COMMUTATORS OF RELATIVE AND UNRELATIVE ELEMENTARY SUBGROUPS IN CHEVALLEY GROUPS

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ABSTRACT. In the present paper, which is a direct sequel of our papers [10, 11, 35] joint with Roozbeh Hazrat, we achieve a further dramatic reduction of the generating sets for commutators of relative elementary subgroups in Chevalley groups. Namely, let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring and let A, B be two ideals of R. We consider subgroups of the Chevalley group $G(\Phi, R)$ of type Φ over R. The unrelative elementary subgroup $E(\Phi, A)$ of level A is generated (as a group) by the elementary unipotents $x_\alpha(a)$, $\alpha \in \Phi$, $a \in A$, of level A. Its normal closure in the absolute elementary subgroup $E(\Phi, R)$ is denoted by $E(\Phi, R, A)$ and is called the relative elementary subgroup of level A. The main results of [11, 35] consisted in construction of economic generator sets for the mutual commutator subgroups $[E(\Phi, R, A), E(\Phi, R, B)]$, where A and B are two ideals of R. It turned out that one can take Stein—Tits—Vaserstein generators of $E(\Phi, R, AB)$, plus elementary commutators of the form $y_\alpha(a, b) = [x_\alpha(a), x_{-\alpha}(b)]$, where $a \in A$, $b \in B$. Here we improve these results even further, by showing that in fact it suffices to engage only elementary commutators corresponding to *one* long root, and that modulo $E(\Phi, R, AB)$ the commutators $y_\alpha(a, b)$ behave as symbols. We discuss also some further variations and applications of these results.

<div style="text-align: center">

To our distinguished colleague Ivan Panin,
a brilliant mathematician, and a wonderful friend

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In the present paper we continue the study of the mutual commutator subgroups of relative subgroups in Chevalley groups. In the context of the general linear group $\mathrm{GL}(n, R)$ such commutator formulas were first systematically considered in the groundbreaking work by Hyman Bass [1]. Soon thereafter, they were expanded to various more general contexts by a number of experts including Anthony Bak, Michael Stein, Alec Mason, Andrei Suslin, Leonid Vaserstein, Zenon Borewicz and the first-named author, and many others. One can find an outline of that stage in the survey [8].

The present paper continues the same general line of a long series of our joint papers with Roozbeh Hazrat and Alexei Stepanov, where we established similar *birelative* and *multirelative* formulas in various contexts, see, for instance, [24, 32, 33, 14,

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15, 9, 12], etc. For a much broader picture of the area and applications of those resuts we refer the reader to our surveys and conference papers [4, 5, 6, 13].

More specifically, the present paper is a natural sequel of our joint papers with Hazrat, Victor Petrov and Stepanov on relative subgroups and commutator formulas in Chevalley groups, see [7, 3, 25, 10, 11, 21, 22], compare also the pioneering early work by Hong You [39]. There we found, in particular, economic generating sets for such mutual commutator subgroups $[E(\Phi, R, A), E(\Phi, R, B)]$, which were later used by Alexei Stepanov in his oustanding results on commutator width [23].

In 2018–2019 this line of research got an astounding boost, when we noticed that for $\mathrm{GL}(n, R)$ everything works already for the unrelativised groups [29, 34, 36, 37]. In [35] we have partially generalised these results to Chevalley groups, by proving that the third type of generators of $[E(\Phi, R, A), E(\Phi, R, B)]$ that occurred in [11] are redundant. Here, we obtain yet another dramatic improvement, and prove that it suffices to consider the elementary commutators $y_\alpha = [x_\alpha(a), x_{-\alpha}(b)]$ for a single *long* root and, moreover, that the classes of these elementary commutators modulo $E(\Phi, R, AB)$ behave as symbols in algebraic K-theory.

Introduction

Let Φ be a reduced irreducible root system of rank ≥ 2, let R be a commutative ring with 1, and let $G(\Phi, R)$ be a Chevalley group of type Φ over R. For the background on Chevalley groups over rings see [28] or [31], where one can find many further references. We fix a split maximal torus $T(\Phi, R)$ in $G(\Phi, R)$ and consider root unipotents $x_\alpha(\xi)$ elementary with respect to $T(\Phi, R)$. The subgroup $E(\Phi, R)$ generated by all $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, is called the *absolute* elementary subgroup of $G(\Phi, R)$.

Now, let $I \trianglelefteq R$ be an ideal of R. Then the *unrelativised elementary subgroup* $E(\Phi, I)$ of level I is defined as the subgroup of $E(\Phi, R)$, generated by all elementary root unipotents $x_\alpha(\xi)$ of level I,

$$E(\Phi, I) = \langle x_\alpha(\xi) \mid \alpha \in \Phi, \ \xi \in I \rangle.$$

In general, this subgroup has no chances to be normal in $E(\Phi, R)$. Its normal closure $E(\Phi, R, I) = E(\Phi, I)^{E(\Phi, R)}$ is called the *relative elementary subgroup* of level I.

In the rest of this paper we impose the following umbrella assumption:

(*) In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume that R does not have residue fields $\mathbb{F}_2$ of two elements, and in the case $\Phi = \mathrm{C}_l$, $l \geq 2$, assume additionally that any $c \in R$ is contained in the ideal $c^2 R + 2cR$.

This is precisely the condition that arises in the computation of the lower level of the mixed commutator subgroup $[E(\Phi, A), E(\Phi, B)]$, in [10], Lemma 17, and [11], Theorem 3.1, see also further related results, and discussion of this condition in [21, 22]. This condition ensures the inclusion

$$E(\Phi, R, AB) \leq [E(\Phi, A), E(\Phi, B)].$$

Since all vital calculations in the present paper occur modulo $E(\Phi, R, AB)$, we are not trying to remove or weaken this condition. In fact, when structure constants of type Φ are not invertible in R, one should consider in all results more general elementary subgroups, corresponding to admissible pairs, rather than individual ideals anyway.

Let us state the main result of our previous paper [35], Theorem 1.2, which, in turn, is an elaboration of the main result of [11], Theorem 1.3. Below, $z_\alpha(a, c)$ are Stein—Tits—Vaserstein generators, whereas $y_\alpha(a, b)$ are elementary commutators, both are defined in the statement itself, see also §§ 1 and 2 for details.

