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Scattering of acoustic waves from a point source over an impedance wedge

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ABSTRACT

In this work we study diffraction of a spherical acoustic wave due to a point source, by an impedance wedge In the exterior of the wedge the acoustic pressure satisfies the stationary wave (Helmholtz) equation and classical impedance boundary conditions on two faces of the wedge, as well as Meixner's condition at the edge and the radiation conditions at infinity. Solution of the boundary value problem is represented by a Weyl type integral and its asymptotic behavior is discussed. On this way, we derive various components in the far field interpreting them accordingly and discussing their physical meaning.

1. Introduction

Geometrical Theory of Diffraction (GTD) is one of the most powerful and widely exploited asymptotic theories in engineering and research practice. It is based on **a** commonly accepted high frequency localization principle. As is well known, practical use of GTD in various ray-tracing procedures requires knowledge of diffraction or excitation coefficients which should be incorporated into such procedures. These coefficients are responsible for local transformation of the wave field attributed to the rays interacting with some points of the boundary. It is remarkable however that these coefficients are traditionally determined from canonical problems locally describing the process of such interaction. Due to the high frequency localization principle, various canonical problems are of importance in applications and therefore attract attention of researchers.

The canonical problem under consideration plays an important role in numerous applications including first of all Geometrical Theory of Diffraction and its various modifications. In the case of perfect wedges, i.e. those with ideal boundary conditions, some results dealing with the 3D diffraction of waves from a point source are known in the literature [1–5]. To our knowledge, a basic idea is in use of the Weyl type integral representation [6] of solution for the corresponding boundary value problem. This representation is actually a plane wave expansion for the incident field from a point source and a similar representation for the field diffracted by the perfect wedge. Such an expansion is based on solution of the plane wave diffraction problem by a wedge for real angles of incidence, which is then analytically continued for complex-valued angles. An appropriate choice of integration contours leads to a rapidly convergent iterated integral called further Weyl integral representation. For a perfect wedge the integrand of Weyl representation has a simple elementary form, and further analysis deals with asymptotic evaluation of the Weyl integral for large distances from the edge.

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However, only recently some progress has been achieved for the problem of diffraction of the electromagnetic wave field from a Hertzian dipole located over an impedance wedge [7], Chapter 3, [8,9]. Actually, in the present work we appropriately adapt results of the abovementioned works to the acoustic case, taking into account that the acoustic problem admits solution in quadratures with the aid of the Malyuzhinets function. The authors of [8,9] relied upon analytical-numerical solution of the diffraction problem of a skew incident plane electromagnetic wave in a wedge-shaped domain with Leontovich boundary conditions [10,11], see also [12]. (Remark that for some special cases this problem can be solved explicitly, i.e. in quadratures [13,14].) Exploiting, in particular, the Malyuzhinets technique [15], they have reduced the electromagnetic problem to a system of vector functional equations and then to a Fredholm integral equation. Then, making use of some ideas of analytic continuation with respect to the angles of skew incidence, they have asymptotically evaluated the iterated Weyl integral and described the scattered field at far distances from the edge. Some numerical results have been also obtained, which included numerical solution of the integral equation. It should be emphasized that a principal difficulty of the corresponding analysis was in the fact that, as it seems, the electromagnetic problem of diffraction of a skew incident plane wave by an impedance wedge cannot be solved explicitly, i.e. in quadratures.

In the present work we exploit the possibility to solve the acoustic problem explicitly, i.e. the problem of diffraction of an acoustic plane wave which is skew incident at the edge of an impedance wedge. This solution given below is actually some modification of the Malyuzhinets solution [15] and is found explicitly. Having Weyl integral representation of the wave field for the point source illumination of an impedance wedge, we evaluate the integral by means of the saddle point technique (or stationary phase technique). To this end, we deform the integration contours appropriately, taking into account the corresponding singularities captured in the course of the deformation. Then we evaluate various terms asymptotically and discuss physical meaning of the wave components at far distances from the edge. These components are reflected waves from the faces, the space wave from the edge as well as the surface waves from the edge. The amplitudes and phases of these waves depend on position of the point source. Provided that the point source is in some close vicinity of a face of the wedge, additional components in the asymptotics arise. In this case the source additionally excites a surface wave that propagates along the face towards the edge then it is reflected from and transmitted across the edge, giving rise to an additional space wave from the edge.

2. Statement of the problem

An acoustic point source illuminates an impedance wedge of the opening angle 2Φ. The wave field interacts with the faces and with the edge giving rise to the scattered field, which consists of various far-field components having clear physical meaning. The asymptotic description of these components in the far field, mentioned in the Introduction, is one of the main goals of our study.

2.1. Formulation of the problem

In the exterior Ω of **a** wedge with the surface S consisting of two faces S_+ and S_- (Fig. 1) the acoustic wave field u (Green's function) satisfies the Helmholtz equation

$$(\triangle + k^2)u(X,Y,Z) = -\delta(X-x_0)\delta(Y-y_0)\delta(Z)\,, \tag{1}$$

where the point source is located at $(x_0, y_0, 0)$. Together with the Cartesian coordinates, it is also useful to introduce cylindrical coordinates r, φ, z,

$$X = r\cos\varphi, \quad Y = r\sin\varphi, \quad Z = z.$$

In these coordinates the position of the point source is given by $(r_0, \varphi_0, 0)$ and the domain Ω is $\Omega = \{(r,\varphi,z): \ r > 0,\ |\varphi| < \Phi,\ |z| < \infty\}$.

The total acoustic field $u = u(r,\varphi,z)$ fulfills the impedance boundary conditions on the wedge's faces $S_\pm$

$$\left(\pm\frac{1}{r}\frac{\partial u}{\partial\varphi} - \mathrm{i}k\eta_\pm u\right)\Big|_{\varphi=\pm\Phi} = 0\,, \tag{2}$$

where $k > 0$, $\pi/2 < \Phi \le \pi$, $\eta_\pm$ are the surface impedances. It is worth commenting on the real and imaginary parts of the surface impedances. We assume that $\Re\eta_\pm = 0$, which means that the impedance surfaces are not absorbing, and $\Im\eta_\pm < 0$. In acoustics the latter restriction implies that the impedance surfaces $S_\pm$ can support surface waves [5], Chapter 2.[1]

The wave field behavior at the edge satisfies Meixner's condition, as $r \to 0$,

$$u(r,\varphi,z) = C + O(r^\delta)\,, \quad \delta > 0\,, \tag{3}$$

for arbitrary fixed z and uniformly with respect to (w.r.t.) φ.

[1] The time dependence $\exp\{-\mathrm{i}\omega t\}$ is assumed and suppressed throughout the paper.

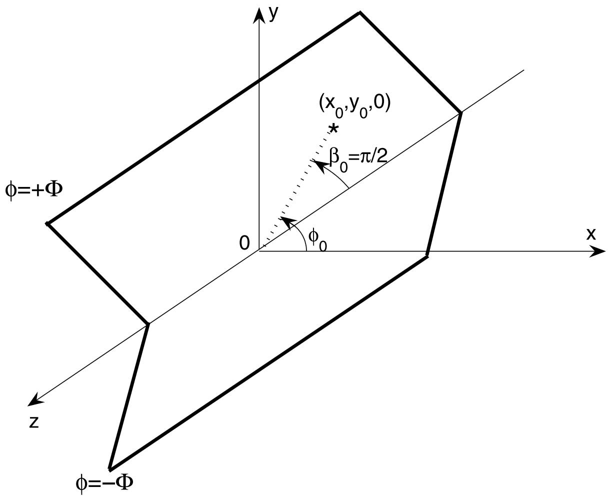

Fig. 1. Diffraction of waves from a point source over an impedance wedge.

An appropriate radiation condition at infinity is implied and it can be formulated in different forms (see also [5], Sect. 2.4). Such a formulation is the simplest provided $\Re\eta_\pm = \epsilon$, $\epsilon > 0$ is small, which means that the faces $S_\pm$ are slightly absorbing,[2] then

$$\int_{S_R} \left|\frac{\partial u}{\partial r} - \mathrm{i}ku\right|^2 \mathrm{d}s \to 0, \quad \text{as } r = R \to \infty\,, \tag{4}$$

where S_R is the part of the sphere $r = R$ which is contained in Ω. The problem (1)–(4) is uniquely solvable with the classical solution u_ϵ. Then, because we are interested in the case $\Re\eta_\pm = 0$, we can specify the desired solution u with the non-absorbing wedge's faces as the limiting one,

$$u = \lim_{\epsilon\to 0} u_\epsilon\,. \tag{5}$$

The existence of the latter limit in (5) can be directly verified for the problem at hand.

2.2. The far-field asymptotics and dependence on parameters of the problem

We intend to describe the far-field behavior of u satisfying the problem (1)–(5). It should be noticed that it strongly depends on the position of the point source. We, first, assume that the dimensionless parameter kr_0 is large, i.e. the source is not close to the edge. In what follows, however, we assume that, instead of kr_0, $k \gg 1$ is the main dimensionless large parameter implying that the dimensionless coordinates r, φ, z are introduced in accordance with the substitution $r/r_0 \to r$, $z/r_0 \to z$. Second, the position of the point source w.r.t. the wedge's faces depends on φ_0, $|\varphi_0| < \Phi$.

