. Thus, any pair of 2-tori is simultaneous conjugate to a pair contained in one of the following subgroups

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

depending on whether $t = 0, 1, 2$.

6. The highest degree orbit

As above, we consider a pair of m-tori X and Y, by U_1, U_2 and by W_1, W_2 we denote their axes and centres, respectively. We fix some bases in these subspaces

$$U_1 = \langle u_1, \dots, u_m \rangle, \quad U_2 = \langle u_{m+1}, \dots, u_{2m} \rangle,$$
$$W_1 = \langle w_1, \dots, w_m \rangle, \quad W_2 = \langle w_{m+1} \dots, , w_{2m} \rangle.$$

For the standard m-torus Q we have $U = \langle e_1, \dots, e_m \rangle$, $W = \langle f_1, \dots, f_m \rangle$.

In the present section we consider the simplest possible type of subgroups generated by two m-tori, viz. the direct sums of m isomorphic linear groups generated by 1-tori.

With this end consider the representation

$$\phi_m : \mathrm{GL}(n,T) \longrightarrow \mathrm{GL}(mn,T), \qquad g \mapsto g \oplus \dots \oplus g = \mathrm{diag}(g, \dots, g),$$

where the number of summands equals m.

Clearly, the image of an 1-torus under ϕ_m is an m-torus. Thus, applying this map to the subgroups listed in [11], Theorem 1 (= Lemma 1 above), we get *some* subgroups generated by m-tori, which we call **replications** of subgroups generated by a pair of 1-tori.

The unique new orbit of $\mathrm{GL}(3m,T)$ on the pairs of m-tori is the orbit obtained by the replication of the unique new $\mathrm{GL}(3,T)$-orbit on the pairs of 1-tori.

Theorem 2. *There exists a unique orbit of* $\mathrm{GL}(3m,T)$ *on pairs of* m*-tori that are not contained in* $\mathrm{GL}(3m-1,T)$*. For this orbit the parameters introduced in § 4 take the following values:* $r = s = 2m$, $p = q = 0$, $t = m$.

Proof. By hypothesis our orbit is not contained in $\mathrm{GL}(3m-1,T)$, so that without loss of generality we can assume that

$$U_1 + U_2 \le \langle e_1, \dots, e_{2m} \rangle, \qquad W_1 + W_2 \le \langle f_{m+1}, \dots, f_{3m} \rangle,$$

we construct the series of conjugations to reduce such a pair to the canonical form.

- Conjugating by appropriate transvections from X_{ij}, where $1 \le i \le m$, $m+1 \le j \le 2m$, we can assume that $u_i = e_{m+i}$, for all $1 \le i \le m$.
- Similarly, conjugating by appropriate transvections from X_{hk}, $m+1 \le h \le 2m$, $2m+1 \le k \le 3m$, we can assume that, moreover, $w_i = f_i$, for all $m+1 \le i \le 2m$.

Then the remaining axes and centres are of the form

$$(u_{m+1}, \dots, u_{2m}) = (e_{m+1}, \dots, e_{2m}) + (u_1, \dots, u_m) g_1,$$

and of the form

$$\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} e_{m+1} \\ \vdots \\ e_{2m} \end{pmatrix} + g_2 \begin{pmatrix} e_{2m+1} \\ \vdots \\ e_{3m} \end{pmatrix},$$

respectively. Since $r = 2m$, the matrix g_1 is invertible, and since $s = 2m$, the matrix g_2 is also invertible.

- Conjugating by g_1^{-1} in the embedding of $\mathrm{GL}(m,T) \longrightarrow \mathrm{GL}(2m,T)$ on the first m positions (the usual stability embedding), we can assume that $u_{m+i} = e_i + e_{m+i}$, for all $1 \le i \le m$.
- Conjugating by g_2 in the embedding of $\mathrm{GL}(m,T) \longrightarrow \mathrm{GL}(2m,T)$ on the last m positions, we can , moreover, assume that $w_i = f_{m+i} + f_{2m+i}$, for all $1 \le i \le m$.

Recall one more piece of notation from [11]. For $u \in T^n$ and $v \in {}^nT$ such that $vu = 1$, we set
$$Q_{uv} = \{e + u(\varepsilon - 1)v \mid \varepsilon \in T^*\}.$$
Then the above means precisely that any such pair of m-tori is conjugate to the image under ϕ_m of the following pair of 1-tori:
$$Q_{e_2, f_2+f_3}, Q_{e_1+e_2, f_2} \in \mathrm{GL}(3, T),$$
as claimed. □

In the forthcoming papers we take it from here for the next simplest case $m = 2$. In [8] the first author considers the most difficult case of pairs of 2-tori in $\mathrm{GL}(4, R)$, and under some assumptions on T identifies their spans. In [9] we consider the remaining case of pairs of 2-tori in $\mathrm{GL}(5, T)$.

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