Theorem A. *Let Φ be a reduced irreducible root system of rank ≥ 2. Further, let A and B be two ideals of a commutative ring R. Assume (*). Then the mixed commutator subgroup $\big[E(\Phi, R, A), E(\Phi, R, B)\big]$ is generated as a group by the elements of the form*

- $z_\alpha(ab, c) = x_{-\alpha}(c)x_\alpha(ab)x_{-\alpha}(-c),$
- $y_\alpha(a, b) = \big[x_\alpha(a), x_{-\alpha}(b)\big],$

where in both cases $\alpha \in \Phi$, $a \in A$, $b \in B$, $c \in R$.

Recall that previous results, including [11], Theorem 1.3, required also a third type of generators for mixed commutator subgroups, viz. $\big[x_\alpha(a), z_\alpha(b, c)\big]$, but the Main Lemma of [35] shows that this type of generators are redundant, and can be expressed as product of elementary conjugates of the generators listed in Theorem A. Since both remaining types of generators sit already in $\big[E(\Phi, A), E(\Phi, B)\big]$, the above theorem immediately implies the following result, [35], Theorem 1.1.

Theorem B. *In conditions of Theorem A*
$$\big[E(\Phi, R, A), E(\Phi, R, B)\big] = \big[E(\Phi, A), E(\Phi, B)\big].$$

Here, we obtain further striking improvements of these results. First of all, it turns out that the set of generators in Theorem A can be further reduced by restricting α for the second type of generators to a single *long* root.

Theorem 1. *In conditions of Theorem A the mixed commutator of elementary subgroups $\big[E(\Phi, R, A), E(\Phi, R, B)\big]$ is generated as a group by the elements of the form*

- $z_\alpha(ab, c) = x_{-\alpha}(c)x_\alpha(ab)x_{-\alpha}(-c),$
- $y_\beta(a, b) = \big[x_\beta(a), x_{-\beta}(b)\big],$

where in both cases $a \in A$, $b \in B$, $c \in R$, whereas $\alpha \in \Phi$ is arbitrary, and $\beta \in \Phi$ is a fixed long root.

Morally, this theorem is also a partial counterpart of [35], Theorem 4.1, which asserts that the relative elementary subgroups $E(\Phi, R, A)$ are themselves generated by *long* root type unipotents.

Many of the auxiliary results are important and interesting in themselves, and we reproduce some of them in the introduction. Firstly, it turns out that the elementary

commutators are central in $E(\Phi, R)/E(\Phi, R, AB)$. The proof of the following result is similar to that of the Main Lemma in [35], and in fact easier.

Theorem 2. *In conditions of Theorem A one has*

$$^x y_\alpha(a, b) \equiv y_\alpha(a, b) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,.$$

for any $\alpha \in \Phi$, all $a \in A$, $b \in B$, and any $x \in E(\Phi, R)$.

This theorem asserts that

$$\big[[E(\Phi, A), E(\Phi, B)], E(\Phi, R)\big] \leq E(\Phi, R, AB).$$

In particular, the quotient $[E(\Phi, A), E(\Phi, B)]/E(\Phi, R, AB)$ is itself abelian, so that we get the following result.

Theorem 3. *In conditions of Theorem A for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ one has*

- $y_\alpha(a_1 + a_2, b) \equiv y_\alpha(a_1, b) \cdot y_\alpha(a_2, b) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$
- $y_\alpha(a, b_1 + b_2) \equiv y_\alpha(a, b_1) \cdot y_\alpha(a, b_2) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$
- $y_\alpha(a, b)^{-1} \equiv y_\alpha(-a, b) \equiv y_\alpha(a, -b) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$
- $y_\alpha(ab_1, b_2) \equiv y_\alpha(a_1, a_2 b) \equiv e \, (\mathrm{mod}\; E(\Phi, R, AB)) \,.$

The following two results afford the advance from Theorem A to Theorem 1. Their proofs are exactly the main novelty of the present paper, the rest are either easy variations of the preceding results, or easily follows.

Theorem 4. *In conditions of Theorem A for all $a \in A$, $b \in B$, $c \in R$, one has*:

- $y_\alpha(a, b) \equiv y_\beta(a, b) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$

for any roots $\alpha, \beta \in \Phi$ of the same length.

- $y_\alpha(a, b) \equiv y_\beta(a, b)^p \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$

if the root $\alpha \in \Phi$ is short, whereas the long $\beta \in \Phi$ is long, while $p = 2$ for $\Phi = \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4$, and $p = 3$ for $\Phi = \mathrm{G}_2$.

Theorem 5. *In conditions of Theorem A for all $a \in A$, $b \in B$, $c \in R$, one has*:

- $y_\alpha(ac, b) \equiv y_\alpha(a, cb) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$

where either $\Phi \neq \mathrm{C}_l$, or α is short.

In the exceptional case when $\Phi = \mathrm{C}_l$ and α is long only the following weaker congruences hold:

- $y_\alpha(ac^2, b) \equiv y_\alpha(a, c^2 b) \, (\mathrm{mod}\; E(\Phi, R, AB)) \,,$
- $y_\alpha(ac, b)^2 \equiv y_\alpha(a, cb)^2 \, (\mathrm{mod}\; E(\Phi, R, AB)) \,.$

For $\mathrm{GL}(n, R)$ over an *arbitrary* associative ring R similar results were established in our recent papers [34, 36, 37]. For Bak's unitary groups $\mathrm{GU}(2n, R, \Lambda)$, again over an *arbitrary* form ring (R, Λ), such similar results are presently under way [38].

The paper is organised as follows. In § 1 we recall necessary notation and background. In § 2 we prove Theorem 2, and thus also Theorem 3. The technical core of the paper are §§ 3–5, where we prove Theorems 4 and 5, for rank 2 root systems, A_2, C_2 (which is again *by far* the most difficult case!) and G_2, respectively. Together, Theorems 2 and 4 imply Theorem 1. Finally, in § 6 we derive some corollaries of Theorem 1 and state some further related problems.

1. NOTATION AND PRELIMINARY FACTS

To make this paper independent of [10, 11, 35], here we recall basic notation and the requisite facts, which will be used in our proofs. For more background information on Chevalley groups over rings, see [28, 31, 7] and references therein.

1.1. **Notation.** Let G be a group. For any $x, y \in G$, ${}^x y = xyx^{-1}$ denotes the left x-conjugate of y. As usual, $[x, y] = xyx^{-1}y^{-1}$ denotes the [left normed] commutator of x and y. We shall make constant use of the obvious commutator identities, such as $[x, yz] = [x, y] \cdot {}^y[x, z]$ or $[xy, z] = {}^x[y, z] \cdot [x, z]$, usually without any specific reference.