A basic analytical tool for study of the problem is the Weyl type integral representation of the solution, [6,7]; Chapter 3. For the point source in the 3D space this is a double contour integral which, in accordance with the physical terminology, is an expansion w.r.t. the plane waves depending on complex angles of incidence α, β. It should be noticed, however, that for real α, β these angles can be really attributed to the direction of propagation of some 'incident' plane wave with its expression contained in the integrand.

Solution of the problem for the point source located over an impedance wedge in 3D space is sought in a similar form of 'the plane wave expansion', i.e. in the form of the double Weyl integral, however, with yet unknown function in the integrand. It turns out that this unknown function can be found, for real α, β, as a solution of the diffraction problem for the plane wave which is skew incident at the edge of the wedge. For real α, β, in the acoustic problem the solution can be determined in an explicit form (as a Malyuzhinets' type solution) and then analytically continued on the complex values so that the Weyl integral solves the problem for a point source over the impedance wedge. Further, our analysis deals with asymptotic evaluation of the Weyl integral representation as $k \gg 1$.

2.3. Overview of the main new results

As already mentioned, in the present work we make use of the Weyl integral representation describing solution of the problem in an explicit form of triple integral. This work deals basically with asymptotic evaluation of the Weyl integral

2 In this case, in particular, the surface waves that outgo from the edge to infinity exponentially vanish as $r \to \infty$.

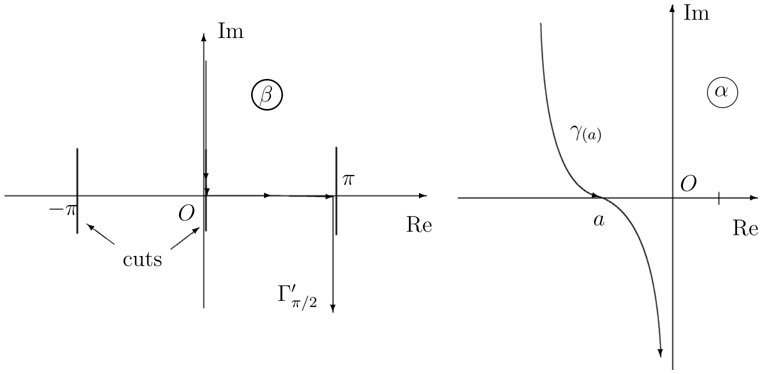

Fig. 2. The contours of integration for the Weyl integrals for the wave field from the point source. The branch cuts for $\theta^{+}(\beta)$ connect the points $\arcsin\eta_{+}+\pi n$ and $-\arcsin\eta_{+}+\pi n$ with $n=0,\pm1,\ldots$ correspondingly.

and with description of various components in the far-field asymptotics. We apply multi-dimensional versions of the saddle point technique. On this way, we are forced to consider two different cases. The first case implies that the point source is located outside a close neighborhood of the wedge's faces. The second one corresponds to the source located near a wegde's face, however, outside some close vicinity of the edge. In the first case we obtain asymptotic expressions of the far field that have clear physical meaning of waves reflected from the faces. The other components correspond to the diffracted wave from the edge as well as the surface waves propagating from the edge. These terms are analogous to those obtained in the electromagnetic version of the problem [7]; Chapter 3.[3] The asymptotic expressions are written in simple geometrical terms, depending on Malyuzhinets function, and can be easily interpreted in the framework of Geometrical Theory of Diffraction (GTD). The latter means that geometrical optics objects like angle of conical diffraction, diffractional eikonals etc. can be easily identified in the far-field expressions. In the second case the results are similar. However, several new terms in the asymptotics arise. The source located near a wedge's face additionally generates the so-called primary surface wave that propagates to the edge and gives rise to reflected and transmitted surface waves. We carefully discuss laws of reflection and transmission of these waves. On this way, an exhaustive physical analysis of wave processes can be given. At the same time, the space wave from the edge is also generated as a result of interaction of the primary surface waves with the edge. Contrary to the result, discussed in [9] that requires solution of an integral equation, we obtain new expressions for the diffraction or excitation coefficients which have clear physical interpretation and can be directly used for numerics. These expressions depend on the eponymous Malyuzhinets function and can be efficiently incorporated in various GTD-type procedures of research and engineering practice. Some important results similar to those in this work were discussed in the paper [16] for the case of surface wave incidence.

3. Weyl integral representation, Malyuzhinets type solution for the amplitude and analytic continuation

We make use of the integral representation for the incident field from an acoustic point source in 3D space located at $M_0=(x_0,y_0,0)$

$$u_0(M)=\frac{\mathrm{i}k}{8\pi^2}\int_{\Gamma_{\pi/2}}\mathrm{d}\beta\int_{\gamma_{\psi-\pi}}\mathrm{d}\alpha\ \sin\beta\,\mathrm{e}^{\mathrm{i}k[z\cos\beta+\sin\beta(r_0\cos(\alpha-\varphi_0)-r\cos(\alpha-\varphi))]} \tag{6}$$

and also $u_0(M)=\frac{\mathrm{e}^{\mathrm{i}kR_0}}{4\pi R_0}$, where $M=(X,Y,Z)$ (or (r,φ,z)), $R_0=|MM_0|$, the contours of integration in (6) are shown in Fig. 2, $(0\le\psi<2\pi)$.[4] It should be noticed that Weyl integral representation for the wave field from the point source in 3D space requires some work to derive. The detailed derivation of the Weyl integral representation is discussed in Sect. 3.1 of [7]. In particular, it is shown that the parameter ψ in (6) is connected with direction from the point source to the observation point. The representation (6) can be interpreted as an expansion of the wave field from a point source with the aid of the plane waves $\mathrm{e}^{\mathrm{i}k[z\cos\beta-r\sin\beta\cos(\alpha-\varphi)]}A_0$ with the amplitude $A_0=\exp(\mathrm{i}kr_0\sin\beta\cos(\alpha-\varphi_0))$.

Remark 1. It is worth noticing that $v=\mathrm{e}^{-\mathrm{i}kr\sin\beta\cos(\alpha-\varphi)}$ is a plane wave solution of the 2D Helmholtz equation, $(\triangle_{r,\varphi}+k^2\sin^2\beta)v=0$, whereas $w=\mathrm{e}^{\mathrm{i}k[z\cos\beta-r\sin\beta\cos(\alpha-\varphi)]}A_0$ satisfies the equation $(\triangle_{r,\varphi,z}+k^2)w=0$.

[3] In this case solution of an integral equation is required because the electromagnetic problem is more complex than acoustic one which is solved in quadratures.

[4] The contour $\Gamma_{\pi/2}$ coincides with $\Gamma'_{\pi/2}$ in Fig. 2., however, the integrand in (6) does not have any branching points.

The total field from the point source over a wedge is sought in an analogous form

$$u(M)=\frac{\mathrm{i}k}{8\pi^2}\int_{\Gamma'_{\pi/2}}\mathrm{d}\beta\int_{\gamma(\varphi_0)}\mathrm{d}\alpha\,\sin\beta\,\mathrm{e}^{\mathrm{i}k[z\cos\beta+\sin\beta r_0\cos(\alpha-\varphi_0)]}\,U(r,\varphi;\alpha,\beta) \tag{7}$$

with yet unknown $U(r,\varphi;\alpha,\beta)$ and with the contours of integration shown in Fig. 2, $\Gamma'_{\pi/2}=(\mathrm{i}\infty,0+]\cup[0+,\pi-0]\cup[\pi-0,\pi-0-\mathrm{i}\infty)$. The contour $\Gamma'_{\pi/2}$ goes along the corresponding sides of the branch cuts. The contour $\gamma_{(\varphi_0)}$ is located in the strip which is parallel to the imaginary axis and has the width π. We assume that $|\varphi_0|<\Phi-\pi/2$ which ensures convergence of the Weyl integral for the incident wave. It should be remarked that this restriction can be actually ignored for the asymptotic expressions to be derived.

Taking into account Remark 1 as a motivation, by analogy we assume that U solves the equation

$$(\triangle_{r,\varphi}+(k')^2)U(r,\varphi;\alpha,\beta)=0, \tag{8}$$

and additionally satisfies the boundary condition

$$\left(\pm\frac{1}{r}\frac{\partial U}{\partial\varphi}-\mathrm{i}k'\sin\vartheta^{\pm}(\beta)U\right)\bigg|_{\varphi=\pm\Phi}=0\,, \tag{9}$$

where we exploited the notations

$$\sin\vartheta^{\pm}(\beta)=\frac{\eta_{\pm}}{\sin\beta} \tag{10}$$

and $k'=k\sin\beta$ which are convenient for further studies.

Let us also assume that for real α and β and such that $-\Phi<\alpha<\Phi$, $0<\beta<\pi$ the yet unknown $U(r,\varphi;\alpha,\beta)$ satisfies Meixner's condition as $r\to 0$ and the condition as $r\to\infty$. This condition for U will be discussed below.

In other words we require that the wave field $\mathrm{e}^{\mathrm{i}k[z\cos\beta+\sin\beta r_0\cos(\alpha-\varphi_0)]}\,U(r,\varphi;\alpha,\beta)$ be the solution of the diffraction problem of the skew incident plane wave $\mathrm{e}^{\mathrm{i}k[z\cos\beta-r\sin\beta\cos(\alpha-\varphi)]}A_0$ with the amplitude $A_0=\exp(\mathrm{i}kr_0\sin\beta\cos(\alpha-\varphi_0))$. The angles α,β specify the direction of the incident ray impinging the axis OZ: β is the angle between the axis OZ and the incident ray and α is the angle between the axis OX and the projection of the ray on the plane $Z=0$ (see also Fig. 1).

It is remarkable that the desired 2D solution $U(r,\varphi;\alpha,\beta)$ of (8), (9) satisfying Meixner's and radiation conditions for arbitrarily fixed values of the parameters α,β $(-\Phi<\alpha<\Phi,\ 0<\beta<\pi)$ can be found explicitly and is actually the Malyuzhinets solution (see [5], Chapter 6, for the wave number $k'(\beta)=k\sin\beta$).

For given angles α,β the desired solution satisfying Eq. (8) and the boundary conditions (9) is found in the form of the Sommerfeld integral

$$U(r,\varphi;\alpha,\beta)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}kr\sin\beta\cos s}f(s+\varphi;\alpha,\beta)\,, \tag{11}$$

where the double-loop Sommerfeld contour of integration γ is shown in Fig. 3 together with the steepest descent paths (SDP), the function f is specified by the expressions

$$f(s;\alpha,\beta)=\frac{\mu\cos\mu\alpha}{\sin\mu s-\sin\mu\alpha}\,\frac{\Psi(s;\beta)}{\Psi(\alpha;\beta)},\quad \mu=\frac{\pi}{2\Phi}$$

and

$$\Psi(s;\beta)=$$

$$\psi_{\Phi}(s-\Phi+\pi/2-\vartheta^{-}(\beta))\psi_{\Phi}(s-\Phi-\pi/2+\vartheta^{-}(\beta))\psi_{\Phi}(s+\Phi+\pi/2-\vartheta^{+}(\beta))\psi_{\Phi}(s+\Phi-\pi/2+\vartheta^{+}(\beta)).$$

The meromorphic function $\psi_{\Phi}(\cdot)$ is the Malyuzhinets function (see [5], Sect. 6.2). The Malyuzhinets function is a meromorphic solution of the functional difference equation

$$\frac{\psi_{\Phi}(z+2\Phi)}{\psi_{\Phi}(z-2\Phi)}=\cot\left(\frac{z}{2}+\frac{\pi}{4}\right)$$

and in the strip $|\Re(z)|<\pi/2+2\Phi$ has an integral representation

$$\psi_{\Phi}(z)=\exp\left[-\frac{1}{2}\int_0^{\infty}\frac{\cosh(z\zeta)-1}{\zeta\cosh(\pi\zeta/2)\sinh(2\Phi\zeta)}\mathrm{d}\zeta\right].$$

In order to be able to substitute the function $U(r,\varphi;\alpha,\beta)$ into the representation (7) one ought to continue analytically w.r.t. α,β from the real values onto complex domains containing the integration contours $\Gamma'_{\pi/2}$ and $\gamma_{(\varphi_0)}$. To this end, in particular, one has to describe the analytic branches of the functions $\vartheta^{\pm}(\beta)$ which are the solutions of the Eqs. (10). This is traditional and is described in [7] pp. 79–80. The branch cuts on the β-plane are shown in Fig. 2.

We are interested in asymptotics as $k\to\infty$ or, in other words, we deduce the far-field asymptotics of the integral (7) because we also assume that $kr\to\infty$. To this end, it is necessary to study the behavior of U and deform the integration

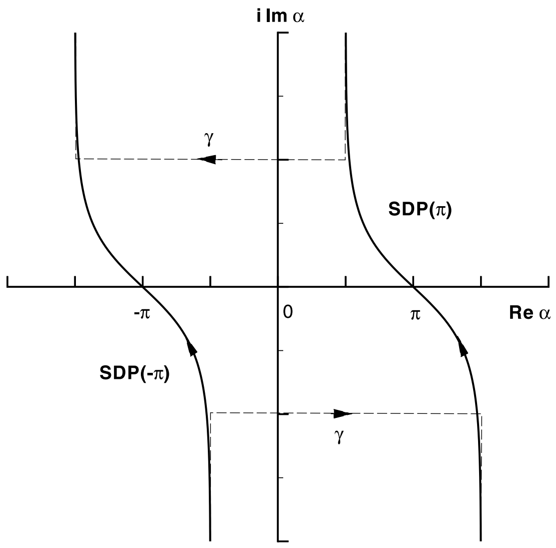

Fig. 3. The plane of the complex variable s with the Sommerfeld double loops γ and the steepest descent paths $\gamma_{\pm\pi} := SDP(\pm\pi)$.

contour γ in the integral into the SDPs. On this way, the singularities of $f(s+\varphi;\alpha,\beta)$ will be captured. Namely the location of the observation point and that of the source specify the singularities captured, see (32) in Appendix. First, we consider the case when the point source is not close to a wedge's face and compute different components in the far field. Then, assuming that the source is near the face $S_+,\ \varphi=\Phi$, we obtain expressions for the additional components that arise in this case.

4. The far-field asymptotics: the point source is not close to the wedge's faces

The main tool in calculation of the asymptotics for the Weyl integral (7) is the saddle point technique, in particular, its multi-dimensional version, [17–21]. We also provide the calculations by appropriate physical interpretations. The latter enables us to describe physical meaning of the components in the scattered field but are also used to control correctness of the derivations.

4.1. Motivation, reductions and asymptotic evaluation of (7)

We evaluate asymptotics of (7) as $k\to\infty$. To this end, consider, first, the phase function

$$\mathcal{S}=\mathrm{i}\left\{z\cos\beta+\sin\beta\left[r_0\cos\left(\alpha-\varphi_0\right)-r\cos s\right]\right\} \tag{12}$$

which is formed by the exponents in the integrands of (7) and (11). For a simple saddle point (α_0,β_0,s_0) we require that

$$\operatorname{grad}\mathcal{S}(\alpha,\beta,s)\big|_{\alpha=\alpha_0,\beta=\beta_0,s=s_0}=0,\quad \det\mathcal{S}''(\alpha,\beta,s)\big|_{\alpha=\alpha_0,\beta=\beta_0,s=s_0}\neq 0$$

is true. As can easily be seen, the following two saddle points are determined

$$(\varphi_0,\beta_0,-\pi)\,,\quad(\varphi_0,\beta_0,\pi) \tag{13}$$

with $\beta_0=\arctan\left[(r+r_0)/z\right]$ and $r=\sqrt{x^2+y^2}$.

The moduli of det S'' at these two saddle points (13) are identical and given by

$$\left|\det\,\mathcal{S}''\right|_{\alpha=\varphi_0,\beta=\beta_0,s=\pm\pi}=r_0r\left[l^{\mathrm{i}}+l\right]\sin^2\!\beta_0, \tag{14}$$

with

$$l^{\mathrm{i}}=r_0/\sin\beta_0,\quad l=r/\sin\beta_0,\quad \sqrt{(r+r_0)^2+z^2}=l^{\mathrm{i}}+l.$$

For the moduli of det $\mathcal{S}''$ differ from zero, the two points given in (13) are indeed the sought-for simple saddle points. Then deformation of the integration surface to that of the steepest descent $S(\beta_0)\times S(\varphi_0)\times S_{\arg\sin\beta_0}(\pm\pi)$ is required [17], see below the definitions of $S(a)$. In this way, some singularities of the integrand are captured which additionally complicates the analysis.

It is reasonable, however, to proceed in a different way considering the integral (7) as iterated formally substituting U from (11). Consider the innermost integration with respect to s. The rapidly varying exponential factor $\exp(-ikr\sin\beta\cos s)$ of the Sommerfeld integrals has the saddle points $s_0 = \pm\pi$ as well as the steepest descent paths $S_{\arg\sin\beta}(\pm\pi)$:

$$\mathrm{Re}\, s = \pm\pi - \mathrm{Gd}(\mathrm{Im} s, \arg\sin\beta),$$

where $\mathrm{Gd}(\xi,\psi)$ is the generalized Gudermann function defined as a solution of the equation

$$\cos\mathrm{Gd}(\xi,\psi)\cosh\xi - \tan\psi\sin\mathrm{Gd}(\xi,\psi)\sinh\xi = 1,$$

$$\mathrm{Gd}(x,y) = \arctan\left(\frac{\sinh(x)\cos(y)}{1+\cosh(x)\sin(y)}\right)$$

satisfying the conditions $\mathrm{Gd}(\xi,0) = \mathrm{gd}(\xi)$ and $\mathrm{Gd}(\xi,\pi/2) = 0$, where $\mathrm{gd}(x) = \mathrm{sign}(x)\mathrm{arcccos}(1/\cosh(x))$ is the traditional Gudermann function (see also [9] and Appendix). The curve $S_{\arg\sin\beta}(a)$ intersects the real axis at the point a. As $\arg\sin\beta = 0$ we also denote the contour $S_{\arg\sin\beta}(a)|_{\beta=\pi/2}$ by $S(a)$.

Then we deform the integration paths γ to those of steepest descent in one-dimension, namely, to $S_{\arg\sin\beta}(\pm\pi)$. Recall that for $\beta = \pi/2$ one has $S_{\arg\sin\beta}(+\pi)|_{\beta=\pi/2} = S(\pi) = \gamma_{+\pi}$. During this process, several polar singularities (see the Table A.1 in Appendix) may be captured. The contours $\Gamma'_{\pi/2}$ and $\gamma_{(\varphi_0)}$ in (7) are also deformed to those of steepest descent $S(\beta_0)$ and $S(\varphi_0)$ correspondingly.