Let Φ be a reduced irreducible root system of rank $l = \mathrm{rk}(\Phi)$. We denote by Φ_s the subset Φ consisting of short roots, and by Φ_l the subsystem of Φ consisting of long roots. Fix an order on Φ with Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ being the sets of *positive*, *negative* and *fundamental* roots, respectively. Further, let $W = W(\Phi)$ be the Weyl group of Φ.

As in the introduction, we denote by $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, the elementary generators of the (absolute) elementary Chevalley subgroup $E(\Phi, R)$. For a root $\alpha \in \Phi$ we denote by X_α the corresponding [elementary] root subgroup $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$. Recall that any conjugate ${}^g x_\alpha(\xi)$ of an elementary root unipotent, where $g \in G(\Phi, R)$ is called *root element* or *root unipotent*, it is called *long* or *short*, depending on whether the root α itself is long or short.

Let, as in the introduction, I be an ideal of R. We denote by $X_\alpha(I)$ the intersection of X_α with the principal congruence subgroup $G(\Phi, R, I)$. Clearly, $X_\alpha(I)$ consists of all elementary root elements $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in I$, of level I:

$$X_\alpha(I) = \{x_\alpha(\xi) \mid \xi \in I\}.$$

By definition, $E(\Phi, I)$ is generated by $X_\alpha(I)$, for all roots $\alpha \in \Phi$. The same subgroups generate $E(\Phi, R, I)$ as a *normal* subgroup of the absolute elementary group $E(\Phi, R)$. Generators of $E(\Phi, R, I)$ *as a group* are recalled in the next subsection.

1.2. **Generation of elementary subgroups.** Apart from Theorem A we shall extensively use the two following generation results. The first one is a classical result by Michael Stein [19], Jacques Tits [26] and Leonid Vaserstein [27].

Lemma 1. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let I be an ideal of a commutative ring R. Then as a group $E(\Phi, R, I)$ is generated by the elements of the form*

$$z_\alpha(a, c) = x_{-\alpha}(c) x_\alpha(a) x_{-\alpha}(-c),$$

where $a \in I$, $c \in R$, and $\alpha \in \Phi$.

The following result on levels of mixed commutator subgroups is [11], Theorem 4, which in turn is a sharpening of [10], Lemmas 17–19.

Lemma 2. *Let Φ be a reduced irreducible root system of rank ≥ 2 and let R be a commutative ring. Then for any two ideals A and B of the ring R one has the following inclusion*

$$E(\Phi, R, AB) \leq \big[E(\Phi, A), E(\Phi, B\big] \leq \big[E(\Phi, R, A), E(\Phi, R, B)\big]$$
$$\leq \big[G(\Phi, R, A), G(\Phi, R, B)\big] \leq G(\Phi, R, AB).$$

1.3. **Structure constants.** All results of the present paper are based on the Steinberg relations among the elementary generators, which will be repeatedly used without any specific reference. Especially important for us is the Chevalley commutator formula

$$[x_\alpha(a), x_\beta(b)] = \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(N_{\alpha\beta ij}a^i b^j),$$

where $\alpha \neq -\beta$ and $N_{\alpha\beta ij}$ are the structure constants which do not depend on a and b. However, for $\Phi = \mathrm{G}_2$ they may depend on the order of the roots in the product on the right hand side. See [2, 19, 20, 31] for more details regarding the structure constants $N_{\alpha\beta ij}$.

In the proof of Theorems 4 and 5 we need somewhat more specific information about the structure constants. For $\Phi = \mathrm{A}_2$ and $\Phi = \mathrm{C}_2$ this is easy, since the corresponding simply connected Chevalley groups can be identified with $\mathrm{SL}(3, R)$ and $\mathrm{Sp}(4, R)$, respectively, and we select the usual parametrisation of the elementary root subgroups therein.

For $\Phi = \mathrm{C}_2$ the most complicated instance of the Chevalley commutator formula is when α and β are the long and short fundamental roots, respectively. We will choose the parametrisation of root subgroups for which

$$[x_\alpha(a), x_\beta(b)] = x_{\alpha+\beta}(ab)x_{\alpha+2\beta}(ab^2).$$

The case of $\Phi = \mathrm{G}_2$ is somewhat more tricky. Let α and β be the short and long fundamental roots, respectively. Then it is known that the parametrisation of the root subgroups can be chosen in such a way that

$$[x_\alpha(a), x_\beta(b)] = x_{\alpha+\beta}(ab)x_{2\alpha+\beta}(a^2 b)x_{3\alpha+\beta}(a^3 b)x_{3\alpha+2\beta}(2a^3 b^2)$$
$$[x_\alpha(a), x_{\alpha+\beta}(b)] = x_{2\alpha+\beta}(2ab)x_{3\alpha+\beta}(3a^2 b)x_{3\alpha+2\beta}(-3ab^2),$$
$$[x_\alpha(a), x_{2\alpha+\beta}(b)] = x_{3\alpha+\beta}(3ab),$$
$$[x_\beta(a), x_{3\alpha+\beta}(b)] = x_{3\alpha+2\beta}(ab),$$
$$[x_{\alpha+\beta}(a), x_{2\alpha+\beta}(b)] = x_{3\alpha+2\beta}(-3ab),$$

these are precisely the signs you get for the *positive* Chevalley base. See, for instance [2, 20, 31].

Our initial proof of Theorems 4 and 5 in the case $\Phi = \mathrm{G}_2$ relied on an explicit knowledge of the structure constants also in some further instances of the Chevalley commutator formula. Initially, we used a `Mathematica` package `g2.nb` by Alexander Luzgarev, to compute the structure constants. However, later we noticed that pairs of short roots do not require a separate analysis. This is precisely the shortcut presented in § 5 below.

1.4. **Parabolic subgroups.** As in [36] an important part in the proof of Theorems 2, 4 and 5 is played by the Levi decomposition for [elementary] parabolic subgroups. Oftentimes, it allows us to discard factors in the unipotent radicals, to limit the number of instances, where we have to explicitly invoke precise forms of the Chevalley commutator formula.

Classical Levi decomposition asserts that any parabolic subgroup P of $G(\Phi, R)$ can be expressed as the semi-direct product $P = L_P \ltimes U_P$ of its unipotent radical $U_P \trianglelefteq P$ and a Levi subgroup $L_P \leq P$. However, as in [11, 35] we do not have to recall the general case.