The integral (7) reduces to

$$u(M) = u^e(M) + \frac{ik}{8\pi^2}\sum_{n=1}^{5}\int_{S(\beta_0)} d\beta \int_{S(\varphi_0)} d\alpha\ \sin\beta\, e^{ik[z\cos\beta+\sin\beta(r_0\cos(\alpha-\varphi_0)-r\cos s_n(\alpha,\beta))]}\, r_n(\alpha,\beta), \tag{15}$$

$r_n(\alpha,\beta) = \mathcal{H}_{\sigma(\beta)}(A_n)R_n(\alpha,\beta)$ and

$$u^e(M) = \frac{ik}{8\pi^2}\int_{S(\beta_0)} d\beta \int_{S(\varphi_0)} d\alpha \int_{S_{\arg\sin(\beta)}(0)} \frac{ds}{2\pi i}\ \sin\beta\, e^{ik[z\cos\beta+\sin\beta(r_0\cos(\alpha-\varphi_0)+r\cos s)]}\times \\ [f(s+\pi+\varphi;\alpha,\beta) - f(s-\pi+\varphi;\alpha,\beta)], \tag{16}$$

where $R_n(\alpha,\beta)$, $\mathcal{H}_{\sigma(\beta)}$ are defined in Appendix and the boundaries of the curvilinear strip $\sigma(\beta)$ are $S_{\arg\sin\beta}(\pm\pi)$. Remark that in (16) we made the change of the integration variable s in accordance with $s \to s \pm \pi$ so that the contours $S_{\arg\sin(\beta)}(\mp\pi)$ are transformed to $S_{\arg\sin(\beta)}(0)$.

4.2. The incident field from the source, the reflected waves

Consider the summands with $n = 1, 2, 3$ for the double integral in (15) and apply the 2D saddle point technique directly following the derivations in Sect. 3.3.3 in [7] (see also (13), (14)) which are only briefly described in this Section.

As a result of simple calculations we find the expressions for the direct (incident) and waves reflected from the faces. For the incident wave we have

$$u_0(M) = \frac{e^{ikR_0}}{4\pi R_0} \tag{17}$$

as $|\varphi - \varphi_0| < \pi$, otherwise, there is no contribution from the saddle point, $u_0(M) = 0$.

We notice that $(\alpha_0^\pm, \beta_0^\pm)$ are the saddle points of the double integral $(k \to \infty)$

$$\frac{ik}{8\pi^2}\sum_{n=2}^{3}\int_{S(\beta_0)} d\beta \int_{S(\varphi_0)} d\alpha\ \sin\beta\, e^{ik[z\cos\beta+\sin\beta(r_0\cos(\alpha-\varphi_0)-r\cos s_n(\alpha,\beta))]},$$

where $n = 2$ corresponds to (α_0^+, β_0^+) and $n = 3$ to (α_0^-, β_0^-),

$$\alpha_0^\pm = \arctan\frac{y_0 - r\sin(\pm 2\Phi - \varphi)}{x_0 - r\cos(\pm 2\Phi - \varphi)}, \quad \beta_0^\pm = \arccos\frac{z}{\psi^{r\pm}}.$$

$\psi^{r\pm}$ denote the distances that the reflected waves travel from the source point via the respective points of reflection to the point of observation

$$\psi^{r\pm} = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\pm 2\Phi - \varphi - \varphi_0) + z^2}.$$

As a result of derivations analogous to those in Sect. 3.3.3 in [7], we arrive at the non-uniform asymptotic expressions for the reflected waves

$$u^r_\pm(M) = R^\pm(\alpha_0^\pm, \beta_0^\pm)\frac{e^{ik\psi^{r\pm}}}{4\pi\psi^{r\pm}} \tag{18}$$

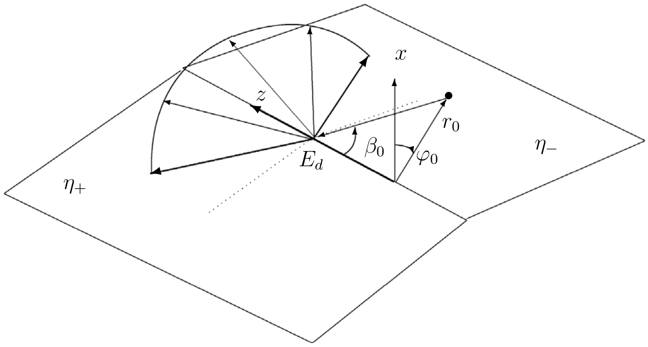

Fig. 4. The edge wave and Keller's cone.

as $|\pm 2\Phi - \varphi - \varphi_0| < \pi$, otherwise, $u^r_\pm(M) = 0$. The reflected waves can be interpreted as the waves emanated by the imaginary sources which are the mirror images of the real source at $(x_0, y_0, 0)$ w.r.t. the wedge's faces $S_\pm$, $R^\pm(\alpha_0^\pm, \beta_0^\pm)$ are the reflection coefficients, see Appendix. The eikonals of these waves are $\psi^{r\pm}$ correspondingly.

It is worth remarking that the expressions for the incident and reflected waves in (17), (18) are in full agreement with the Geometrical Optics.

4.3. The space wave, excited by the incident space wave, from the edge

We intend to apply the multi-dimensional saddle point technique [17] to the triple integral (16). In the integral expression for u^e in (16) we consider the phase function of rapidly varying exponent

$$\mathcal{S}^e = \mathrm{i}\,\{z\cos\beta + \sin\beta\,[r_0\cos(\alpha - \varphi_0) + r\cos s]\}$$

and deform integration contours to those of the steepest descent. To this end, we study the saddle points.

The equations for the saddle point

$$\begin{aligned}
&\mathcal{S}^e_\alpha = -\mathrm{i}\sin\beta r_0 \sin(\alpha - \varphi_0) = 0,\\
&\mathcal{S}^e_\beta = \mathrm{i}\,\{-z\sin\beta + \cos\beta\,[r_0\cos(\alpha - \varphi_0) + r\cos s]\} = 0,\\
&\mathcal{S}^e_s = -\mathrm{i}r\sin\beta\sin s = 0
\end{aligned}$$

are satisfied by $w_0 = (\varphi_0, \beta_0, 0)$, where β_0 is a solution of the equation $\cot\beta_0 = \frac{z}{r+r_0}$ from the segment $(0, \pi)$.

It is instructive to discuss the geometrical meaning of the saddle point w_0. The point E_d on the edge with the Cartesian coordinates $(0, 0, z_d)$ (with $z_d = \frac{z r_0}{r+r_0}$) is called diffraction point, Fig. 4. The incident ray goes from the source to the point E_d under the angle of incidence $\pi/2 - \beta_0$[5] and gives rise to the right circular cone of the diffracted rays i.e. to the so called Keller cone with the vertex at E_d and with the axis $E_d z$ having the opening angle $\pi/2 - \beta_0$ and composed by the diffracted rays. The observation point $M = (r, \varphi, z)$ belongs to this cone so that one of the diffracted rays arrives at this point.

We can compute the second derivatives and then $\det[\{\mathcal{S}^e\}'']$ is assumed to be non-degenerate,

$$\det[\{\mathcal{S}^e\}''] = \mathrm{i}r r_0 \sin^2\beta_0\{z\cos\beta_0 + (r + r_0)\sin\beta_0\} \neq 0.$$

The latter means that the observation point $M = (r, \varphi, z)$ is located outside the pernumbral domains for the incident or reflected waves.

Applying the formula for the leading term in the 3D steepest descent technique to the integral

$$\begin{aligned}
u^e(M) = \frac{\mathrm{i}k}{8\pi^2}\int_{S(\beta_0)} \mathrm{d}\beta \int_{S(\varphi_0)} \mathrm{d}\alpha \int_{S_{\arg\sin(\beta)}(0)} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \sin\beta\, \mathrm{e}^{\mathrm{i}k[z\cos\beta + \sin\beta(r_0\cos(\alpha-\varphi_0) + r\cos s)]} \times\\
[f(s + \pi + \varphi; \alpha, \beta) - f(s - \pi + \varphi; \alpha, \beta)],
\end{aligned}$$

we arrive at[6]

$$u^e(M) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{z^2+(r+r_0)^2}}}{\sqrt{(z^2 + (r + r_0)^2)}} \left\{\frac{\sqrt{z^2 + (r + r_0)^2}}{2\pi k r r_0}\right\}^{1/2} \mathcal{D}(\varphi, \varphi_0, \beta_0)\left(1 + O\left(\frac{1}{k}\right)\right), \tag{19}$$

5 This is the angle between the incident ray and the plane orthogonal to the edge and conducted at the diffraction point E_d.

6 The formula for the leading term of the asymptotics is given in [17], see also pp. 89–92 in [7].

where $\mathcal{D}(\varphi, \varphi_0, \beta_0) = f(-\pi + \varphi; \varphi_0, \beta_0) - f(\pi + \varphi; \varphi_0, \beta_0)$ is the diffraction coefficient of the edge wave. As we have already mentioned the asymptotics of the edge wave in (19) is non-uniform w.r.t. φ, however, the corresponding uniform version in the framework of Uniform Asymptotic Theory (UAT) of diffraction can easily be developed in the line with Sect. 3.4.2, [7], i.e. with the aid of the Fresnel type transition functions. Remark that the non-uniform expression (19) can also be written in terms of the incident and diffracted rays introducing the corresponding eikonals l and l^i as it is described in Sect. 3.4.1 of [7].