- Since we calculate inside $E(\Phi, R)$, we can limit ourselves to the *elementary* parabolic subgroups, spanned by some root subgroups X_α.

- Since we can choose the order on Φ arbitrarily, we can always assume that α is fundamental and, thus, limit ourselves to *standard* parabolic subgroups.

- Since the proofs of our main results reduces to groups of rank 2, we could only consider rank 1 parabolic subgroups, which *in this case* are maximal parabolic subgroups.

Thus, we consider only *elementary* rank 1 parabolics, which are defined as follows. Namely, we fix an order on Φ, and let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. Further, let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the corresponding fundamental system. For any r, $1 \leq r \leq l$, and define the r-th rank 1 *elementary* parabolic subgroup as

$$P_{\alpha_r} = \langle U, X_{-\alpha_r} \rangle \leq E(\Phi, R).$$

Here $U = \prod X_\alpha$, $\alpha \in \Phi^+$, is the unipotent radical of the standard Borel subgroup B. Then the unipotent radical of P_{α_r} has the form

$$U_{\alpha_r} = \prod X_\alpha, \quad \alpha \in \Phi^+, \ \alpha \neq \alpha_r,$$

whereas $L_{\alpha_r} = \langle X_{\alpha_r}, X_{-\alpha_r} \rangle$ is the [standard] Levi subgroup of P_r. Clearly, L_{α_r} is isomorphic to the elementary subgroup $E(2, R)$ in $\mathrm{SL}(2, R)$, or to its projectivised version $\mathrm{PE}(2, R)$ in $\mathrm{PGL}(2, R)$. In the sequel we usually (but not always!) abbreviate $P_{\alpha_r}, U_{\alpha_r}, L_{\alpha_r}$, etc., to P_r, U_r, L_r, etc.

Levi decomposition (which in the case of elementary parabolics immediately follows from the Chevalley commutator formula) asserts that the group P_r is the semi-direct product $P_r = L_r \ltimes U_r$ of $U_r \trianglelefteq P_r$ and $L_r \leq P_r$. The most important part is the [obvious] claim is that U_r is normal in P_r.

Simultaneously with P_r one considers also the opposite parabolic subgroup P_r^- defined as

$$P_r^- = \langle U^-, X_{\alpha_r} \rangle \le E(\Phi, R).$$

Here $U^- = \prod X_\alpha$, $\alpha \in \Phi^-$, is the unipotent radical of the Borel subgroup B^- opposite to the standard one. Clearly, P_r and P_r^- share the common [standard] Levi subgroup L_r, whereas the unipotent radical U_r^- of P_r^- is opposite to that of P_r, and has the form

$$U_r^- = \prod X_\alpha, \quad \alpha \in \Phi^-, \ \alpha \ne -\alpha_r.$$

Now, Levi decomposition takes the form $P_r^- = L_r \ltimes U_r^-$ with $U_r^- \trianglelefteq P_r^-$. In other words, U_r and U_r^- are both normalised by L_r.

Actually, we need a slightly more precise form of this last statement. Namely, let I be an ideal of R. Denote by $L_r(I)$ the principal congruence subgroup of level I in L_r and by $U_r(I)$ and $U_r^-(I)$ the respective intersections of U_r and U_r^- with $G(\Phi, R, I)$ — or, what is the same, with $E(\Phi, R, I)$:

$$U_r(I) = U_r \cap E(\Phi, R, I), \qquad U_r^-(I) = U_r^- \cap E(\Phi, R, I).$$

Obviously, $U_r(I), U_r^-(I) \le E(\Phi, I)$ are normalised by L_r.

The following fact is well known, and obvious.

Lemma 3. *Let A and B be two ideals of R. Then*

$$[L_r(A), U_r(B)] \le U_r(AB), \qquad [L_r(A), U_r^-(B)] \le U_r^-(AB).$$

In particular, both commutators are contained in $E(\Phi, AB) \le E(\Phi, R, AB)$.

2. Proof of Theorems 2 and 3

This section is devoted to the proof of Theorem 2. It is a calculation of the same type as the proof of the Main Lemma in [7], and actually easier than that, since now we can expand the exponent level, rather than the ground level, so that there are no protruding factors that have to be taken care of and the elementary commutators do not procreate.

2.1. **Idea of the proof.** Consider the elementary conjugate ${}^x y_\alpha(a, b)$. We argue by induction on the length of $x \in E(\Phi, R)$ in elementary generators. Let $x = y x_\beta(c)$, where $y \in E(\Phi, R)$ is shorter than x, whereas $\beta \in \Phi$, $c \in R$. First of all, recall that under the action of the Weyl group $W(\Phi)$ the root α is conjugate to a fundamental root of the same length. Thus, we could from the very start choose an ordering of Φ such that $\alpha = \alpha_r \in \Pi$ is a fundamental root for some r, $1 \le r \le l$.

If $\beta \ne \pm\alpha$, then $x_\beta(c)$ belongs either to U_r, or to U_r^-. By Lemma 3 in each case $[x_\beta(c), y_\alpha(a, b)] \in E(\Phi, R, AB)$ and thus

$${}^{x_\beta(c)} y_\alpha(a, b) = [x_\beta(c), y_\alpha(a, b)] \cdot y_\alpha(a, b) \equiv y_\alpha(a, b) \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

2.2. **Expansion of the exponent.** It remains only to consider the case, where $\beta = \pm\alpha$. In each case we will express $x_\beta(c)$ as a product of root elements satisfying the conditions of the previous item. One of the four following possibilities may occure. Since we are only looking at *one* instance of the Chevalley commutator formula at a time, the parametrisation of the corresponding root subgroups can be chosen in such a way that all the resulting structure constants are positive (see [2, 20] or [31] and references there.

• First, assume that α can be embedded into a subsystem of type A_2. This already proves Theorem 1 for simply laced Chevalley groups, and for the Chevalley group of type F_4. It also proves necessary congruences for a *short* root α in Chevalley groups of type C_l, $l \geq 3$, for a *long* root α in Chevalley groups of type B_l, $l \geq 3$, and for a *long* root α in the Chevalley group of type G_2.

In this case there exist roots $\gamma, \delta \in \Phi$, of the same length as α such that $\beta = \gamma + \delta$ and $N_{\gamma\delta 11} = 1$. Express $x_\beta(c)$ in the form $x_\beta(c) = [x_\gamma(1), x_\delta(c)]$ and plug this expression in the exponent. We get

$$^{x_\beta(c)}y_\alpha(a, b) = {}^{x_\gamma(1)x_\delta(c)x_\gamma(-1)x_\delta(-c)}y_\alpha(a, b) \equiv y_\alpha(a, b) \,(\mathrm{mod}\ E(\Phi, R, AB))\,,$$

by the first item in the proof.

• Next, assume that α can be embedded into a subsystem of type C_2 as a *short* root. In this case we express β as $\beta = \gamma + \delta$, where γ is long and δ is short. By the above we may $x_\beta(c)$ in the form

$$x_\beta(c) = [x_\gamma(c), x_\delta(1)] \cdot x_{\gamma+2\delta}(-c).$$

Plugging this expression in the exponent, we get

$$^{x_\beta(c)}y_\alpha(a, b) = {}^{x_\gamma(1)x_\delta(c)x_\gamma(-1)x_\delta(-c)x_{\gamma+2\delta}(-c)}y_\alpha(a, b) \equiv y_\alpha(a, b) \,(\mathrm{mod}\ E(\Phi, R, AB))\,,$$

where again $\gamma, \delta, \gamma + 2\delta \neq \pm\alpha$, so that we can invoke the first item.

• Next, assume that α can be embedded into a subsystem of type C_2 as a *long* root. In this case we express β as $\beta = \gamma + 2\delta$, with the same γ, δ as above. so that the formula takes the form

$$x_\beta(c) = [x_\gamma(c), x_\delta(1)] \cdot x_{\gamma+\delta}(-c).$$

Plugging this expression in the exponent, we get

$$^{x_\beta(c)}y_\alpha(a, b) = {}^{x_\gamma(1)x_\delta(c)x_\gamma(-1)x_\delta(-c)x_{\gamma+\delta}(-c)}y_\alpha(a, b) \equiv y_\alpha(a, b) \,(\mathrm{mod}\ E(\Phi, R, AB))\,,$$

where again $\gamma, \delta, \gamma + \delta \neq \pm\alpha$.

• Finally, when α is a *short* root in G_2, β as $\beta = \gamma + \delta$, where γ is long and δ is short. By the above, we can rewrite the Chevalley commutator formula in the form

$$x_\beta(c) = [x_\gamma(c), x_\delta(1)] \cdot x_{\gamma+2\delta}(-c)x_{\gamma+3\delta}(-c)x_{2\gamma+3\delta}(-2c^2).$$

Plugging this expression in the exponent, we get

$$^{x_\beta(c)}y_\alpha(a,b) = {}^{x_\gamma(1)x_\delta(c)x_\gamma(-1)x_\delta(-c)x_{\gamma+2\delta}(-c)x_{\gamma+3\delta}(-c)x_{2\gamma+3\delta}(-2c^2)}y_\alpha(a,b) \equiv$$

$$y_\alpha(a,b) \,(\mathrm{mod}\ E(\Phi,R,AB)),$$

where again $\gamma, \delta, \gamma+2\delta, \gamma+3\delta, 2\gamma+3\delta \neq \pm\alpha$.

2.3. **Proof of Theorems 2 and 3.** Summarising the above, we see that for all elementary generators $x_\beta(c)$ one has $^{x_\beta(c)}y_\alpha(a,b) \equiv y_\alpha(a,b)\,(\mathrm{mod}\ E(\Phi,R,AB))$ and thus

$$^{x}y_\alpha(a,b) \equiv {}^{y}y_\alpha(a,b) \,(\mathrm{mod}\ E(\Phi,R,A\mathrm{B})),$$

where the length of y in elementary generators is smaller than the length of x. By induction we get that $^{x}y_\alpha(a,b) \equiv y_\alpha(a,b)\,(\mathrm{mod}\ E(\Phi,R,A\mathrm{B}))$, as claimed. This proves Theorem 2.

It is clear that Theorem 3 immediately follows. Indeed, to derive the first item, observe that

$$y_\alpha(a_1+a_2,b) = [x_\alpha(a_1+a_2), x_{-\alpha}(b)] = [x_\alpha(a_1)x_\alpha(a_2), x_{-\alpha}(b)].$$

Using multiplicativity of the commutator w. r.'t. the first argument, we see that

$$y_\alpha(a_1+a_2,b) = {}^{x_\alpha(a_1)}[x_\alpha(a_2), x_{-\alpha}(b)] \cdot [x_\alpha(a_1), x_{-\alpha}(b)] = {}^{x_\alpha(a_1)}y_\alpha(a_2,b) \cdot y_\alpha(a_1,b).$$

It remains to apply Theorem 2. The second item is similar, and the third item follows. The last item is obvious from the definition.

3. Proof of Theorems 4 and 5: the case A_2

We are now all set to take up the proof of Theorems 4 and 5. In the present section we prove Theorems 4 and 5 for simply laced systems.

3.1. **Structure of the proof.** The proof will be subdivided into a sequence of five lemmas, which either simultaneously establish congruences in Theorems 4 and 5, for some pairs of roots of the same length, or reduce elementary commutators for short roots to elementary commutators for long roots. These five cases are:

- Two roots α and β that can be embedded into A_2, Lemma 4,
- Two short roots in C_2, Lemma 5,
- Two long roots in C_2, Lemma 6,
- A short and a long root in C_2, Lemma 7,
- A short and a long root in G_2, Lemma 8.

Already Lemma 4 suffices to establish Theorem 4, and thus also Theorem 1, for the case of simply-laced systems. It also reduces both long and short elementary commutators in F_4, long elementary commutators in B_l, $l \geq 3$, and G_2 and short elementary commutators in C_l, $l \geq 3$, to such elementary commutators for a single root of that length. After that, Lemmas 5–7 completely settle the case of doubly

laced root systems. Finally, Lemma 8 is only needed for G_2. Observe that together with Lemma 4 it immediately implies also the necessary congruences for pairs of short roots in G_2.

Warning. A similar strategy does not work for C_2 since in this case the congruences for long roots in Theorem 5 are *weaker*, than the desired congruences for short roots. This compels us to derive the congruences for pairs of short roots and for pairs of long roots independently, *before* comparing elementary commutators for short roots with those for long roots. This makes C_2 the most exacting case of all.

3.2. **Two roots in** A_2. The first of these lemmas was essentially contained already in [34], Lemma 5, and [36], Lemma 11. Of course, there we used matrix language. For the sake of completeness, and also as a template for the following more difficult lemmas, below we reproduce its proof in the language of roots.