It is worth commenting on the expression (19) in a limiting situation as $z = 0$ and also $kr_0 \to \infty$. This will enable us to verify the result of calculation by its reduction to a formula known from literature. Expression for the cylindrical wave is well known in the far field asymptotics for the plane wave which is normally incident at the edge. It is also known that the expression for the plane wave can be obtained from that for the point source if the latter is moved to infinity ($kr_0 \to \infty$) and its amplitude is appropriately normalized. It is, therefore, natural to compare the expression for the edge wave (19) with that in the limiting case $z = 0$ and $kr_0 \to \infty$. To this end, we set $\beta_0 \to \pi/2$, $r/r_0 \to 0$ and find

$$u^e(M) \sim \frac{\mathrm{e}^{ikr_0}}{4\pi r_0} \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{\mathrm{e}^{ikr_0}}{\sqrt{kr}} \mathcal{D}(\varphi, \varphi_0, \pi/2) \left(1 + O\left(\frac{1}{k}\right)\right). \tag{20}$$

In these limiting conditions for the incident wave field we have the expression

$$u_0(M) = \frac{\mathrm{e}^{ikr_0}}{4\pi r_0} \mathrm{e}^{-\mathrm{i}kr\cos[\varphi-\varphi_0]} \left(1 + O\left(\frac{1}{kr_0}\right)\right)$$

which represents the 2D plane incident wave with the amplitude $\mathcal{A} = \frac{\mathrm{e}^{ikr_0}}{4\pi r_0}$. As a result, the formula (20) gives the correct expression for the 2D circular wave from the vertex excited by the 2D plane wave $\mathcal{A}\,\mathrm{e}^{-\mathrm{i}kr\cos[\varphi-\varphi_0]}$, compare with the formula (6.53) in [5].

4.4. Surface waves propagating from the edge

Now we turn to the summands with $n = 4, 5$ in the expression (15), the asymptotics of which describe the surface waves propagating from the edge. They are excited by the incident wave from the point source interacting with the edge. We shall apply the 2D saddle point technique and, to this end, it is useful to make the change of the integration variable β in accordance with $\tau = \cos\beta$. The latter mapping transforms the strip $|\Re\beta - \pi/2| \le \pi/2$ onto the complex plane τ with the cuts conducted from $-\infty$ to -1 and the from 1 to $+\infty$. The integration contour $\Gamma'_{\pi/2}$ is reduced to the contour R' which passes from $+\infty$ along the lower side of the branch cut to $+1+0$ then comprises it along the small arc in the lower half plane and goes to $-1+0$ along the segment $(-1, 1)$ and in a similar way compassing the point -1 in the upper half plane arriving at infinity along the upper side of the branch cut $(-\infty, -1]$. Remark that $-1 < \tau < 1$ provided $0 < \beta < \pi$.

Thus we have[7]

$$u^{sw}_{\pm}(M) = \mathrm{e}^{-\mathrm{i}kr\sin(\Phi\mp\varphi)\eta^{\pm}} \times$$
$$\frac{-\mathrm{i}k}{8\pi^2}\int_{R'} d\tau \int_{S(\varphi_0)} d\alpha\; \mathcal{H}_{\sigma(\beta(\tau))}(A_{\pm})C^{\pm}(\alpha, \beta(\tau))\, \mathrm{e}^{\mathrm{i}k[z\tau+\sqrt{1-\tau^2}r_0\cos(\alpha-\varphi_0)+\sqrt{1-\eta_{\pm}^2-\tau^2}\, r\cos(\Phi\mp\varphi)]}, \tag{21}$$

where

$$\mathcal{S}^{sw}_{\pm} = \mathrm{i}[z\tau + \sqrt{1-\tau^2}r_0\cos(\alpha-\varphi_0) + \sqrt{1-\eta_{\pm}^2-\tau^2}\, r\cos(\Phi\mp\varphi)]$$

is the phase function of rapidly varying exponent as $k \to \infty$,

$$C^{\pm}(\alpha, \beta) = R_{\vartheta^{\pm}}(\beta)\cdot f(\Phi - \pi - \vartheta^{\pm}(\beta); \alpha, \beta)$$

is the slowly varying factor. The non-degenerate saddle points are found from the equations

$$(\mathcal{S}^{sw}_{\pm})'_{\alpha} = 0, \quad (\mathcal{S}^{sw}_{\pm})'_{\tau} = 0,$$

or

$$-\mathrm{i}r_0\sin(\alpha-\varphi_0)\sqrt{1-\tau^2} = 0,$$

$$\mathrm{i}\left(z - r_0\cos(\alpha-\varphi_0)\frac{\tau}{\sqrt{1-\tau^2}} - r\cos[\Phi\mp\varphi]\frac{\tau}{\sqrt{1-\eta_{\pm}^2-\tau^2}}\right) = 0$$

so that the saddle points $(\varphi_0, \tau_0^{\pm})$ of $\mathcal{S}^{sw}_{\pm}$ are correspondingly specified by

$$\alpha = \varphi_0, \quad \tau = \tau_0^{\pm}.$$

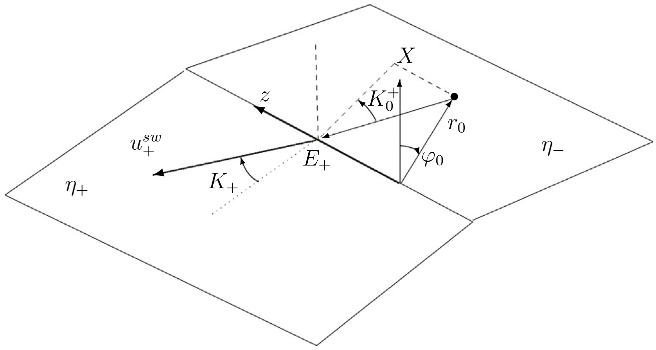

Fig. 5. Surface waves from the edge.

We take $\tau_0^+ = \cos\beta_0^+$ for the upper signs and $\tau_0^- = \cos\beta_0^-$ for the lower signs in the equation

$$z = r_0 \frac{\tau_0^\pm}{\sqrt{1-(\tau_0^\pm)^2}} + r\cos[\Phi \mp \varphi]\frac{\tau_0^\pm}{\sqrt{1-\eta_\pm^2-(\tau_0^\pm)^2}}$$

correspondingly. It is useful to introduce the geometrical characteristics that can be attributed to the latter equation for $\tau_0^\pm$ and to clarify the geometrical optics meaning of the equation, Fig. 5. Let $K_0^\pm$ be correspondingly the angle between the incident ray (of the length $L_\pm = r_0/\cos K_0^\pm$), that impinges the edge at the point $E_\pm$ of diffraction, and the orthogonal to the edge plane conducted through the point $E_\pm$ of diffraction on the edge. (Remark that in general case E_+ and E_- are different points.) Then, along $S_\pm$ it goes to the point of normal projection of the observation point M on $S_\pm$ (which is assumed to be close to $S_\pm$ correspondingly) under the angle of transmission $K_\pm$. The lengths of the diffracted rays along the surface $S_\pm$ are $r\cos[\Phi\mp\varphi]/\cos K_\pm$ correspondingly. As a result, we have an equivalent of the equation in geometrical optics form

$$z = r_0\tan K_0^\pm + r\cos[\Phi\mp\varphi]\tan K_\pm$$

with

$$\tan K_0^\pm = \frac{\tau_0^\pm}{\sqrt{1-(\tau_0^\pm)^2}}, \quad \tan K_\pm = \frac{\tau_0^\pm}{\sqrt{1-\eta_\pm^2-(\tau_0^\pm)^2}}.$$

We can easily compute the second derivatives of $\mathcal{S}_\pm^{sw}$ at the stationary points,

$$\det \mathcal{S}_\pm^{sw} = -\left(\frac{r_0^2}{1-(\tau_0^\pm)^2} + \frac{r_0 r\cos[\Phi\mp\varphi]}{1-\eta_\pm^2-(\tau_0^\pm)^2}\frac{\sqrt{1-(\tau_0^\pm)^2}}{\sqrt{1-\eta_\pm^2-(\tau_0^\pm)^2}}\right)$$

and $\mathcal{S}_\pm^{sw}|_{(\varphi_0,\tau_0^\pm)} = \mathrm{i}[z\tau_0^\pm + \sqrt{1-(\tau_0^\pm)^2}r_0 + \sqrt{1-\eta_\pm^2-(\tau_0^\pm)^2}\, r\cos(\Phi\mp\varphi)]$.