Lemma 4. *Assume that the roots* $\alpha, \beta \in \Phi$ *of the same length can be embedded into a subsystem of type* A_2. *Then for all* $a \in A$, $b \in B$, $c \in R$, *one has*:
$$y_\alpha(ac, b) \equiv y_\beta(a, cb) \, (\mathrm{mod} \, E(\Phi, R, AB)).$$

Proof. First, assume that β is such that $\alpha = \beta + \gamma$, with $N_{\beta\gamma11} = 1$ and rewrite the elementary commutator $y_\alpha(ac, b) = [x_\alpha(ac), x_{-\alpha}(b)]$ as
$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)} x_\alpha(-ac) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)} [x_\beta(a), x_\gamma(-c)].$$
Expanding the conjugation by $x_{-\alpha}(b)$, we see that
$$y_\alpha(ac, b) = x_\alpha(ac) \cdot [{}^{x_{-\alpha}(b)} x_\beta(a), {}^{x_{-\alpha}(b)} x_\gamma(-c)] = x_\alpha(ac) \cdot [x_{-\gamma}(ba) x_\beta(a), x_\gamma(-c) x_{-\beta}(cb)].$$
Now, the first factor $x_{-\gamma}(ba)$ of the first argument in this last commutator already belongs to the group $E(\Phi, AB)$ which is contained in $E(\Phi, R, AB)$. Thus, as above,
$$y_\alpha(ac, b) \equiv x_\alpha(ac) \cdot [x_\beta(a), x_\gamma(-c) x_{-\beta}(cb)] \, (\mathrm{mod} \, E(\Phi, R, AB)).$$
Using multiplicativity of the commutator w. r. t. the second argument, cancelling the first two factors of the resulting expression, and then applying Theorem 2 we see that for a pair of roots α, β at angle $\pi/3$, one has
$$y_\alpha(ac, b) \equiv {}^{x_\gamma(-c)} [x_\beta(a), x_{-\beta}(cb)] \equiv [x_\beta(a), x_{-\beta}(cb)] \equiv y_\beta(a, cb) \, (\mathrm{mod} \, E(\Phi, R, AB)),$$
as claimed. Obviously, one can pass from any root in A_2 to any other such root in not more than 3 such elementary steps. $\square$

Joining two roots of the same length by a sequence of roots where every two neighbours sit in a subsystem of type A_2, we obtain the following corollary.

Corollary. *Assume that the roots* $\alpha, \beta \in \Phi$ *of the same length and one of the following holds*:

- $\Phi = A_l, D_l, E_l, F_4$,
- $\Phi = B_l$, $l \geq 3$, *and* α, β *are long*,
- $\Phi = C_l$, $l \geq 3$, *and* α, β *are short*.

- $\Phi = \mathrm{G}_2$, and α, β are long,

Then for all $a \in A$, $b \in B$, $c \in R$, one has:

$$y_\alpha(ac, b) \equiv y_\beta(a, cb) \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

The remaining cases have to be considered separately, in the same style, as Lemma 4. However, in these cases the roots β and γ in the proof of this lemma would have different lengths, so that it does matter, whether we put parameter a in the above calculation in the short root unipotent, or the long root unipotent. In fact, by choosing one way, or the other, one gets different congruences! Also, in the case $\Phi = \mathrm{G}_2$ the structure constants have to be chosen in consistent way.

4. Proof of Theorems 4 and 5: the case C_2

In this section we prove Theorems 4 and 5 for doubly laced systems. This is by far the most difficult case of all, since in this case we have to consider short roots and long roots separately.

4.1. **Two short roots.** The following lemma settles the case of short roots in B_l, $l \geq 2$.

Lemma 5. *Assume that the roots $\alpha, \beta \in \Phi$ can be embedded as short roots into a subsystem of type C_2. Then for all $a \in A$, $b \in B$, $c \in R$, one has:*

$$y_\alpha(ac, b) \equiv y_\beta(a, cb) \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Proof. First assume that α and β are linearly independent. Then there exists a long root γ such that $\alpha = \beta + \gamma$ and we can choose parametrisation of root subgroups such that $N_{\beta\gamma11} = N_{\beta\gamma21} = 1$. Actually, the signs mostly do not play any role here, apart from one position. Namely, we should eventually get that $y_\alpha(ac, b)$ is equivalent to $y_\beta(a, cb)$, and not to $y_\beta(a, cb)^{-1}$. They were calculated in $\mathrm{Sp}(4, R)$.

Expanding the elementary commutator $y_\alpha(ac, b)$ as in Lemma 4 and plugging in $x_\alpha(-ac) = x_{\alpha+\beta}(a^2c)[x_\beta(a), x_\gamma(-c)]$, we get

$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_\alpha(-ac) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(a^2c) \cdot {}^{x_{-\alpha}(b)}\big[x_\beta(a), x_\gamma(-c)\big].$$

Expanding the conjugation by $x_{-\alpha}(b)$, we see that

$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(a^2c) \cdot \big[x_{-\gamma}(ba)x_\beta(a), x_\gamma(-c)x_{-\beta}(cb)x_{-\alpha-\beta}(cb^2)\big].$$

Now, the first factor $x_{-\gamma}(ba)$ of the first argument in this last commutator already belongs to the group $E(\Phi, AB)$ which is contained in $E(\Phi, R, AB)$. Also,

$$ {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(a^2c) \equiv x_{\alpha+\beta}(a^2c) \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Thus, as above,

$$y_\alpha(ac, b) \equiv x_\alpha(ac)x_{\alpha+\beta}(a^2c) \cdot \big[x_\beta(a), x_\gamma(-c)x_{-\beta}(cb)x_{-\alpha-\beta}(cb^2)\big] \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Using multiplicativity of the commutator w. r. t. the second argument, cancelling the first commutator of the resulting expression, we see that

$$y_\alpha(ac, b) \equiv \left[x_\beta(a), x_{-\beta}(cb)x_{-\alpha-\beta}(cb^2)\right] =$$
$$y_\beta(a, cb) \cdot {}^{x_{-\beta}(cb)}\left[x_\beta(a), x_{-\alpha-\beta}(cb^2)\right] \equiv y_\beta(a, cb) \,(\mathrm{mod}\ E(\Phi, R, AB)) \,.$$

Obviously, one can pass from a short root α in C_2 to the opposite root $-\alpha$ in two such elementary steps. $\qquad\square$

4.2. **Two long roots.** The following lemma settles the case of long roots in C_l, $l \geq 2$. This case is exceptional, since here, unlike all other cases, the arguments of an elementary commutator are only balanced up to squares. In the following lemma we establish the first related congruence in Theorem 5.