Applying the formula for the leading term of the asymptotics of the double integral (21), we find ($\tau_0^\pm = \cos\beta_0^\pm = \sin K_0^\pm$)

$$u_\pm^{sw}(M) = \sin\beta_0^\pm C^\pm(\varphi_0,\beta_0^\pm)\frac{\mathrm{e}^{-\mathrm{i}kr\sin(\Phi\mp\varphi)\,\eta^\pm}}{4\pi}\times$$
$$\frac{\exp(\mathrm{i}k[z\cos\beta_0^\pm + \sin\beta_0^\pm r_0\cos(\alpha-\varphi_0) + \sqrt{\sin^2\beta_0^\pm-\eta_\pm^2}\, r\cos(\Phi\mp\varphi)])}{\left\{\frac{r_0^2}{\sin^2\beta_0^\pm} + \frac{r_0 r\cos[\Phi\mp\varphi]}{\sin^2\beta_0^\pm-\eta_\pm^2}\frac{\sin\beta_0^\pm}{\sqrt{\sin^2\beta_0^\pm-\eta_\pm^2}}\right\}^{1/2}}\left(1+O\left(\frac{1}{k}\right)\right). \tag{22}$$

The expressions in (22) are really present in the asymptotics if the observation point is close to the wedge's face $\varphi = \pm\Phi$ correspondingly or, more exactly, as $0 < \Phi\mp\varphi < -\mathrm{gd}(\mathrm{Im}\vartheta^\pm(\beta_0^\pm))$. It is obvious from the expressions (22) that wave field is exponentially small outside some close vicinities of $S_\pm$ as $kr \gg 1$ due to the factor $\mathrm{e}^{-\mathrm{i}kr\sin(\Phi\mp\varphi)\,\eta^\pm}$.

[7] The analysis can be given for surface waves both on S_+ and on S_- simultaneously.

5. The far-field asymptotics: the point source is near one face of the wedge

In this section we assume that the point source is located in some close vicinity of the wedge's face S_+ so that $0 < \Phi - \varphi_0 < -\mathrm{gd}(\mathrm{Im}\vartheta^+(\pi/2))$. In this case some additional components arise in the asymptotics as $k \to \infty$. Indeed, from the physical point of view the point source located near the impedance surface S_+ excites the so-called primary surface wave that propagates at the edge then, interacting with the edge, gives rise to the edge wave that differs from that described. At the same time the surface waves reflected from and transmitted across the edge discussed in [16] are added to the surface waves discussed in Section 4.4. We begin the analysis with the primary surface wave from the point source.

In the adopted assumption about the position of the point source an additional singularity $\alpha = \Phi + \vartheta^+(\beta)$ of $R^+(\alpha, \beta)$ is captured when deforming the contour of integration w.r.t. α in the integral (15) with $n = 2$ to the SD path $S(\varphi_0)$ and the corresponding residue contribution reads

$$u_0^{sw} = -\frac{k}{4\pi}\int_{\Gamma'_{\pi/2}} \mathrm{d}\beta\ \sin\beta\, \mathrm{res}_{\Phi+\vartheta^+(\beta)} R^+(\alpha,\beta)\, \mathrm{e}^{\mathrm{i}k\left\{z\cos\beta+\sin\beta\left[r_0\cos(\Phi+\vartheta^+(\beta)-\varphi_0)-r\cos(\vartheta^+(\beta)-\Phi+\varphi)\right]\right\}}.$$

As in the previous section we make use of the new integration variable $\tau = \cos\beta$ and arrive at the integral

$$u_0^{sw} = \frac{k}{4\pi}\, \mathrm{e}^{\mathrm{i}k[r_0\sin(\varphi_0-\Phi)\eta_+ + r\sin(\varphi-\Phi)\eta_+]}\int_{R'} \mathrm{d}\tau\, r^+(\tau)\, \mathrm{e}^{\mathrm{i}k\left\{z\tau+\rho_0\sqrt{1-\eta_+^2-\tau^2}\right\}}, \tag{23}$$

where $\rho_0 = r_0\cos[\Phi-\varphi_0] - r\cos[\Phi-\varphi] > 0$, $r^+(\tau) = \mathrm{res}_{\Phi+\vartheta^+(\beta)} R^+(\alpha,\beta)|_{\tau=\cos\beta}$. Applying the stationary phase technique to the integral (23), we arrive at the leading term of the primary surface wave from a point source

$$u_0^{sw}(M) = r^+(\tau_*)\times \frac{k\mathrm{e}^{\mathrm{i}3\pi/4}}{2\sqrt{2\pi}}\,\frac{\mathrm{e}^{\mathrm{i}k[r_0\sin(\varphi_0-\Phi)\eta_+ + r\sin(\varphi-\Phi)\eta_+]}}{\sqrt{(1-\eta_+^2)k\rho_0}}\left\{\frac{1-\eta_+^2}{1+z^2/\rho_0^2}\right\}^{3/4}\mathrm{e}^{\mathrm{i}k\sqrt{(1-\eta_+^2)(z^2+\rho_0^2)}}\left(1+O\left(\frac{1}{k}\right)\right), \tag{24}$$

where $\tau_* = z\sqrt{\frac{1-\eta_+^2}{z^2+\rho_0^2}}$.

5.1. Reflection and transmission of the primary surface wave at the edge of the wedge

The primary surface wave (24) propagates **to** the edge and gives rise to the edge wave and to the reflected and transmitted surface wave. Derivation of the expressions for those waves is the main goal of our simple calculations in this section. From the analytical point of view the reflected surface wave is described by the additional contribution of the polar singularity at $\alpha = \Phi + \vartheta^+(\beta)$ of $C^+(\alpha,\beta) = R_{\vartheta^+}(\beta)\cdot f(\Phi-\pi-\vartheta^+(\beta);\alpha,\beta)$ that is captured when deforming the integration contour $\gamma_{(\varphi_0)}$ into $S(\varphi_0)$ for the summand in (15) with $n = 4$ (see also (21)). The residue contribution then reads

$$u_r^{sw}(M) = -\frac{k}{4\pi}\int_{\Gamma'_{\pi/2}} \mathrm{d}\beta\ \sin\beta\, R_{\vartheta^+}(\beta)\mathrm{res}_{\Phi+\vartheta^+(\beta)}\ f(\Phi-\pi-\vartheta^+(\beta);\alpha,\beta)\times \mathrm{e}^{\mathrm{i}k\left\{z\cos\beta+\sin\beta\left[r_0\cos(\Phi+\vartheta^+(\beta)-\varphi_0)+r\cos(\vartheta^+(\beta)+\Phi-\varphi)\right]\right\}}.$$

We make use of the new integration variable $\tau = \cos\beta$ and arrive at the integral

$$u_r^{sw}(M) = \frac{k}{4\pi}\, \mathrm{e}^{-\mathrm{i}k[r_0\sin(\Phi-\varphi_0)\eta_+ + r\sin(\Phi-\varphi)\eta_+]}\int_{R'} \mathrm{d}\tau\, r_{\vartheta^+}(\tau)\, \mathrm{e}^{\mathrm{i}k\left\{z\tau+\rho\sqrt{1-\eta_+^2-\tau^2}\right\}}, \tag{25}$$

$r_{\vartheta^+}(\tau) = R_{\vartheta^+}(\beta)\mathrm{res}_{\Phi+\vartheta^+(\beta)}\ f(\Phi-\pi-\vartheta^+(\beta);\alpha,\beta)|_{\tau=\cos\beta}$ and $\rho = r_0\cos[\Phi-\varphi_0] + r\cos[\Phi-\varphi]$. The integral in (25) is computed asymptotically as $k \to \infty$. The stationary point τ_0 of the phase function

$$\mathcal{S}_+(\tau) = z\tau + \rho\sqrt{1-\eta_+^2-\tau^2}$$

satisfies the equation

$$\mathcal{S}'_+(\tau_+) = z - \frac{\rho\,\tau_+}{\sqrt{1-\eta_+^2-\tau_+^2}} = 0,$$

and $\tau_+ = z\sqrt{\frac{1-\eta_+^2}{z^2+\rho^2}}$. It is non-degenerate with

$$\mathcal{S}''_+(\tau_+) = -\rho(1-\eta_+^2)\left\{\frac{1+z^2/\rho^2}{1-\eta_+^2}\right\}^{3/2} < 0.$$

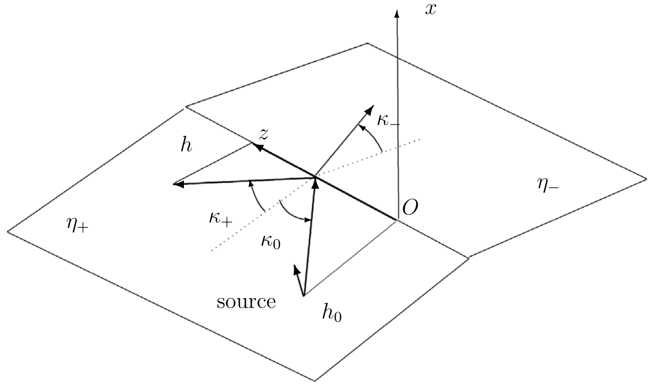

Fig. 6. Reflection and transmission of the primary surface wave at the edge, $h = r\cos[\Phi - \varphi]$, $h_0 = r_0\cos[\Phi - \varphi_0]$.

The stationary point τ_+ has simple geometrical meaning. We introduce the point of reflection $(0, 0, z_r)$ on the edge and the angle of reflection κ_+ by the equality (see Fig. 6)

$$\tan\kappa_+ = \frac{z - z_r}{h}, \quad h = r\cos[\Phi - \varphi].$$

In the same manner, we can introduce the angle of incidence of the primary surface wave (see (24)) by the equality

$$\tan\kappa_0 = \frac{z_r}{h_0}, \quad h_0 = r_0\cos[\Phi - \varphi_0].$$

As a result, the equation for the stationary point is written as

$$z = h_0\tan\kappa_0 + h\tan\kappa_+, \quad \tan\kappa_0 = \frac{\tau_*}{\sqrt{1-\eta_+^2-\tau_*^2}} = \tan\kappa_+$$

with $\kappa_+ = \kappa_0$ which is the Geometrical Optics law of reflection of the surface wave at the edge, see Fig. 6. In the leading approximation, for the reflected surface wave we find

$$u_r^{sw}(M) = r_{\vartheta^+}(\tau_+)\times \frac{k e^{i3\pi/4}}{2\sqrt{2\pi}}\,\frac{e^{ik[r_0\sin(\varphi_0-\Phi)\eta_+ + r\sin(\varphi-\Phi)\eta_+]}}{\sqrt{(1-\eta_+^2)k\rho}}\left\{\frac{1-\eta_+^2}{1+z^2/\rho^2}\right\}^{3/4} e^{ik\sqrt{(1-\eta_+^2)(z^2+\rho^2)}}\left(1+O\left(\frac{1}{k}\right)\right). \tag{26}$$

In a similar manner we deal with the transmitted surface wave that originates from the analysis of the contribution of the polar singularity at $\alpha = \Phi + \vartheta^+(\beta)$ of $C^-(\alpha,\beta) = R_{\vartheta^-}(\beta)\cdot f(-\Phi+\pi+\vartheta^-(\beta);\alpha,\beta)$ that is captured when deforming the integration contour $\gamma(\Phi-\pi/2)$ into $S(\varphi_0)$ for the summand in (15) with $n = 5$. After the change of variable $\tau = \cos\beta$ we arrive at the integral

$$u_t^{sw}(M) = \frac{k}{4\pi}\,e^{-ik[r_0\sin(\Phi-\varphi_0)\eta_- + r\sin(\Phi+\varphi)\eta_-]}\int_{R'} d\tau\, r_{\vartheta^-}(\tau)\, e^{ik\left\{z\tau+\rho\gamma^+\sqrt{1-\eta_+^2-\tau^2}+\rho\gamma^-\sqrt{1-\eta_-^2-\tau^2}\right\}}, \tag{27}$$

where $\gamma^+ = r_0\cos(\Phi-\varphi_0)/\rho$, $\gamma^- = r\cos(\varphi+\Phi)/\rho$, $r_{\vartheta^-}(\tau) = R_{\vartheta^+}(\beta)\mathrm{res}_{\alpha=\Phi+\vartheta^+(\beta)}\, f(-\Phi+\pi+\vartheta^-(\beta);\alpha,\beta)|_{\tau=\cos\beta}$.

The phase function $\mathcal{S}_-(\tau) = z\tau + \rho\gamma^+\sqrt{1-\eta_+^2-\tau^2} + \rho\gamma^-\sqrt{1-\eta_-^2-\tau^2}$ of the integrand in (27) has the stationary point τ_- solving the equation

$$\mathcal{S}_-'(\tau_-) = z - \frac{\rho\,\gamma^+\tau_-}{\sqrt{1-\eta_+^2-\tau_-^2}} - \frac{\rho\,\gamma^-\tau_-}{\sqrt{1-\eta_-^2-\tau_-^2}} = 0$$

which is non-degenerate and

$$\mathcal{S}_-''(\tau_-) = -\rho\left(\frac{\gamma^+(1-\eta_+^2)\tau_-}{[\sqrt{1-\eta_+^2-\tau_-^2}]^3} + \frac{\gamma^-(1-\eta_-^2)\tau_-}{[\sqrt{1-\eta_-^2-\tau_-^2}]^3}\right) < 0.$$

We can attribute some geometrical meaning to the equation for the stationary point introducing the angles of incidence κ_+ and of refraction κ_- of the surface wave (Fig. 6)

$$\tan\kappa_\pm = \frac{\tau_-}{\sqrt{1-\eta_\pm^2-\tau_-^2}}$$

and writing the equation for the stationary point as

$$z = r\cos(\varphi+\Phi)\tan\kappa_- + r_0\cos(\Phi-\varphi_0)\tan\kappa_+.$$

The Snell's type law of refraction of the primary surface wave across the edge of two impedance halfplanes reads

$$\sqrt{1-\eta_+^2}\sin\kappa_+ = \sqrt{1-\eta_-^2}\sin\kappa_-.$$

Remark 2. In the assumption $|\eta_-| < |\eta_+|$ the phenomenon of the total internal reflection of the primary surface wave arriving at the edge may occur.[8] In this case the stationary point approaches one of the branch point $\pm\sqrt{1-\eta_-^2}$. (Recall that a contribution from the branch cut is usually attributed to the so called head surface wave.) The corresponding critical angle of the total internal reflection is given by ($\kappa_- \to \pi/2$)

$$\kappa_+^* = \arcsin\left(\sqrt{\frac{1-\eta_-^2}{1-\eta_+^2}}\right).$$

For the critical angle $\kappa_+ = \kappa_+^*$ the transmitted surface wave propagates along the edge ($\kappa_- = \pi/2$) and also rapidly vanishes as $kr \to \infty$.

The leading term of the transmitted surface wave takes the form

$$u_t^{sw}(M) = r_{\vartheta^-}(\tau_-)\frac{k\mathrm{e}^{\mathrm{i}3\pi/4}}{2\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}k[r_0\sin(\varphi_0-\Phi)\eta_- + r\sin(\varphi-\Phi)\eta_-]}}{\sqrt{k\rho}}\times$$
$$\left\{\frac{\gamma^+(1-\eta_+^2)\tau_-}{[\sqrt{1-\eta_+^2-\tau_-^2}]^3} + \frac{\gamma^-(1-\eta_-^2)\tau_-}{[\sqrt{1-\eta_-^2-\tau_-^2}]^3}\right\}^{-1/2} \mathrm{e}^{\mathrm{i}k[z\tau_- + \rho\gamma^+\sqrt{1-\eta_+^2-\tau_-^2}+\rho\gamma^-\sqrt{1-\eta_-^2-\tau_-^2}]}\left(1+O\left(\frac{1}{k}\right)\right). \quad (28)$$

The expression (28) can be also written with the aid of the introduced angles $\kappa_\pm$ in order to underline the geometrical nature of refraction of the surface waves.

A similar case, when the surface wave interacts with the edge, however, coming along the surface from infinity, is discussed in [16].

It is worth commenting on the reciprocity principle because it is easily manifested in the formulas for the surface waves. Indeed, in the asymptotic formulas (26), (28) interchange of the source and observation points leads to a simple substitutions of $r \to r_0$, $\varphi \to \varphi_0$ and to other simple modifications. The expressions have obviously symmetric form and do not alter its form while interchanging. Such a demonstration of reciprocity for the other components requires more work, however, there is no doubt that it could be done.

5.2. The edge wave generated by the primary surface wave interacting with the edge

The expression for this wave is also obtained by means of the asymptotic evaluation of the integral that originates from the residue of $f(s+\pi+\varphi;\alpha,\beta) - f(s-\pi+\varphi;\alpha,\beta)$ of the integrand in (16). The corresponding singularity at $\alpha = \Phi + \vartheta^+(\beta)$ is captured provided that the source is close to the face S_+ when deforming the integration contour $\gamma_{(\varphi_0)}$ into $S(\varphi_0)$ in (16). We find

$$u_e^{sw}(M) = -\frac{k}{4\pi}\int_{\Gamma'_{\pi/2}}\mathrm{d}\beta\int_{S_{\arg\sin(\beta)}(0)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\sin\beta\,\mathrm{e}^{\mathrm{i}k[z\cos\beta+\sin\beta(r_0\cos(\Phi+\vartheta^+(\beta)-\varphi_0)+r\cos s)]}\times$$
$$\mathrm{res}_{\alpha=\Phi+\vartheta^+(\beta)}[f(s+\pi+\varphi;\alpha,\beta)-f(s-\pi+\varphi;\alpha,\beta)]. \quad (29)$$

Then introduce the new variable of integration $\tau = \cos\beta$, from (29) thus obtain

$$u_e^{sw}(M) = \frac{k}{4\pi}\mathrm{e}^{-\mathrm{i}kr_0\sin(\Phi-\varphi_0)\eta_+}\int_{R'}\mathrm{d}\tau\int_{S_{\arg\sin\beta(\tau)}(0)}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\times$$
$$\mathrm{e}^{\mathrm{i}k[z\tau+\sqrt{1-\tau^2}r\cos s+r_0\cos(\Phi-\varphi_0)\sqrt{1-\eta_+^2-\tau^2}]}d(\tau,s;\varphi), \quad (30)$$

where $d(\tau,s;\varphi) = \mathrm{res}_{\alpha=\Phi+\vartheta^+(\beta)}[f(s+\pi+\varphi;\alpha,\beta)-f(s-\pi+\varphi;\alpha,\beta)]|_{\tau=\cos\beta}$ is computed explicitly.

[8] Recall that $\eta_\pm$ are assumed to be purely imaginary.