Lemma 6. *Assume that the roots $\alpha, \gamma \in \Phi$ can be embedded as long roots into a subsystem of type C_2. Then for all $a \in A$, $b \in B$, $c \in R$, one has:*

$$y_\alpha(ac^2, b) \equiv y_\gamma(a, c^2 b) \,(\mathrm{mod}\ E(\Phi, R, AB)) \,.$$

Proof. First, let α and γ be linearly independent long roots. As in the previous lemma we choose a short root β such that $\alpha = 2\beta + \gamma$ and specify the same choice of signs.

Expanding the elementary commutator $y_\alpha(ac^2, b)$ as in Lemma 4 and plugging in $x_\alpha(-ac^2) = x_{\gamma+\beta}(ac)[x_\beta(c), x_\gamma(-a)]$, we get

$$y_\alpha(ac^2, b) = x_\alpha(ac^2) \cdot {}^{x_{-\alpha}(b)}x_\alpha(-ac^2) = x_\alpha(ac^2) \cdot {}^{x_{-\alpha}(b)}x_{\gamma+\beta}(ac) \cdot {}^{x_{-\alpha}(b)}\left[x_\beta(c), x_\gamma(-a)\right].$$

Expanding the conjugation by $x_{-\alpha}(b)$, we see that

$$y_\alpha(ac^2, b) = x_\alpha(ac^2) \cdot {}^{x_{-\alpha}(b)}x_{\gamma+\beta}(ac) \cdot \left[x_\beta(c)x_{-\beta-\gamma}(cb)x_{-\gamma}(c^2 b), x_\gamma(-a)\right].$$

As usual,

$${}^{x_{-\alpha}(b)}x_{\gamma+\beta}(ac) \equiv x_{\gamma+\beta}(ac) \,(\mathrm{mod}\ E(\Phi, R, AB)) \,.$$

so that the first two factors of the above expression are the inverse of $[x_\beta(c), x_\gamma(-a)]$. Thus, up to a congruence modulo $E(\Phi, R, AB)$ we get

$$y_\alpha(ac^2, b) \equiv \left[x_{-\beta-\gamma}(cb)x_{-\gamma}(c^2 b), x_\gamma(-a)\right] \equiv$$
$$y_{-\gamma}(c^2 b, -a) \equiv y_\gamma(a, c^2 b) \,(\mathrm{mod}\ E(\Phi, R, AB)) \,.$$

Obviously, one can pass from a long root α in C_2 to the opposite root $-\alpha$ in two such elementary steps. $\qquad\square$

4.3. **A short root and a long root.** The following lemma establishes connection between the classes of short and long elementary commutators in doubly laced systems.

Lemma 7. *Assume that the roots $\alpha, \gamma \in \Phi$ can be embedded as a short root and a long root into a subsystem of type C_2. Then for all $a \in A$, $b \in B$, $c \in R$, one has:*

$$y_\alpha(ac, b) \equiv y_\gamma(a, cb)^2 \,(\mathrm{mod}\ E(\Phi, R, AB)) \,.$$

Proof. First, assume that α and γ form an angle $\pi/4$. We choose a short root β such that $\alpha = \beta + \gamma$ and specify the same choice of signs.

Expanding the elementary commutator $y_\alpha(ac, b)$ as in Lemma 5 and plugging in $x_\alpha(-ac) = x_{\alpha+\beta}(-ac^2)[x_\beta(-c), x_\gamma(a)]$, we get

$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_\alpha(-ac) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(-ac^2) \cdot {}^{x_{-\alpha}(b)}[x_\beta(-c), x_\gamma(a)].$$

Expanding the conjugation by $x_{-\alpha}(b)$, we see that

$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(-ac^2) \cdot \left[x_{-\gamma}(-2cb)x_\beta(-c), x_\gamma(a)x_{-\beta}(-ab)x_{-\alpha-\beta}(-ab^2)\right].$$

Now, the last two factors $x_{-\beta}(-ab)x_{-\alpha-\beta}(-ab^2)$ of the second argument in this last commutator already belong to the group $E(\Phi, AB)$ which is contained in $E(\Phi, R, AB)$. Also,

$$ {}^{x_{-\alpha}(b)}x_{\alpha+\beta}(-ac^2) \equiv x_{\alpha+\beta}(-ac^2) \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Thus, as above,

$$y_\alpha(ac, b) \equiv x_\alpha(ac)x_{\alpha+\beta}(-ac^2) \cdot \left[x_{-\gamma}(-2cb), x_\gamma(a)\right] \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Using multiplicativity of the commutator w. r. t. the first argument, and cancelling the first commutator of the resulting expression, we see that

$$y_\alpha(ac, b) \equiv y_{-\gamma}(-2cb, a) \equiv y_{-\gamma}(cb, a)^{-2} \equiv y_\gamma(a, cb)^2 \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Obviously, combined with the previous lemma this gives necessary inlcusions for all pairs of a short and a long root. $\square$

Corollary. *Assume that the roots $\alpha, \gamma \in \Phi$ can be embedded as long roots into a subsystem of type* C_2. *Then for all $a \in A$, $b \in B$, $c \in R$, one has:*

$$y_\alpha(ac, b)^2 \equiv y_\beta(a, cb)^2 \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

Proof. Indeed, let γ be any short root. Then by the previous lemma and Lemma 5 one has

$$y_\alpha(ac, b)^2 \equiv y_\gamma(ac, b) \equiv y_\gamma(a, cb) \equiv y_\beta(a, cb)^2 \,(\mathrm{mod}\ E(\Phi, R, AB)).$$

$\square$

This completes the proof of Theorems 4 and 5 for doubly laced root systems.

5. Proof of Theorems 4 and 5: the case G_2

In this section we finish the proof of Theorems 4 and 5 for the only remaining case $\Phi = \mathrm{G}_2$. Since in this case long roots themselves form a root system of type A_2, the corresponding elementary commutators are balanced with respect to all elements of R, which makes the proof quite a bit easier.

The following lemma establishes connection between the classes of short and long elementary commutators in G_2.