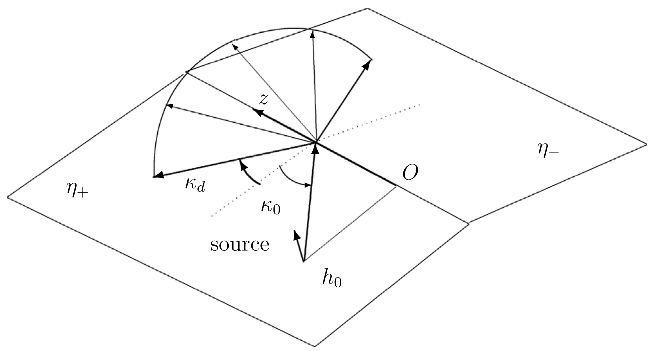

Fig. 7. The edge wave generated by the primary surface wave interacting with the edge.

Asymptotic evaluation of the double integral (30) as $k \to \infty$ is conducted in a traditional manner. We specify the saddle point as a solution of the equations

$$(\mathcal{S}_{swe})'_s = 0\,, \quad (\mathcal{S}_{swe})'_\tau = 0,$$

with $\mathcal{S}_{swe} = \mathrm{i}[z\tau + r\cos s\sqrt{1-\tau^2} + r_0\cos(\Phi-\varphi_0)\sqrt{1-\eta_+^2-\tau^2}]$. Actually, our calculations are straightforward. However, we consider some details because, in this way, the geometrical law of diffraction of the primary surface wave by the edge is directly manifested. The latter equations are written as

$$-r\sqrt{1-\tau^2}\sin s = 0,$$

$$z - r\cos s\frac{\tau}{\sqrt{1-\tau^2}} - r_0\cos[\Phi-\varphi_0]\frac{\tau}{\sqrt{1-\eta_+^2-\tau^2}} = 0$$

and have a solution $(\tau_e, s=0)$, where τ_e ($\cos\beta_e = \tau_e < 1$) is real. It satisfies the **equation**

$$z = r\tan\kappa_d + r_0\cos[\Phi-\varphi_0]\tan\kappa_0,$$

where the angles of diffraction and of incidence of the surface wave are defined by

$$\tan\kappa_d = \frac{\tau_e}{\sqrt{1-\tau_e^2}}\,, \quad \tan\kappa_0 = \frac{\tau_e}{\sqrt{1-\eta_+^2-\tau_e^2}}.$$

The geometrical meaning of the equation for τ_e is obvious from Fig. 7, where the diffractional (Keller) cone of the edge wave is specified by the angle κ_d, see also [22]. The Geometrical Theory of Diffraction (of Keller with abbreviation GTD) traditionally operates with various types of rays: incident, reflected, diffracted or others. In our case, the diffracted rays in Fig. 7 compose the Keller cone the opening of which is $\pi/2 - \kappa_d$, where κ_d is determined from the transcendental equation

$$\frac{\sin\kappa_d}{\sqrt{\cos^2\kappa_d - \eta_+^2}} = \tan\kappa_0$$

if the angle of incidence κ_0 is given. In GTD the latter equation can naturally be called the law of conical diffraction of the surface wave at the edge of an impedance wedge. It obviously has an asymptotic nature and can be exploited together with high-frequency localization principle. It is worth noticing the existence of the critical angle κ_0^* for edge wave which corresponds to $\kappa_d = \pi/2$,

$$\kappa_0^* = \arctan\left(\frac{1}{|\eta_+|}\right).$$

For this angle the edge wave collapses to be concentrated near the edge. The point τ_e goes to the branch points of $\sqrt{1-\tau^2}$ and disappears through the cut. The asymptotics of the integral, in this case, requires a special study. For the electromagnetic case some additional details can be found in [23].

The saddle point $(\tau_e, 0)$ is non-degenerate with

$$\det\{\mathcal{S}''_{swe}\} = \frac{r^2}{1-\tau_e^2} + rr_0\cos[\Phi-\varphi_0]\frac{(1-\eta_+^2)\sqrt{1-\tau_e^2}}{\sqrt{1-\eta_+^2-\tau_e^2}}.$$

Table A.1
Polar singularities and residues of $f(s+\varphi;\alpha,\beta)$.

n	Singularity $s_n(\varphi,\alpha,\beta)$	Residue $R_n(\alpha,\beta)$	Argument A_n
1	$\alpha-\varphi$	1	$\pi-[\alpha-\varphi]$
2	$2\Phi-\alpha-\varphi$	$R^+(\alpha,\beta)$	$\pi-2\Phi+\alpha+\varphi$
3	$-2\Phi-\alpha-\varphi$	$R^-(\alpha,\beta)$	$\pi-2\Phi-\alpha-\varphi$
4	$\pi+\Phi+\vartheta^+(\beta)-\varphi$	$R_{\vartheta^+}(\beta)\cdot f(\Phi-\pi-\vartheta^+(\beta);\alpha,\beta)$	$-\Phi-\varphi-\mathrm{gd}\left(\mathrm{Im}\,\vartheta^+(\beta)\right)$
5	$-\pi-\Phi-\vartheta^-(\beta)-\varphi$	$R_{\vartheta^-}(\beta)\cdot f(-\Phi+\pi+\vartheta^-(\beta);\alpha,\beta)$	$-\Phi-\varphi-\mathrm{gd}\left(\mathrm{Im}\,\vartheta^-(\beta)\right)$

The leading term of the asymptotics of (30) reads ($\tau_e=\cos\beta_e$)

$$u_e^{sw}(M)=\frac{d(\tau_e,0;\varphi)}{4\pi}\left\{\frac{r^2}{\sin^2\beta_e}+rr_0\cos[\Phi-\varphi_0]\frac{(1-\eta_+^2)\sin\beta_e}{\sqrt{\sin^2\beta_e-\tau_e^2}}\right\}^{-1/2}$$
$$\mathrm{e}^{-ikr_0\sin(\Phi-\varphi_0)\eta_+}\,\mathrm{e}^{ik[z\cos\beta_e+\sin\beta_e r+r_0\cos(\Phi-\varphi_0)\sqrt{\sin^2\beta_e-\eta_+^2}]}\left(1+O\left(\frac{1}{k}\right)\right). \tag{31}$$

Remark that the formula (31) can also be written in terms of the angles κ_0 and κ_d.

6. Conclusion

In this paper, we applied recently developed results [7] (Chapter 3), [8,9], obtained in the case of electromagnetic problem, to the acoustic one. A principal difference of the acoustic case is that, contrary to the electromagnetic problem which requires solution of an integral equation, it is explicitly solvable. The corresponding non-uniform asymptotic results are written in terms of the Malyuzhinets' and elementary functions. The Weyl integral representation played a crucial role, whereas the integrand was found explicitly in terms of the Malyuzhinets' solution of an auxiliary problem.

We could obtain asymptotic components of the total field as $k\to\infty$. In this way, as a result of asymptotic evaluation of the integrals we also clarified physical meaning of the wave components computed. In particular, the laws of the Geometrical Theory of Diffraction describing the interaction of the primary surface wave with the edge were discussed.

One of the further prospects is in study of possible excitation of the edge waves, i.e. the waves whose energy is concentrated near the edge. Such localized waves, together with the other waves, might actually propagate along the edge in the opposite directions from the source located near the edge. To our mind, existence of such phenomenon can be expected, for instance, provided the impedances of the faces coincide and the wedge's opening 2Φ is less than π. Remark that the existence and excitation of the edge waves in elasticity is also of great practical importance.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Deformation of the Sommerfeld contour and contribution of the captured singularities, $-\Phi<\alpha<\Phi$, $0<\beta<\pi$

Recall that we can deform the Sommerfeld contours into SDPs $\gamma_{\pm\pi}$ in the expression (11) and make use of the theorem on residues thus formally writing

$$U(r,\varphi;\alpha,\beta)=\sum_{n=1}^{5}\mathcal{H}_\sigma(A_n)R_n(\alpha,\beta)\exp\{-ikr\sin\beta\cos s_n\}+$$
$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\pi\cup\gamma_{-\pi}}\mathrm{d}s\,\mathrm{e}^{-ikr\sin\beta\cos s}f(s+\varphi;\alpha,\beta), \tag{32}$$

where the residues are described in the Table A.1. The Heaviside type function $\mathcal{H}_\sigma(A_n)$ is defined as follows. By definition σ is the strip on the complex plane s between the curves γ_π and $\gamma_{-\pi}$ then $\mathcal{H}_\sigma(A_n)=1$ provided $A_n\in\sigma$, otherwise, it is zero.

In the Table A.1 we use the following notations

$$R^{\pm}(\alpha,\beta)=\frac{\sin(\Phi\mp\alpha)-\sin\vartheta^{\pm}(\beta)}{\sin(\Phi\mp\alpha)+\sin\vartheta^{\pm}(\beta)},\quad R_{\vartheta^{\pm}}(\beta)=\pm 2\tan\vartheta^{\pm}(\beta),$$

$\mathrm{gd}(x)=\mathrm{sign}(x)\mathrm{arcccos}(1/\cosh(x))$. For the real-valued α,β ($-\Phi<\alpha<\Phi$, $0<\beta<\pi$) the residues in the Table A.1 have clear physical meaning describing incident, from $S_{\pm}$ reflected or surface waves (see [7], Sect 2.6.2 for details) in the 2D problem of diffraction by a wedge.

It is worth mentioning that the expressions above admit analytic continuation for the complex values of α,β implying that the branch of $\vartheta^{\pm}(\beta)$ is properly specified.

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