Lemma 8. *Assume that $\alpha, \gamma \in G_2$, where α is short and γ is long. Then for all $a \in A$, $b \in B$, $c \in R$, one has*:

$$y_\alpha(ac, b) \equiv y_\gamma(a, cb)^3 \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$$

Proof. First, assume that α and γ form an angle $\pi/6$. We choose a short root β such that $\alpha = \beta + \gamma$ and specify the same choice of signs as in Section 1.3.

Expanding the elementary commutator $y_\alpha(ac, b)$ as in Lemma 4 and plugging in $x_\alpha(-ac) = u \cdot [x_\beta(-c), x_\gamma(a)]$, where $u = x_{\alpha+\beta}(-ac^2)x_{\alpha+2\beta}(ac^3)x_{2\alpha+\beta}(2a^2c^3)$, we get

$$y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}x_\alpha(-ac) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}u \cdot {}^{x_{-\alpha}(b)}\big[x_\beta(-c), x_\gamma(a)\big].$$

Clearly, ${}^{x_{-\alpha}(b)}u \equiv u \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$

Expanding the conjugation by $x_{-\alpha}(b)$, we see that $y_\alpha(ac, b) = x_\alpha(ac) \cdot {}^{x_{-\alpha}(b)}u \cdot v$, where

$$v = \big[x_{-\gamma}(-3cb)x_\beta(-c), x_\gamma(a)x_{-\alpha-2\beta}(-a^2b^3)x_{-2\alpha-\beta}(ab^3)x_{-\alpha-\beta}(ab^2)x_{-\beta}(ab)\big].$$

Clearly, the last four factors of the second argument in this last commutator already belong to the group $E(\Phi, AB)$ which is contained in $E(\Phi, R, AB)$.

Thus, by the same token, as above,

$$y_\alpha(ac, b) \equiv x_\alpha(ac) \cdot u \cdot \big[x_{-\gamma}(-3cb)x_\beta(-c), x_\gamma(a)\big] \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$$

Using multiplicativity of the commutator w. r. t. first argument, cancelling the first commutator of the resulting expression, we see that

$$y_\alpha(ac, b) \equiv y_{-\gamma}(-3cb, a) \equiv y_{-\gamma}(cb, a)^{-3} \equiv y_\gamma(a, cb)^3 \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$$

Obviously, combined with Lemma 4 this gives necessary inlcusions for all pairs of a short and a long root. $\qquad\square$

Corollary. *Assume that the roots $\alpha, \beta \in G_2$. Then for all $a \in A$, $b \in B$, $c \in R$, one has*:

$$y_\alpha(ac, b) \equiv y_\beta(a, cb) \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$$

Proof. Indeed, let γ be any long root. Then by the previous lemma and Lemma 4 one has

$$y_\alpha(ac, b) \equiv y_\gamma(ac, b)^3 \equiv y_\gamma(a, cb)^3 \equiv y_\beta(a, cb) \,(\mathrm{mod}\; E(\Phi, R, AB))\,.$$

$$\square$$

This completes the proof of Theorems 4 and 5 for the only remaining case $\Phi = G_2$, and thus also the proof of Theorem 1, for all cases.

6. FINAL REMARKS

Theorem 1 implies surjective stability for the abelian quotients

$$\big[E(\Phi, A), E(\Phi, B)\big]/E(\Phi, R, AB)$$

described in Theorem 2, without any stability conditions. This is a generalisation of the first half of [13], Lemma 15, to all Chevalley groups. Indeed, in view of Theorems 1 and 2 as a normal subgroup of $E(\Phi, R)$ the group $[E(\Phi, A), E(\Phi, B)]$ is generated by a similar commutator for a rank 2 subsystem. This can be restated as follows.

Theorem 6. *Let R be any commutative ring with 1, and let A and B be two sided ideals of R. Further, assume that $\Delta \leq \Phi$ is a root subsystem containing A_2 on long roots or C_2. Then the stability map*

$$\big[E(\Delta, A), E(\Delta, B)\big]/E(\Delta, R, AB) \longrightarrow \big[E(\Phi, A), E(\Phi, B)\big]/E(\Phi, R, AB)$$

is surjective.

According to Theorem 4 modulo $E(\Phi, R, AB)$ the elementary commutators $y_\alpha(a, b)$ behave as symbols. Theorems 3 and 5 list some relations satisfied by these symbols. However, looking at the examples for which $[E(\Phi, A), E(\Phi, B)]$ was explicitly calculated, such as Dedekind rings of arithmetic type, [18, 17, 30], it is easy to see that there must be further relations.

Problem 1. *Give a presentation of $\big[E(\Phi, A), E(\Phi, B)\big]/E(\Phi, R, AB)$ by generators and relations.*

In the present paper we have generalised the main results of [35] to all Chevalley groups. It is natural to ask, whether the same can be done also for the results of [36, 37]. For the results of [36] this does not have much sense, since for *commutative* rings they already follow from the birelative standard commutator formula, and are already contained in [10, 23, 13]. The fact that they can be proven by elementary calculations alone, without any use of localisation methods, is amusing, but does not have any tangible implications.

However, the analogues of results of [37] would be markedly new, and would have vital consequences. It is not even totally clear, whether the triple congruences for subgroups of $\mathrm{GL}(n, R)$, such as established in [37], Theorem 1, hold in this form in more general contexts, or should be replaced by fancier and longer ones.

Problem 2. *Prove analogues of [37], Theorem 1, for Chevalley groups.*

The partially relativised group $E(\Phi, B, A) = E(\Phi, A)^{E(\Phi, B)}$ is the smallest $E(\Phi, B)$-normalised subgroup containing $E(\Phi, A)$. It is easy to derive from Theorem 1 that $E(\Phi, B, A)$ is generated by the elementary conjugates $z_\alpha(a, b) = {}^{x_{-\alpha}(b)}x_\alpha(a)$, where $\alpha \in \Phi$, $a \in A$, $b \in B$. It is natural to ask, whether this result can be improved further. Namely, can one limit the roots α here to roots in the special part of some parabolic set of roots, as was done for $E(\Phi, R, A)$ by van der Kallen and Stepanov, see [16, 21, 22].

Problem 3. *Prove that $E(\Phi, B, A)$ is generated by $E(\Phi, R)$ together with the elementary conjugates $z_\alpha(a, b) = {}^{x_{-\alpha}(b)}x_\alpha(a)$, where $a \in A$, $b \in B$, while α runs over the special part of a fixed parabolic set of roots in Φ.*

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